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Exercise sheet 1 for Algebraic curves and the Weil conjectures

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Exercise 1.1. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements and fix an algebraic closure $\bar{\mathbb{F}}_q$. Let $X/\mathbb{F}_q$ be a quasi-projective variety/$\mathbb{F}_q$ with $X \subset \mathbb{P}^n(\bar{\mathbb{F}}_q)$. Recall that we defined in the lecture the sets $X(\mathbb{F}_{q^n})$ of $\mathbb{F}_{q^n}$-rational points and $(X/\mathbb{F}_q)_0$ the set of closed points. For a closed point $x_0 \in (X/\mathbb{F}_q)_0$ we define $\deg(x_0) = [k(x_0) : \mathbb{F}_q]$(= vector space dimension of $k(x_0)$ over $\mathbb{F}_q$), where $k(x_0)$ is the residue field associated with x_0. The aim of this exercise is to is show

$$|X(\mathbb{F}_{q^n})| = \sum_{\substack{x_0 \in (X/\mathbb{F}_q)_0 \\ \deg(x)|n}} \deg(x_0). \tag{1.1}$$

To this end proceed as follows:

(1) Show (recall) that $k(x_0)/\mathbb{F}_q$ is a finite Galois extension, for all $x_0 \in (X/\mathbb{F}_q)_0$. In particular, $k(x_0) = \mathbb{F}_{q^d}$ with $d = \deg(x_0)$.
(2) For x_0 and d as above, show that $|x_0| = d$ (here we view x_0 as a subset of $X(\bar{\mathbb{F}}_q)$).
(3) Given d, n, then: $\mathbb{F}_{q^d} \subset \mathbb{F}_{q^n} \iff d|n$.
(4) Show (1.1).

Exercise 1.2 (*)**.** Let k be a perfect field with algebraic closure $\bar{k}$ and X/k an affine variety/k, i.e. $X = Z(I) = \{a \in \bar{k}^n \mid f(a) = 0 \,\forall\, f \in I\}$, where $I \subset k[x_1, \ldots, x_n]$. Set $A := k[x_1, \ldots, x_n]/I$ and $\bar{A} = A \otimes_k \bar{k}$ $= \bar{k}[x_1, \ldots, x_n]/I \cdot \bar{k}[x_1, \ldots, x_n]$. Denote by $\varphi : A \hookrightarrow \bar{A}$ the natural inclusion and by $\mathrm{Max}(A)$ the set of maximal ideals of A.

(1) Show that there is a well defined homomorphism $\varphi^{-1} : \mathrm{Max}(\bar{A}) \to \mathrm{Max}(A)$. (*Hint:* Note that $A \hookrightarrow \bar{A}$ is an integral extension and use the going-up theorem from commutative algebra.)
(2) Let $x = (a_1, \ldots, a_n) \in X$. Hence $< x_1 - a_1, \ldots, x_n - a_n > \in \mathrm{Max}\ (A)$ and denote $\mathfrak{m}_x := \varphi^{-1}(< x_1 - a_1, \ldots, x_n - a_n >)$. Show that $A/\mathfrak{m}_x$ is the residue field $k(x)$ of x as defined in the lecture.

(3) Show that the map $X \to \mathrm{Max}(A)$, $x \mapsto \mathfrak{m}_x$ (notation as above) induces a bijection

$$(X/k)_0 \xrightarrow{1:1} \mathrm{Max}(A).$$

Exercise 1.3. Give a product formula for $\zeta(\mathbb{A}^n/\mathbb{F}_q, s)$ and $\zeta(\mathbb{P}^n/\mathbb{F}_q, s)$.

Exercise 1.4. Let $X/\mathbb{F}_2$ be the affine variety/$\mathbb{F}_2$ given by

$$X = Z(x^2 + x + y^2 + y + 1) \subset \mathbb{A}^2(\bar{\mathbb{F}}_2)$$

and denote by $\bar{X}$ its closure in $\mathbb{P}^2(\bar{\mathbb{F}}_2)$. Let $Z(X/\mathbb{F}_2, t) \in \mathbb{Q}[[t]]$ be the power series defined by $Z(X/\mathbb{F}_2, 2^{-s}) = \zeta(X/\mathbb{F}_2, s)$. Show

(1) $Z(X/\mathbb{F}_2, t) = 1 + 4t^2 + \text{higher terms} \ldots$
(2) $Z(\bar{X}/\mathbb{F}_2, t) = 1 + t + 5t^2 + \text{higher terms} \ldots$