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Exercise sheet 2 for Algebraic curves and the Weil conjectures

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Exercise 2.1. Let k be a perfect field, $\bar{k}$ a fixed algebraic closure and X/k be an affine k-variety with coordinate ring $A = k[X]$. Show

(1) X/k is connected $\Longleftrightarrow$ the only idempotents in A are 1 and 0, i.e., if $e \in A$ satisfies $e^2 = e$, then $e \in \{0, 1\}$. (*Hint:* Show that X is not connected iff there are two proper non-trivial radical ideals $I, J \subset A$, which are comaximal, i.e., $I + J = A$, and use the Chinese Remainder Theorem.)

(2) X/k is irreducible $\Longleftrightarrow$ A is a domain. (*Hint:* Show that X is not irreducible iff there are two proper non-trivial radical ideals $I, J \subset A$ with $I \cap J = 0$ iff A is not a domain.)

(3) Let $x \in X = X(\bar{k})$ be a point and $x_0 \in (X/k)_0$ the associated closed point as defined in the lecture (in particular x_0 is a finite subset of X). Show that the set x_0 is closed in the Zariski topology of X/k and that it is the closure of x. (*Hint:* Use Exercise 1.2, (3).)

Exercise 2.2. Let k be a perfect field and fix an algebraic closure $\bar{k}$

(1) Set $X := Z(x^2 + y^2) \subset \mathbb{A}^2(\bar{k})$. Show that the k-variety X is irreducible if $\mathrm{char}(k) = 2$. If $\mathrm{char}(k) \neq 2$, then it is irreducible iff -1 is not a square in $k[X]$. In particular, it is never irreducible over $\bar{k}$.

(2) Set $Y := Z(x^2 + y^2 + z^2)$. Show that the k-variety Y is always irreducible. (*Hint:* Show that Y is irreducible over $\bar{k}$ by using the Eisenstein irreducibility criterion from algebra. Note that $k[x, y]$ is a UFD.)

Exercise 2.3. Let k be a perfect field of characteristic $\neq 2$ with algebraic closure $\bar{k}$. Consider the polynomials

(1) $f_1 = x^2 - (x^4 + y^4)$;

(2) $f_2 = x^2y + xy^2 - (x^4 + y^4)$.

Set $X_i := Z(f_i) \subset \mathbb{A}^2(\bar{k})$ and denote by $\overline{X} \subset \mathbb{P}^2(\bar{k})$ the closure of X. Decide whether X_i or $\overline{X}_i$ is smooth and if not compute the singular locus, i.e., the set of singular points.

Exercise 2.4. Recall that a *discrete valuation ring (DVR)* is a domain A with a discrete valuation on its function field K, i.e., a map $v : K^\times \to \mathbb{Z}$, with $v(ab) = v(a)+v(b)$, $v(a+b) \geq \min\{v(a), v(b)\}$, $a, b \in K^\times$, such that $A = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$.

Let C be a smooth affine curve$/k$ (i.e. it is an irreducible smooth 1-dimensional affine k-variety). Let $x \in (C/k)_0$ be a closed point and $\mathcal{O}_{C,x_0} := k[C]_{\mathfrak{m}_{x_0}}$ the localization of the coordinate ring of C at the maximal ideal $\mathfrak{m}_{x_0} \subset k[C]$ corresponding to x_0 (see Exercise 1.2). Show that $\mathcal{O}_{C,x_0}$ is a DVR. (*Hint:* Use that $\mathcal{O}_{C,x_0}$ is a regular local ring.)