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Exercise sheet 3 for Algebraic curves and the Weil conjectures

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Exercise 3.1. Let $k = \bar{k}$ be an algebraically closed field and $X \subset \mathbb{P}^n(k)$ a projective variety over k. Denote by $I(X) \subset k[T_0, \dots, T_n]$ the ideal of X. Let $F_1, \dots, F_r \in I(X)$ be homogeneous polynomials generating $I(X)$. Consider the $(n+1) \times r$-matrix $J := (\partial F_i/\partial T_j)_{1 \le i \le r, 0 \le j \le n}$; it is a matrix with coefficients homogeneous polynomials and hence we can evaluate it at any point $a \in k^{n+1}$.

(1) For $x \in X$ define $\mathrm{rk}(J(x))$ as the rank of the matrix $J(a_0, \dots, a_n)$, where $(a_0, \dots, a_n) \in k^{n+1}$ is some representative of x. Show that $\mathrm{rk}(J(x))$ is a well defined number, i.e., independent of the choice of the representative of x.

(2) Show that if $F \in k[T_0, \dots, T_n]$ is a homogeneous polynomial of degree s, then we have $s \cdot F = \sum_i T_i \partial F/\partial T_i$ (Euler's identity).

(3) Show that X is smooth at x (as defined in the lecture) if and only if $\mathrm{rk} J(x) = n - \dim \mathcal{O}_{X,x}$.

Exercise 3.2. Let k be a field and fix an algebraic closure $\bar{k}$. On the projective n-space over k $\mathbb{P}^n/k$ we defined Serre's twisted sheaves $\mathcal{O}_{\mathbb{P}^n}(r)$, for all $r \in \mathbb{Z}$. Show

$$\Gamma(\mathbb{P}^n/k, \mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} 0 & \text{if } r < 0 \\ k[T_0, \dots, T_n]_r & \text{if } r \ge 0, \end{cases}$$

where $k[T_0, \dots, T_n]_r$ is the k-vector space of homogeneous polynomials of degree r in $k[T_0, \dots, T_n]$.

Exercise 3.3. Let X be a topological space. An open cover $\mathcal{U}$ of X is a tuple $\mathcal{U} = (U_i)_{i \in I}$ of open subsets $U_i \subset X$ indexed by some set I, such that $X = \cup_i U_i$. We say that an open cover $\mathcal{V} = (V_j)_{j \in J}$ is a *refinement* of $\mathcal{U}$ and write $\mathcal{V} \prec \mathcal{U}$, if there exists a set map $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$. Show that $\prec$ induces the structure of a directed set on the set of open covers of X (i.e. $\mathcal{U} \prec \mathcal{U}$; $\mathcal{W} \prec \mathcal{V}$ and $\mathcal{V} \prec \mathcal{U} \Rightarrow \mathcal{W} \prec \mathcal{U}$; any two open covers have a common refinement).

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Exercise 3.4. Let X be a topological space and F a sheaf of abelian groups on X (written additively). Given an open cover $\mathcal{U} = (U_i)_{i\in I}$ define the two maps (where $U_{i,j} = U_i \cap U_j$ etc.)

$$\delta_0 : \prod_{i\in I} F(U_i) \to \prod_{(i,j)\in I^2} F(U_{i,j}), \quad (a_i)_i \mapsto (a_{i|U_{i,j}} - a_{j|U_{i,j}})_{(i,j)},$$

and

$$\delta_1 : \prod_{(i,j)\in I^2} F(U_{i,j}) \to \prod_{(i,j,k)\in I^3} F(U_{i,j,k}),$$

$$(a_{i,j})_{(i,j)} \mapsto (a_{i,j|U_{i,j,k}} - a_{i,k|U_{i,j,k}} + a_{j,k|U_{i,j,k}})_{(i,j,k)}.$$

(1) Show $\mathrm{Ker}(\delta_1) \supset \mathrm{Im}(\delta_0)$. Set

$$H^1(\mathcal{U}, F) := \mathrm{Ker}(\delta_1)/\mathrm{Im}(\delta_0).$$

(2) Let $\mathcal{V} = (V_j)_{j\in J} \prec \mathcal{U}$ be a refinement. Take $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$. Define $\tau^* : \prod_{(r,s)\in I^2} F(U_{r,s}) \to \prod_{(i,j)\in J^2} F(V_{i,j})$ via

$$\tau^*(a)_{i,j} := a_{\tau(i),\tau(j)|V_{i,j}}.$$

Show that τ^* induces a morphism $\tau^* : H^1(\mathcal{U}, F) \to H^1(\mathcal{V}, F)$.

(3) In the situation above assume we pick another $\tau' : J \to I$ such that $V_j \subset U_{\tau'(j)}$. Show that $\tau^* = \tau'^* : H^1(\mathcal{U}, F) \to H^1(\mathcal{V}, F)$. (*Hint:* For $a = (a_{r,s}) \in \mathrm{Ker}(\delta_1^{\mathcal{U}})$ show first that $a_{\tau(i),\tau(j)|V_{i,j}} - a_{\tau'(i),\tau'(j)|V_{i,j}} = a_{\tau(i),\tau'(i)|V_i|V_{i,j}} - a_{\tau(j),\tau'(j)|V_j|V_{i,j}}$.)

(4) Conclude that a refinement $\mathcal{V} \prec \mathcal{U}$ induces a well-defined morphism $H^1(\mathcal{U}, F) \to H^1(\mathcal{V}, F)$. Hence we can define

$$\check{H}^1(X, F) := \varinjlim H^1(\mathcal{U}, F),$$

where the limit is over the directed set of open covers (see Exercise 3.3).

Exercise 3.5. Let X be a quasi-projective variety over a field k and denote by $\mathcal{O}_X^\times$ the sheaf of abelian groups given by $U \mapsto \Gamma(U, \mathcal{O}_X)^\times$.

(1) Let L be an invertible sheaf on X/k. Thus there exists a open cover $\mathcal{U} = (U_i)$ such that $L_{|U_i} \cong \mathcal{O}_{U_i}$. Then the $\mathcal{O}_{U_{i,j}}$-linear isomorphism $\mathcal{O}_{U_{i,j}} \cong L_{|U_i|U_{i,j}} \cong L_{|U_j|U_{i,j}} \cong \mathcal{O}_{U_{i,j}}$ is induced by multiplication with a section $g_{j,i} \in \Gamma(U_{i,j}, \mathcal{O}_X^\times)$. Show that $(g_{i,j})_{(i,j)}$ defines a well defined element $\gamma(\mathcal{U}, L) \in H^1(\mathcal{U}, \mathcal{O}_X^\times)$.

(2) For a line bundle L which is trivialized on $\mathcal{U}$ denote by $\gamma(L)$ the image of $\gamma(\mathcal{U}, L)$ under the natural map $H^1(\mathcal{U}, \mathcal{O}_X^\times) \to \check{H}^1(X/k, \mathcal{O}_X^\times)$. Show that there is a well defined group isomorphism

$$\mathrm{Pic}(X/k) \xrightarrow{\simeq} \check{H}^1(X/k, \mathcal{O}_X^\times), \quad [L] \mapsto \gamma(L).$$