VANISHING OF WITT-VECTOR COHOMOLOGY AND ACTION OF CORRESPONDENCES

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Abstract. This text essentially is the introduction of my Habilitation thesis (with only small changes) and is thus meant to be an overview of the articles [CR11], [CR12], [BER12] and [IR]. The point of view is a little bit changed from the introductions in the respective articles. Nevertheless I don't claim any originality in this notes.

Contents

1. Rational resolutions

Let k be a perfect field and X an integral k-scheme, which we always assume to be separated and of finite type over k. In order to study the singularities of X - and as we see later also certain arithmetic properties- it is natural and classical to compare it to a smooth k-scheme by choosing a resolution, i.e. a proper and birational k-morphism $\pi : Y \to X$ from a smooth scheme Y to X. One drawback is that resolutions in positive characteristic are known to exist only if X has dimension at most 3 (see [CP09]). Thus in general the existence of a resolution is an extra assumption. We know that X is normal if and only if $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ for some (or any) resolution $\pi : Y \to X$. Therefore in this case the global sections of a locally free sheaf $\mathcal{E}$ on X are the global sections of the locally free sheaf $\pi^*\mathcal{E}$ on the smooth scheme Y, $H^0(X, \mathcal{E}) = H^0(Y, \pi^*\mathcal{E})$. The wish to control also the higher cohomology groups and their duality theory in terms of Y, leads to the following definition (see [KKMS73, p. 50]).

Definition 1.2. Let X be an integral k-scheme of dimension d. Then we say a morphism $\pi : Y \to X$ is a *rational resolution* if the following conditions are satisfied:

(i) π is a resolution, i.e. it is proper and birational and Y is smooth.
(ii) $\pi_*\mathcal{O}_Y = \mathcal{O}_X$.
(iii) $R^i\pi_*\mathcal{O}_Y = 0$ for all $i \geq 1$.
(iv) $R^i\pi_*\omega_Y = 0$ for all $i \geq 1$, where $\omega_Y = \Omega^d_{Y/k}$.

Notice that the existence of a rational resolution of X already implies that X is normal and Cohen-Macaulay (CM). Indeed normality is equivalent to (ii) and CM follows by applying duality theory to the isomorphism $R\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$ and using (iv). In particular the reflexive hull of $\Omega^d_{X/k}$ is the canonical dualizing sheaf ω_X and the following equality holds

$$\omega_X = \pi_*\omega_Y.$$

Furthermore if X is proper and $\mathcal{F}$ is a coherent sheaf on X then we have a commutative diagram of isomorphisms

$$\begin{tikzcd} H^i(X,\mathcal{F})^\vee \arrow[r, "\simeq", "\text{duality}"'] & \mathrm{Ext}^{d-i}(\mathcal{F},\omega_X) \\ H^i(Y,L\pi^*\mathcal{F})^\vee \arrow[u, "\simeq"] \arrow[r, "\simeq", "\text{duality}"'] & \mathrm{Ext}^{d-i}(L\pi^*\mathcal{F},\omega_Y), \arrow[u, "\simeq"'] \end{tikzcd}$$

where $(-)^\vee$ denotes the duality functor on finite dimensional k-vector spaces. Thus once X admits a rational resolution, we can describe the coherent cohomology of X plus its duality theory in terms of a smooth scheme. For example, if the characteristic of k is zero and $\mathcal{L}$ is a big and nef invertible sheaf on X and if we assume that X has a rational resolution $\pi : Y \to X$, then we have $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < d$. This follows from the Kawamata-Viehweg vanishing theorem applied to the nef and big invertible sheaf $\pi^*\mathcal{L}$ on the smooth scheme Y.

But it is not clear that the existence of one rational resolution characterizes an intrinsic property of X. This motivates the following definition:

Definition 1.3. Let X be an integral k-scheme. Then we say that X has *rational singularities* if X admits at least one resolution and all resolutions are rational.

The problem with this definition is that it is not easy to find examples of rational resolutions. It is even not clear whether a smooth scheme has rational resolutions.

2. Rational singularities in characteristic zero

In this section we assume that the characteristic of k is zero. Then the theory of rational singularities is well developed and there are many important classes of singularities which are known to be rational. First of all, resolutions of singularities always exist in characteristic zero, by the fundamental work of Hironaka. Further we have the following two vanishing results

(i) (Grauert-Riemenschneider vanishing) Let X be any reduced k-scheme and $\pi : Y \to X$ a resolution, then $R^i\pi_*\omega_Y = 0$, for all $i > 0$.
(ii) Let $\pi : Y \to X$ be a proper and birational morphism between *smooth* k-schemes, then we have $R\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$.

Here (i) was first proven in [GR70, Satz 2.3] using analytic methods, alternatively it also follows from the Kawamata-Viehweg vanishing theorem together with Serre vanishing (see e.g. [KM98, Cor. 2.68]). Part (ii) was proven by Hironaka (see [Hir64, p. 144, (2)]). Let us recall Hironaka's proof: Since $\pi : Y \to X$ is birational, we have a rational map $X \dashrightarrow Y$. Now by [Hir64, p. 144, (1)] we can eliminate the indeterminacies by successively blowing up smooth loci in X. We obtain a commutative diagram of smooth k-schemes

$$\begin{tikzcd} Y' \arrow[r, "\pi'"] \arrow[d, "f"'] & X' \arrow[d, "g"] \arrow[dl, "\phi"'] \\ Y \arrow[r, "\pi"] & X, \end{tikzcd}$$

in which f and g are successive blow ups in smooth loci; in particular it is straightforward to compute that $Rf_*\mathcal{O}_{Y'} = \mathcal{O}_Y$ and $Rg_*\mathcal{O}_{X'} = \mathcal{O}_X$. Now the composition

$$R\pi_*\mathcal{O}_Y \xrightarrow{\phi^*} R\pi_*R\phi_*\mathcal{O}_{X'} = Rg_*\mathcal{O}_{X'} \xrightarrow{\pi'^*} Rg_*R\pi'_*\mathcal{O}_{Y'} = R\pi_*Rf_*\mathcal{O}_{Y'}$$

is an isomorphism since it equals $R\pi_*$ of the isomorphism $f^* : \mathcal{O}_Y \xrightarrow{\simeq} Rf_*\mathcal{O}_{Y'}$. But it factors over $Rg_*\mathcal{O}_{X'} = \mathcal{O}_X$, which implies $R^i\pi_*\mathcal{O}_Y = 0$ for all $i > 0$. Notice that the proofs of (i) and (ii) use characteristic zero in an essential way.

Since any two resolutions of an integral k-scheme X can be dominated by a third one, we obtain that once X has one rational resolution, then all its resolutions have to be rational. In particular: *An integral k-scheme X has rational singularities if and only if it is normal and there exists one resolution $\pi : Y \to X$, such that $R^i\pi_*\mathcal{O}_Y = 0$ for all $i > 0$.* Using duality theory and Grauert-Riemenschneider vanishing one easily sees that this is also equivalent to: *X is CM and there exists a resolution $\pi : Y \to X$, such that $\pi_*\omega_Y \cong \omega_X$.*

Examples of rational singularities are the singularities of toric varieties (see [KKMS73, I, Thm 14, c)]), quotient singularities (see e.g. [Vie77, Prop. 1]) and log terminal singularities (see e.g. [Elk81], [Kov00]). Let us also mention two special cases:

Isolated singularities of surfaces. Assume k is algebraically closed and let X be a normal surface with one isolated singularity $x_0 \in X$. Let $\pi : Y \to X$ be a resolution and let $E = \pi^{-1}(x_0)_{\mathrm{red}}$ be the reduced preimage of x_0 with irreducible components $E = \cup_i E_i$. In [Art66] the fundamental cycle of E is defined to be the smallest positive cycle Z supported on E such that $(Z.E_i) \le 0$ for all i. It always satisfies $Z \ge \sum_i E_i$. Denote by $p(Z) = 1 - (\dim H^0(Z, \mathcal{O}_Z) - \dim H^1(Z, \mathcal{O}_Z))$ the arithmetic genus of Z (viewed as a closed subscheme of X). Then by [Art66, Thm 3] X has rational singularities if and only if $p(Z) = 0$. Also if X has rational singularities and the irreducible components E_i are smooth then $E_i \cong \mathbb{P}^1$.

Cone singularities. Let k be algebraically closed and X_0 a smooth connected projective k-scheme and $\mathcal{L}$ an ample line bundle on X_0. Let $X = \mathrm{Proj}(S[z])$, with $S = \oplus_{n\ge 0} H^0(X_0, \mathcal{L}^{\otimes n})$, be the projective cone of $(X_0, \mathcal{L})$, which has an isolated singularity at the vertex $v \in X$. Assume $\mathcal{L}$ is sufficiently ample (more precisely $H^i(X_0, \mathcal{L}^{\otimes n}) = 0$, for all $i, n \ge 1$). Then X has rational singularities if and only if $H^i(X_0, \mathcal{O}_{X_0}) = 0$. (This is well-known.)

3. Rational singularities in characteristic $p > 0$

Now we assume that k is a perfect field of characteristic $p > 0$.

Definition 3.2. We say that an integral and normal k-scheme X is a *finite quotient* if there exists a finite and surjective morphism $X' \to X$ from a smooth and integral k-scheme X' to X. We say that X is a *tame finite quotient* if there exists such a morphism whose degree is prime to p.

Notice that if X is integral and normal and $\pi : X' \to X$ is finite and surjective of degree prime to p with X' smooth and integral and if in addition the field extension $k(X')/k(X)$ is normal, then π is a Galois covering and we have $X = X'/G$, with $G = \mathrm{Aut}_X(X')$.

Definition 3.3. Let S be a k-scheme and X and Y integral k-schemes, which map to S. Then we say that *X and Y are properly birational over S* if there exists an integral k-scheme Z over S and proper and birational S-morphisms $Z \to X$ and $Z \to Y$. In this case Z is called a *proper birational correspondence between X and Y over S.*

For example, if X and Y are integral k-schemes which are proper over S and with non-empty open subsets $U \subset X$ and $V \subset Y$ which are isomorphic over S, then X and Y are properly birational over S. (Indeed the closure in $X \times_S Y$ of the graph of $U \cong V$ with its reduced scheme structure is a proper birational correspondence.)

Now the main result of Chapter 2 is the following:

Theorem 3.4 ([CR11, Thm 4.3.1]). *Let S be a k-scheme. Let X and Y be two tame finite quotients over S, which are properly birational over S. Denote by $\pi_X : X \to S$ and $\pi_Y : Y \to S$ the structure maps. Then any proper birational correspondence Z between X and Y over S induces isomorphisms of $\mathcal{O}_S$-modules*

$$R^i\pi_X\mathcal{O}_X \cong R^i\pi_*\mathcal{O}_Y \quad \text{and} \quad R^i\pi_X\omega_X \cong R^i\pi_*\omega_Y, \quad \text{for all } i \geq 0.$$

Furthermore, these isomorphisms depend only on the isomorphism $k(X) \cong k(Y)$ induced by Z.

We will give the main idea and explain the main technical tool needed to prove this theorem in Section 5 of this introduction. The proof is actually also valid in characteristic 0 and gives a new proof of the above statement without using resolutions of singularities or Grauert-Riemenschneider vanishing.

The theorem implies:

(i) Let X be an integral k-scheme, which has a resolution. Then one resolution of X is rational if and only if all its resolutions are rational if and only if X has rational singularities.

(ii) Let $f : X \xrightarrow{\simeq} Y$ be a proper and birational morphism between integral schemes with rational singularities. Then

$$Rf_*\mathcal{O}_X \cong \mathcal{O}_Y, \quad Rf_*\omega_X \cong \omega_Y.$$

(iii) Tame finite quotients, which have some resolution, have rational singularities.

(iv) Toric varieties have rational singularities (by (i) and [KKMS73, I, Thm 14, c)]).

Let us give an arithmetic application of (ii) above. For each k-scheme X we have the sheaves of Witt vectors of length n, $W_n\mathcal{O}_X$ and of infinite length

$$W\mathcal{O}_X = \varprojlim_n W_n\mathcal{O}_X$$

at our disposal. We set

$$K_0 = \mathrm{Frac}(W(k)).$$

We have the following general statement.

Proposition 3.5. *Let $\pi : X \to Y$ be a proper morphism between two k-schemes and assume $R^i\pi_*\mathcal{O}_X = 0$ for all $i \geq 1$. Let $Y' \to Y$ be any morphism of k-schemes and denote by $\pi' : X' = X \times_Y Y' \to Y'$ the projection. Then*

$$R^i\pi'_*W\mathcal{O}_{X'} \otimes K_0 = 0 \quad \text{for all } i \geq 1.$$

Proof. For all $n \geq 1$ we have an exact sequence of sheaves of abelian groups

$$0 \to W_{n-1}\mathcal{O}_X \xrightarrow{V} W_n\mathcal{O}_X \to \mathcal{O}_X \to 0,$$

where V is the Verschiebung, $V(a_0, \ldots, a_{n-2}) = (0, a_0, \ldots, a_{n-2})$ and the map on the right is the restriction $(a_0, \ldots, a_{n-1}) \mapsto a_0$. Hence $R^i\pi_*W_n\mathcal{O}_X = 0$, for all $n, i \geq 1$, by induction. Further we have exact sequences for all $i \geq 1$

$$0 \to R^1 \varprojlim_n R^{i-1}\pi_*W_n\mathcal{O}_X \to R^i\pi_*W\mathcal{O}_X \to \varprojlim_n R^i\pi_*W_n\mathcal{O}_X \to 0.$$

Thus also $R^i\pi_*W\mathcal{O}_X = 0$ for all $i \geq 1$. (For the case $i = 1$ notice that the restriction maps $\pi_*W_n\mathcal{O}_X \to \pi_*W_{n-1}\mathcal{O}_X$ are surjective, which implies the vanishing of $R^1 \varprojlim_n \pi_*W_n\mathcal{O}_X$.)

Now assume $i : Y' \hookrightarrow Y$ is a closed immersion. Hence $X' \hookrightarrow X$ is a closed immersion and we denote by $\mathcal{I}$ its ideal sheaf. We obtain a long exact sequence

$$\cdots \to R^i\pi_*W\mathcal{O}_X \otimes K_0 \to i_*R^i\pi'_*W\mathcal{O}_{X'} \otimes K_0 \to R^{i+1}\pi_*W\mathcal{I} \otimes K_0 \to \cdots .$$

By the above the term on the left vanishes and the term on the right vanishes by [CR12, Prop 4.6.1], which is a slight modification of [BBE07, Thm 2.4, (i)]. This gives the statement in this case. In the general case the statement follows by factoring $Y' \to Y$ into a closed immersion followed by a flat morphism, e.g. $Y' \hookrightarrow Y' \times Y \to Y$, and flat base change. □

Assume $k = \mathbb{F}_q$ is the field with $q = p^r$ elements. Then the r-th power of the absolute Frobenius on X yields a K_0-linear endomorphism ϕ on the vector space $H^i(X, W\mathcal{O}_X) \otimes_{W(\mathbb{F}_q)} K_0$ and on Berthelot's rigid cohomology $H^i_{\mathrm{rig}}(X/K_0)$. Further if X is proper both spaces are finite dimensional K_0-vector spaces. Denote by $H^i_{\mathrm{rig}}(X/K_0)^{<1}$ the part of rigid cohomology on which the eigenvalues of ϕ in the algebraic closure of K_0 have p-adic valuation less than 1 (where the valuation is normalized by $v(q) = 1$). Then it follows from the main result of [BBE07] that we have a ϕ-equivariant isomorphism

$$H^i(X, W\mathcal{O}_X) \otimes K_0 \cong H^i_{rig}(X/K_0)^{<1},$$

for all $i \geq 0$ and for all proper k-schemes X. In the case where X is smooth and proper rigid cohomology coincides with crystalline cohomology and this result was proven by Bloch in the case $p \neq 2$ and $\dim X < p$ (see [Blo77]) and by Illusie in general (see [Ill79]). In particular the Lefschetz trace formula for rigid cohomology yields

$$|X(k)| \equiv \sum_{i\geq 0}(-1)^i \mathrm{Tr}(\phi|H^i(X, W\mathcal{O}_X) \otimes K_0) \mod q. \tag{3.5.1}$$

We have the following application of (ii) above.

Corollary 3.6. *Assume k is a finite field and let S be a k-scheme. Let X and Y be S-schemes with rational singularities and assume that there exists a proper and birational S-morphism $\pi : X \to Y$. Then*

$$|X_s(k')| \equiv |Y_s(k')| \quad mod\ |k'|$$

for all finite field extensions k' of k and all k'-rational points s in S, where X_s and Y_s denote the fibers of X and Y over s, respectively.

Proof. Let k' be a finite field extension of k. Then the base change of π over k', $\pi_{k'} : X_{k'} \to Y_{k'}$, is a proper and birational morphism between $S_{k'}$-schemes with rational singularities. Hence it suffices to prove the statement for $k' = k$. Furthermore it clearly suffices to show, that $|\pi^{-1}(y)| \equiv 1 \bmod |k|$ for all k-rational points $y \in Y$. Set $A = H^0(\pi^{-1}(y), \mathcal{O}_{\pi^{-1}(y)})$, for some $y \in Y(k)$. Since π is proper, surjective and geometrically connected, $\operatorname{Spec} A \to y$ is finite, surjective and geometrically connected, hence radical. The perfectness of k thus yields, that A is an artinian local k-algebra with residue field k. In particular

$$H^0(\pi^{-1}(y), W\mathcal{O}_{\pi^{-1}(y)}) \otimes K_0 = W(A) \otimes K_0 = K_0,$$

where the second equality follows from $F \circ V = p = V \circ F$ on $W(A)$, where $F : W(A) \to W(A)$, $(a_0, a_1, \ldots) \mapsto (a_0^p, a_1^p, \ldots)$ is the Frobenius morphism on the Witt vectors. Further (ii) above and Proposition 3.5 give

$$H^i(\pi^{-1}(y), W\mathcal{O}_{\pi^{-1}(y)}) \otimes K_0 = 0 \quad \text{for all } i \geq 1.$$

Thus the statement follows from (3.5.1). □

Actually the formulation of the above corollary is a bit complicated, as we see in the proof it would suffice to say $|\pi^{-1}(y)| \equiv 1 \bmod |k|$, without any reference to a base scheme S. The reason we wrote it this way is to indicate what the general statement should be. Namely in case X and Y are properly birational equivalent over S, but there exists no map π, the statement should still be the same. This clearly holds in case we have resolutions of singularities, but we cannot prove it at the moment in the general case.

The result above should be compared to [FR05, Thm 1.1, Rmk 3.2], where the authors obtain the same conclusion under the assumption that S is smooth, X and Y have quotient singularities and are proper and dominant over S and the condition $R\pi_*\mathcal{O}_X \cong \mathcal{O}_Y$ (the condition, which is really needed in the corollary above) is replaced by a cycle theoretic condition.

3.6.1. In positive characteristic rational singularities behave less convenient than in characteristic zero. For example, the existence of a resolution is an extra assumption (at least in dimension ≥ 4), there are finite quotients $\mathbb{A}^n_k/G$ with $G = \mathbb{Z}/p^r\mathbb{Z}$, which are not CM (see e.g. [ES80]) and hence also don't have rational singularities. Furthermore Kodaira vanishing and hence also Grauert-Riemenschneider vanishing is wrong in positive characteristic. These problems lead to the search of a wider class of singularities and in view of Proposition 3.5 it seems a natural try to replace the structure sheaf by the sheaf of Witt-vectors modulo torsion.

4. Witt-rational singularities

In this section we assume that k is a perfect field of characteristic $p > 0$. Further *k-schemes are assumed to be quasi-projective over k*. In the following if $\mathcal{A}$ is an abelian category and $A \in \mathcal{A}$ is an object we denote by $A_{\mathbb{Q}}$ the image of A in the category $\mathcal{A}_{\mathbb{Q}}$, which has the same objects as $\mathcal{A}$ and the homomorphisms are given by $\mathrm{Hom}_{\mathcal{A}_{\mathbb{Q}}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$. (If the reader wishes he can replace the subscript $(-)_{\mathbb{Q}}$ with $(-) \otimes \mathbb{Q}$, but in general the statements for $(-)_{\mathbb{Q}}$ are finer, e.g. if A is an abelian group then $A_{\mathbb{Q}} = 0$ means that there exists a non-zero integer n with $nA = 0$.)

We want to define a version of rational singularities where we replace $\mathcal{O}$ by $W\mathcal{O}$. First we need a replacement for the canonical sheaf. Recall from [Ill79] that the de Rham-Witt pro-complex $W_\bullet\Omega^\bullet_X$ of a k-scheme X is a projective system $W_{n+1}\Omega^\bullet_X \to W_n\Omega^\bullet_X$, $n \geq 1$, of differential graded algebras (dga's), which satisfies $W_n\Omega^0_X = W_n\mathcal{O}_X$ for all $n \geq 1$, and is equipped with a map of graded pro-rings

$$F : W_\bullet\Omega^\bullet_X \to W_{\bullet-1}\Omega^\bullet_X,$$

called the Frobenius and a map of graded pro-groups,

$$V : W_\bullet\Omega^\bullet_X \to W_{\bullet+1}\Omega^\bullet_X,$$

called the Verschiebung, which in degree 0 are compatible with the Frobenius and Verschiebung morphisms on $W_\bullet\mathcal{O}$ and satisfy

$$FV = p, \quad FdV = d, \quad V(xF(y)) = V(x)y, \quad \text{for all } x \in W_\bullet\Omega^\bullet_X, y \in W_{\bullet+1}\Omega^\bullet_X$$

$$Fd[a] = [a]^{p-1}d[a], \quad \text{for all } a \in \mathcal{O}_X,$$

where $d : W_\bullet\Omega^\bullet_X \to W_\bullet\Omega^{\bullet+1}_X$ is the differential and $[-] : \mathcal{O}_X \to W_\bullet\mathcal{O}_X$, $a \mapsto [a] = (a, 0, \ldots)$ is the Teichmüller lift. Furthermore, $W_\bullet\Omega^\bullet_X$ is universal with the above property, i.e. it maps uniquely to any such pro-complex. We have

$$W_1\Omega^\bullet_X = \Omega^\bullet_X.$$

The *de Rham-Witt complex of* X is defined to be the limit of the pro-complex,

$$W\Omega^\bullet_X := \varprojlim W_n\Omega^\bullet_X;$$

it is a dga equipped with Frobenius and Verschiebung satisfying the relations above. If X is smooth and proper it was proven by Bloch in the case $\dim X < p$ and in general by Illusie (see [Blo77], [Ill79]) that the hypercohomology of the complex $W\Omega^\bullet_X$ is canonically isomorphic to the crystalline cohomology of X, $H^i_{\mathrm{crys}}(X/W(k)) = H^i(X, W\Omega^\bullet_X)$. Furthermore, in this case the spectral sequence induced by the naïve filtration of the de Rham-Witt complex

$$E_1^{i,j} = H^i(X, W\Omega^j_X)_{\mathbb{Q}} \Rightarrow H^*(X, W\Omega^\bullet_X)_{\mathbb{Q}}$$

degenerates at E_1 and gives a canonical decomposition

$$H^n(X, W\Omega^\bullet_X)_{\mathbb{Q}} = \bigoplus_{i+j=n} H^i(X, W\Omega^j_X)_{\mathbb{Q}}.$$

This is reminiscent of the Hodge decomposition in characteristic zero and one is tempted to view $W\Omega^{\dim X}_{X,\mathbb{Q}}$ as a replacement of ω_X. This is supported by the following result of Ekedahl: Let X be a smooth k-scheme of equidimension d and with structure map $\pi : X \to \operatorname{Spec} k$. We obtain a morphism of finite type $\pi_n : W_nX = \operatorname{Spec} W_n\mathcal{O}_X \to \operatorname{Spec} W_n(k)$. Then by [Eke85, I, Thm 4.1] there is a canonical isomorphism

$$\pi_n^! W_n(k) \cong W_n\Omega^d_X[d]$$

in the derived category $D^b_c(W_n\mathcal{O}_X)$ of bounded complexes of sheaves of $W_n\mathcal{O}_X$-modules with coherent cohomology groups. Here $\pi_n^! : D^b_c(W_n) \to D^b_c(W_n\mathcal{O}_X)$ is the extraordinary inverse image constructed in [Har66, VII, Cor 3.4]. This leads to the following definition:

Definition 4.2 (cf. [CR12, Def 4.1.2])**.** Let $\pi : X \to \operatorname{Spec} k$ be a k-scheme of pure dimension d. Then for $n \geq 1$ we set (with above notation)

$$W_n\omega_X := H^{-d}(\pi_n^! W_n(k)).$$

One can show that the $(W_n\omega_X)_n$ form a projective system equipped with Frobenius and Verschiebung which is called the *Witt canonical system of* X and is denoted by $W_\bullet\omega_X$. We set

$$W\omega_X := \varprojlim W_\bullet\omega_X.$$

The Witt canonical system has various nice properties, e.g. if X is normal $W_1\omega_X = \omega_X$ is the canonical sheaf of X, if $j : U \hookrightarrow X$ is the inclusion of a smooth open subset which contains all 1-codimensional points of X, we have $W_\bullet\omega_X = j_*W_\bullet\Omega^d_U$, if $f : X \to Y$ is a projective morphism between schemes of pure dimension d, then there is a pushforward $f_* : f_*W_\bullet\omega_X \to W_\bullet\omega_Y$ and finally we have an exact sequence for all $n \geq 1$

$$0 \to W_{n-1}\omega_X \xrightarrow{p} W_n\omega_X \xrightarrow{F^{n-1}} \omega_X,$$

which is surjective on the right if X is CM.

Definition 4.3. We say that an integral normal scheme is a *topological finite quotient* if there exists a universal homeomorphism $u : X \to X'$ with X' a finite quotient in the sense of Definition 3.2.

Recall, that a map between integral and normal k-schemes is a universal homeomorphism if and only if it is finite, surjective and purely inseparable.

Definition 4.4. A morphism between two integral k-schemes $f : X \to Y$ is a *quasi-resolution* if X is a topological finite quotient and f is projective, surjective, generically finite and generically purely inseparable.

Notice that in particular any resolution $f : X \to Y$ is a quasi-resolution. By a result of de Jong (cf. [dJ97, Cor. 5.15]) quasi-resolutions always exist. The main result of [CR12] is the following theorem:

Theorem 4.5 ([CR12, Thm 4.3.3])**.** *Let Y be a topological finite quotient and $f : X \to Y$ a quasi-resolution. Then we have isomorphisms in $D^b(W\mathcal{O}_{Y,\mathbb{Q}})$*

$$W\mathcal{O}_{Y,\mathbb{Q}} \xrightarrow{f^* \simeq} Rf_*W\mathcal{O}_{X,\mathbb{Q}}, \quad Rf_*W\omega_{X,\mathbb{Q}} \cong f_*W\omega_{X,\mathbb{Q}}[0] \xrightarrow{f_* \simeq} W\omega_{Y,\mathbb{Q}},$$

which are compatible with Frobenius and Verschiebung.

The idea of the proof is similar to one for the proof of Theorem 3.4; it is explained in Section 5 of this introduction.

Definition 4.6. Let X be an integral k-scheme.

(i) We say that X has *Witt-rational singularities* if for all quasi-resolutions $f : Y \to X$ the following conditions are satisfied:
  (a) $f^* : W\mathcal{O}_{X,\mathbb{Q}} \xrightarrow{\simeq} f_*W\mathcal{O}_{Y,\mathbb{Q}}$ is an isomorphism.
  (b) $R^if_*W\mathcal{O}_{Y,\mathbb{Q}} = 0$, for all $i \geq 1$.
  (c) $R^if_*W\omega_{Y,\mathbb{Q}} = 0$, for all $i \geq 1$.

(ii) We say that X has *$W\mathcal{O}$-rational singularities* if for all quasi-resolutions only (a) and (b) above are satisfied.

(iii) We say X has BE-Witt-rational singularities if for any alteration $g : Z \to X$ with Z smooth the pullback morphism

$$g^* : W\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Q} \to Rg_*W\mathcal{O}_Z \otimes_{\mathbb{Z}} \mathbb{Q}$$

splits in the derived category of sheaves of abelian groups on X.

BE-Witt-rational singularities were first defined in [BE08] (and there they were called Witt-rational singularities). This definition is motivated by the following result of Kovács (see [Kov00]): An integral scheme X over a field of characteristic zero has rational singularities if and only if there exists an alteration $g : Z \to X$ with Z smooth such that the pullback $g^* : \mathcal{O}_X \to Rg_*\mathcal{O}_Z$ splits in the derived category of $\mathcal{O}_X$-modules.

We have:

- In (i) and (ii) (resp. in (iii)) above it suffices to consider a single quasi-resolution (resp. alteration). Since any two quasi-resolutions (resp. alterations) can be dominated by a third one this follows from Theorem 4.5 (resp. from $f_* \circ f^* = \deg f$ for an alteration f between smooth schemes).
- If $u : X \to X'$ is a universal homeomorphism between normal k-schemes, then X has Witt-rational singularities (resp. $W\mathcal{O}$-rational singularities, resp. BE-Witt-rational singularities) if and only if X' has.
- Topological finite quotients have Witt-rational singularities as follows immediately from Theorem 4.5.
- We have the following chain of implications (see [CR12, Prop 4.4.17]):

$$\begin{aligned} \text{rational singularities} &\Rightarrow \text{Witt-rational singularities} \\ &\Rightarrow W\mathcal{O}\text{-rational singularities} \\ &\Rightarrow \text{BE-Witt-rational singularities} \end{aligned}$$

  Notice that the first implication is strict, e.g. finite quotients in positive characteristic are in general not CM. We conjecture that the second implication is in fact an equivalence, i. e. some version of Grauert-Riemenschneider vanishing for the Witt canonical sheaf modulo torsion should hold.

- If X has Witt-rational singularities and $f : Y \to X$ is a quasi-resolution we also have $f_* W\omega_{Y,\mathbb{Q}} \cong W\omega_{X,\mathbb{Q}}$ (see [CR12, Lem 4.3.4]).

Using Theorem 4.5 and the main result of [BBE07] as explained in section 3 we obtain:

Corollary 4.7 ([CR12, Cor 4.4.15, Cor 4.4.16])**.** *Let X and Y be two integral and projective k-schemes, which have Witt-rational singularities. Assume that X and Y are quasi-birational, i.e. there exists an integral k-scheme Z with two quasi-resolutions $\pi_X : Z \to X$, $\pi_Y : Z \to Y$. Then we have Frobenius equivariant isomorphisms*

$$H^i_{\mathrm{rig}}(X/K_0)^{<1} \cong H^i_{\mathrm{rig}}(Y/K_0)^{<1} \quad \textit{for all } i > 0.$$

In particular if k is a finite field and k' is a finite extension of k we have

$$|X(k')| \equiv |Y'(k')| \quad \textit{mod } |k'|.$$

We refer to [CR12, Section 4] for some more elaborations on the notion of Witt-rational singularities, e.g. a description of Witt rational singularities using alterations from smooth schemes (see Theorem 4.5.6 *ibid.*), or using morphisms with smooth and rationally connected generic fiber (see Theorem 4.8.1 *ibid.*).

Finally we want to give two more examples of $W\mathcal{O}$-rational singularities to compare with the situation in characteristic zero (see the end of Section 2 of this introduction):

Isolated singularities. Assume k is algebraically closed and let X be a normal surface over k, which is defined over a finite field and has one isolated singularity $x_0 \in X$. Let $\pi : Y \to X$ be a resolution such that $E = \pi^{-1}(x_0)_{\mathrm{red}}$ is a strict normal crossing divisor. Then X has $W\mathcal{O}$-rational singularities if and only if E is a tree of $\mathbb{P}^1$'s. Notice that this condition only depends on the *set-theoretic* exceptional divisor in contrast to what happens if we consider $\mathcal{O}$ instead of $W\mathcal{O}$ (see [Art66]). In [CR12, Thm 4.6.7] we also give a necessary and sufficient condition for an isolated singularity in a higher dimensional normal scheme over a finite field to be $W\mathcal{O}$-rational in case a nice resolution exists.

Cone singularities. Let X_0 be a smooth, projective and geometrically connected k-scheme and $\mathcal{L}$ and ample line bundle on X_0. Then the projective cone of $(X_0, \mathcal{L})$ has $W\mathcal{O}$-rational singularities if and only if $H^i(X_0, W\mathcal{O}_{X_0})_{\mathbb{Q}} = 0$ for all $i \geq 1$ (see [CR12, Thm 4.7.4]). Notice that in contrast to what happens in characteristic zero we don't need to assume that $\mathcal{L}$ is *sufficiently* ample.

5. Action of correspondences

In this section we assume that all our scheme are quasi-projective over a perfect ground field k.

Let S be a k-scheme. We denote by $\mathcal{C}_S$ the category with objects the S-schemes, which are smooth over k and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}_S}(X/S, Y/S) = \varinjlim_{V} \mathrm{CH(V)},$$

where the limit is over all reduced closed subschemes $V \subset X \times_S Y$, such that the projection to Y restricts to a proper morphism $V \to Y$, and $\mathrm{CH}(V) = \oplus_i \mathrm{CH}_i(V)$ denotes the graded Chow groups of cycles modulo rational equivalence. Here the composition is defined as follows (cf. [CR11, Sec 1]): For $X, Y, Z \in \mathcal{C}_S$ and $V \subset X \times_S Y$ and $W \subset Y \times_S Z$ closed subschemes, which are proper over Y and Z, respectively, we have the following cartesian diagram,

$$\begin{tikzcd}
(V \times_k Z) \cap (X \times_k W) \arrow[r] \arrow[d] & (V \times_k Z) \times_k (X \times_k W) \arrow[d] \\
X \times_k Y \times_k Z \arrow[r, "\Delta"] & (X \times_k Y \times_k Z) \times (X \times_k Y \times_k Z),
\end{tikzcd}$$

where the lower horizontal map is the diagonal embedding and the right vertical map is the natural closed immersion. The intersection $(V \times_k Z) \cap (X \times_k W) = V \times_Y W$ is proper over Z a fortiori it is proper over $X \times_S Z$ and we have Fulton's refined Gysin homomorphism (see [Ful98, 6])

$$\Delta^! : \mathrm{CH}((V \times_k Z) \times_k (X \times_k W)) \to \mathrm{CH}(V \times_Y W).$$

Then the composition in $\mathcal{C}_S$ is defined via

$$[W] \circ [V] = p_{1,3_*}(\Delta^!([V \times_k Z] \times [X \times_k W])),$$

where $p_{1,3} : V \times_Y W \to X \times_S Z$ is induced by the projection from $X \times_S Y \times_S Z$.

Now for an object $f : X \to S$ in $\mathcal{C}_S$ we denote

$$\mathcal{H}_\bullet(X/S) = \bigoplus_{i,j \geq 0} R^i f_* W_\bullet \Omega^j_X;$$

it is a projective system of $W_\bullet\mathcal{O}_S$-modules. Notice that the $\mathcal{O}_S$-module $\mathcal{H}_1(X/S)$ equals $\bigoplus_{i,j} R^i f_* \Omega^j_X$. Further we define the $W\mathcal{O}_S$-module

$$\mathcal{H}(X/S) = \bigoplus_{i,j \geq 0} R^i f_* W\Omega^j_X.$$

Notice that the Frobenius F, the Verschiebung V and the differential d naturally act on $\mathcal{H}_\bullet(X/S)$ and $\mathcal{H}(X/S)$. The main technical tool for proving the results in sections 3 and 4 is the following theorem which is a recollection of [CR11, Sec 1.3.18, Lem 1.3.19, Thm 3.1.8, Sec 3.2.3 and Prop 3.2.4] for the case $\mathcal{H}_1$ and [CR12, Sec 3.5, in particular Prop 3.5.4] in the general case.

Theorem 5.2. *There exist functors*

$$\mathcal{H}_\bullet : \mathcal{C}_S \to (W_\bullet\mathcal{O}_S - \text{modules}), \quad X/S \mapsto \mathcal{H}_\bullet(X/S)$$

and

$$\mathcal{H} : \mathcal{C}_S \to (W\mathcal{O}_S - \text{modules}), \quad X/S \mapsto \mathcal{H}(X/S),$$

such that for any morphism $h : X \to Y$ of S-schemes which are smooth over k, we have

$$\mathcal{H}_\bullet([\Gamma^t_h]) = h^*, \quad \mathcal{H}([\Gamma^t_h]) = h^*,$$

where $[\Gamma^t_h]$ denotes the transpose of the graph of h viewed as a morphism $Y/S \to X/S$ in the category $\mathcal{C}_S$ and h^* denotes the natural pullback map in both cases. Furthermore, for any $\alpha : Y/S \to X/S$ in $\mathcal{C}_S$, the morphisms $\mathcal{H}_\bullet(\alpha)$ and $\mathcal{H}(\alpha)$ are compatible with F, V and d.

In the case $\mathcal{H}_1$ the assumption on the quasi-projectivity of the schemes in question made at the beginning of this section is in fact not necessary and it suffices to work with schemes separated and of finite type over k. Let $f : X \to S$ and $g : Y \to S$ be two objects in the category $\mathcal{C}_S$. The morphisms $\mathcal{H}_\bullet([Z])$ and $\mathcal{H}([Z])$ for an integral closed subscheme $Z \subset X \times_S Y$ of codimension c in $X \times_k Y$ which is proper over Y is in fact induced by the following morphisms for $j \geq 0$ in the derived category of $W_\bullet(k)$-modules (see [CR12, Lem 5.1.9]):

$$
\begin{aligned}
(5.2.1) \qquad R[Z]^j_\bullet : Rf_* W_\bullet \Omega^j_X &\xrightarrow{p_1^*} R(f \circ p_1)_* W_\bullet \Omega^j_{X \times_k Y} \\
&\xrightarrow{\cup cl([Z])} R(f \circ p_1)_* R\underline{\Gamma}_Z W_\bullet \Omega^{j+c}_{X \times_k Y}[c] \\
&\xrightarrow{\simeq} R(g \circ p_2)_* R\underline{\Gamma}_Z W_\bullet \Omega^{j+c}_{X \times_k Y}[c] \\
&\xrightarrow{p_{2*}} Rg_* W_\bullet \Omega^{j+c-\dim X}_Y [c - \dim X].
\end{aligned}
$$

We refer to the Sections 2, 3 and 5 of [CR12] for the notation and references and just remark here the following:

- The maps $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ denote the projections.
- The first arrow p_1^* is induced by the natural pullback map.
- The second arrow is induced by cup product with the cycle class $cl([Z])$ constructed by Gros in [Gro85].
- The third arrow is only a $W_\bullet(k)$-linear isomorphism. (It is an isomorphism since $Z \subset X \times_S Y$).
- The forth map is the pushforward, the construction of which highly relies on results of Ekedahl (see [Eke84]) and duality theory. Also that this map is compatible with F, V, and d is a non-trivial fact due to general constructions of Ekedahl. It is here where we need that Z is proper over Y.

The maps $\mathcal{H}_\bullet([Z])$ and $\mathcal{H}([Z])$ are then defined by taking H^i and $H^i \circ R\varprojlim$ respectively and summing over the various $i, j \geq 0$. One checks by hand that these morphisms become $W\mathcal{O}_S$-linear. Also notice that we do not know whether the map in the derived category is compatible with composition, we know this only after taking cohomology.

In order to obtain the vanishing results from the Sections 3 and 4 we also need the following proposition.

Proposition 5.3 ([CR11, Prop 3.2.2], [CR12, Lem 3.6.1, Lem 3.6.2])**.** *In the above situation assume that the dimension of Z is equal to the dimension of Y (i.e. $c = \dim X$) and fix an integer $r \geq 1$. Then the restriction of $\mathcal{H}_1([Z])$ to $R^i f_* \Omega^j_X \to R^i g_* \Omega^j_Y$ and the restriction of $\mathcal{H}([Z])_{\mathbb{Q}}$ to $R^i f_* W\Omega^j_{X,\mathbb{Q}} \to R^i g_* W\Omega^j_{Y,\mathbb{Q}}$ is the zero map in the following cases:*

(i) $\mathrm{codim}_X\, p_1(Z) = r$, $j > \dim X - r$ *and* $i \geq 0$.
(ii) $\mathrm{codim}_Y\, p_2(Y) = r$, $j < r$ *and* $i \geq 0$.

In particular if X and Y are integral and the closures of the projections of Z to X and Y are strict subsets, then

$$\mathcal{H}_1([Z]) = 0 \quad on \quad R^i f_*(\mathcal{O}_X \oplus \omega_X) \to R^i g_*(\mathcal{O}_Y \oplus \omega_Y)$$

and

$$\mathcal{H}([Z])_{\mathbb{Q}} = 0 \quad on \quad R^i f_*(W\mathcal{O}_{X,\mathbb{Q}} \oplus W\omega_{X,\mathbb{Q}}) \to R^i g_*(W\mathcal{O}_{Y,\mathbb{Q}} \oplus W\omega_{Y,\mathbb{Q}}).$$

In case resolutions of singularities hold for all closed subschemes of X and Y of codimension at least r one obtains the same vanishing results for $\mathcal{H}_n([Z])$ for all $n \geq 1$. (The reason we have it for $\mathcal{H}_1([Z])$ is the existence of a Künneth decomposition for $\Omega^j_{X \times_k Y}$, which we do not have for $W_n\Omega^j_{X \times_k Y}$.)

With these techniques at hand it is easy to prove the Theorems 3.4 and 4.5. Let us give a sample:

Proof of Theorem 3.4 for X and Y are smooth. Let $f : X \to S$ and $g : Y \to S$ be two S-schemes, which are integral and smooth over k and let Z be a proper birational correspondence between them, i.e. Z is an integral S-scheme and there exist proper and birational S-morphisms $Z \to X$ and $Z \to Y$. Clearly we may assume that Z is an integral closed subscheme of $X \times_S Y$. We obtain two maps in the derived category of sheaves of k-vector spaces on S (see (5.2.1))

$$R[Z]_1 : Rf_*(\mathcal{O}_X \oplus \omega_X) \to Rg_*(\mathcal{O}_Y \oplus \omega_Y)$$

and

$$R[Z^t]_1 : Rg_*(\mathcal{O}_Y \oplus \omega_Y) \to Rf_*(\mathcal{O}_X \oplus \omega_X).$$

Here $Z^t \subset Y \times_S X$ denotes the transpose of Z. On the other hand using the birationality assumptions and the localization sequence for Chow groups it is straightforward to check that we have

$$[Z] \circ [Z^t] = [\Delta] + E \quad \text{in } \mathrm{Hom}_{\mathcal{C}_S}(Y/S, Y/S),$$

where $\Delta \subset Y \times_S Y$ denotes the diagonal and E is a cycle whose both projections to Y are strict closed subsets. Therefore $\mathcal{H}_1(E) = 0$ on $R^i g_*(\mathcal{O}_Y \oplus \omega_Y)$ for all $i \geq 0$ by the proposition above. We obtain

$$H^i(R[Z]_1 \circ R[Z^t]_1) = \mathcal{H}_1([Z]) \circ \mathcal{H}_1([Z^t]) = \mathcal{H}_1([\Delta] + E) = \mathrm{id}_{R^i g_*(\mathcal{O}_Y \oplus \omega_Y)}.$$

Similar with $H^i(R[Z^t]_1 \circ R[Z]_1)$, which yields the Theorem 3.4 in this case. □

Notice that under the above assumptions we in fact proved that $R[Z]_1$ is a k-linear isomorphism in the derived category

$$R[Z]_1 : Rf_*(\mathcal{O}_X \oplus \omega_X) \cong Rg_*(\mathcal{O}_Y \oplus \omega_Y).$$

This together with the compatibility of $R[Z]_\bullet$ with F, V and d enables us to use Ekedahl's Nakayama Lemma to deduce:

Theorem 5.4 ([CR12, Thm 5.1.10]). *Under the above assumptions we have isomorphism in the derived category of sheaves of $W(k)$-modules on S*

$$Rf_*(W\mathcal{O}_X \oplus W\omega_X) \cong Rg_*(W\mathcal{O}_Y \oplus W\omega_Y).$$

As a corollary we get:

Corollary 5.5 ([CR12, Thm 5.1.12]). *Let X and Y be smooth, projective and birational over k. Then we have isomorphisms for all $i \geq 0$, which are compatible with F, V, d*

$$H^i(X, \mathrm{W}\mathcal{O}_X) \cong H^i(Y, \mathrm{W}\mathcal{O}_Y), \quad H^i(X, \mathrm{W}\omega_X) \cong H^i(Y, \mathrm{W}\omega_Y).$$

Modulo torsion the $W\mathcal{O}$-part of the above corollary was proven before by Ekedahl using ℓ-adic methods, see [Eke83].

6. Rational points over finite fields for regular models of Hodge type ≥ 1

In this section k denotes a perfect field of characteristic $p > 0$. Let A be the local ring at a closed point of a smooth curve over k. Denote by η its generic point and by s its special point. Let X be an integral, regular and projective A-scheme with smooth generic fiber (we could also allow X to have a model with $W\mathcal{O}$-rational singularities and smooth generic fiber) and assume that the degree map induces an isomorphism $\mathrm{CH}_0(X_\eta \times_{k(\eta)} \overline{k(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, where $\overline{k(\eta)}$ is an algebraic closure of $k(\eta)$. Then a particular case of [CR12, Thm 4.8.1] yields the vanishing $H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for all $i > 0$. Now one would like to conclude from this by a kind of base change argument, similar to the reasoning in the proof of Proposition 3.5 the vanishing modulo torsion of the Witt vector cohomology of the special fiber X_s. Unfortunately we cannot make this work at the moment. But using the machinery of p-adic Hodge theory we prove in [BER12] the following (stronger) version of a mixed characteristic analog of the above situation.

Theorem 6.2 ([BER12, Thm 1.3]). *Let R be a discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field k. Denote by η the generic point of $\operatorname{Spec} R$ and by s its closed point. Let X be a regular, proper and flat R-scheme and assume $H^i(X_\eta, \mathcal{O}_{X_\eta}) = 0$ for some $i \geq 1$. Then*

$$H^i(X_s, W\mathcal{O}_{X_s})_{\mathbb{Q}} = 0.$$

Using (3.5.1) we get the following:

Corollary 6.3 ([BER12, Thm 1.1]). *Let R and X be as above. Additionally assume that k is a finite field, X_η is geometrically connected and that we have $H^i(X_\eta, \mathcal{O}_{X_\eta}) = 0$ for all $i \geq 1$. Then for any finite field extension k' of k we have*

$$|X_k(k')| \equiv 1 \quad mod\ |k'|.$$

A particular case of the generalized Hodge conjecture predicts the equivalence of the following two statements for $i \geq 1$ in the case X_η is projective over η:

(i) $H^i(X_\eta, \mathcal{O}_{X_\eta}) = 0$.

(ii) The algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X_\eta)$ is supported in codimension one, i.e. there exists some non-empty open subset $U \subset X_\eta$ such that the restriction $H^i_{dR}(X_\eta) \to H^i_{dR}(U)$ is the zero map.

(It is known that (ii) implies (i), the generalized Hodge conjecture is needed for the other direction.) Denote by $\bar\eta$ a geometric point over η, then by Artin's comparison theorem of singular cohomology with étale cohomology (ii) is also equivalent to:

(ii)' For some prime ℓ, there exists a non-empty open subset $U \subset X_\eta$ such that the restriction map $H^i_{\text{ét}}(X_{\bar\eta}, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(U_{\bar\eta}, \mathbb{Q}_\ell)$ on ℓ-adic étale cohomology vanishes.

Assuming condition (ii)' instead of (i) for all $i \geq 1$ the corollary was already proved before by Esnault in [Esn06] (there R is even allowed to be of equicharacteristic p). Thus the above corollary was already predicted by the generalized Hodge conjecture.

In case the model X has semi-stable reduction Theorem 6.2 follows from standard results of p-adic Hodge theory ("the Newton polygon of the filtered (φ, N)-module $H^i_{\text{log-crys}}(X_s/W(k)) \otimes \mathrm{Frac}(R) \cong H^i_{\mathrm{dR}}(X_\eta/\eta)$ lies above its Hodge polygon"). Using de Jong's alteration theorem and cohomological descent one reduces the proof of Theorem 6.2 to the following theorem (see the introduction of [BER12] for details):

Theorem 6.4 ([BER12, Thm 1.5]). *Let R be as in Theorem 6.2 and $f : Y \to X$ an alteration, i.e. a projective, surjective and generically finite morphism, between regular, flat and finite type R-schemes. Then for all $i \geq 0$ the pullback map*

$$f_s^* : H^i(X_s, W\mathcal{O}_{X_s})_{\mathbb{Q}} \hookrightarrow H^i(Y_s, \mathrm{W}\mathcal{O}_{Y_s})_{\mathbb{Q}}$$

is injective, where $f_s : X_s \to Y_s$ denotes the pullback of f along $s \hookrightarrow \operatorname{Spec} R$.

Let us explain the idea of the proof. By assumption we can factor f as a composition of a regular closed immersion $i : Y \hookrightarrow P := \mathbb{P}^r_X$ with the projection $\pi : P \to X$. Denote by $i_s : Y_s \hookrightarrow P_s$ and $\pi_s : P_s \to X_s$ their pullback along $s \hookrightarrow \operatorname{Spec} R$. Since everything is flat over R the closed immersion i_s is still regular. The theorem now follows from the existence of $W_n\mathcal{O}_{X_s}$-linear maps

$$\tau_{i_s,\pi_s,n} : Rf_{s*}W_n\mathcal{O}_{Y_s} \to W_n\mathcal{O}_{X_s},$$

which are compatible with restriction and have the property that the composition

$$W_n\mathcal{O}_{X_s} \xrightarrow{f_s^*} Rf_{s*}W_n\mathcal{O}_{Y_s} \xrightarrow{\tau_{i_s,\pi_s,n}} W_n\mathcal{O}_{X_s} \tag{6.4.1}$$

is multiplication with the degree of f, for all $n \geq 1$.

First we explain how to define a map $\tau_f : Rf_*\mathcal{O}_Y \to \mathcal{O}_X$ as above: By duality theory to define such a map is equivalent to defining an $\mathcal{O}_Y$-linear morphism

$$\mathcal{O}_Y \to f^!\mathcal{O}_X = i^!\pi^!\mathcal{O}_X \cong (\bigwedge^r \mathcal{I}/\mathcal{I}^2)^\vee \otimes_{\mathcal{O}_Y} i^*\Omega^r_{P/X}, \tag{6.4.2}$$

where $\mathcal{I}$ is the ideal sheaf of the regular closed immersion $Y \hookrightarrow P$. Thus one can take τ_f to be the map induced by $\wedge^r d : \bigwedge^r \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^r_{P/X}$ and check that it is in fact independent of the chosen factorization $f = \pi \circ i$. The analog of property (6.4.1) is proved using computations with the residue symbol. Unfortunately there is no relative duality theory developed so far for $W_n\mathcal{O}$-modules. (The absolute duality theory for the de Rham-Witt complex of smooth schemes over a perfect field developed by Ekedahl and which is heavily used in [CR12] does not suffice in this situation.) The way around is to unravel the duality theory in the definition of τ_f and try to imitate this on the $W_n\mathcal{O}$-level using an ad hoc approach: Recall that the fundamental local isomorphism yields an isomorphism

$$(\bigwedge^r \mathcal{I}/\mathcal{I}^2)^\vee \otimes_{\mathcal{O}_Y} i^*\Omega^r_{P/X} \cong \bar{i}^*\mathcal{E}\mathrm{xt}^r(i_*\mathcal{O}_Y, \Omega^r_{P/X}),$$

where $\bar{i} : (Y, \mathcal{O}_Y) \to (P, i_*\mathcal{O}_Y)$ is the canonical map of locally ringed spaces. Therefore (6.4.2) gives a canonical map

$$\mathcal{O}_Y \to \bar{i}^*\mathcal{E}\mathrm{xt}^r(i_*\mathcal{O}_Y, \Omega^r_{P/X}), \tag{6.4.3}$$

which can be described explicitly using symbols (see [BER12, Sec 4]). Then the map τ_f is given by the composition of Rf_*(6.4.3) with the trace map

$$Rf_*\bar{i}^*\mathcal{E}\mathrm{xt}^r(i_*\mathcal{O}_Y, \Omega^r_{P/X}) \to \mathcal{O}_X,$$

which itself is induced by the composition of $R\pi_*$ applied to the natural map

$$\mathcal{E}\mathrm{xt}^r(i_*\mathcal{O}_Y, \Omega^r_{P/X}) \to \mathcal{E}\mathrm{xt}^r(\mathcal{O}_P, \Omega^r_{P/X}) \cong \Omega^r_{P/X}[r]$$

with the projective trace map

$$R\pi_*\Omega^r_{P/X}[r] \cong \mathcal{O}_X.$$

Now the general approach is to replace in the above construction $\mathcal{O}$ by $W\mathcal{O}$ and $\Omega^r_{P/X}$ by the degree r part of the relative de Rham-Witt complex $W_n\Omega^r_{P_s/X_s}$ constructed in [LZ04] and to imitate the above construction. This occupies Sections 5, 6 and 7 of [BER12]. In this way we obtain the map $\tau_{i_s,\pi_s,n}$ for general n. The property (6.4.1) is then proved in Section 8 of *loc. cit.* by a comparison of the map $\tau_{i_s,\pi_s,n}$ with the reduction of τ_f modulo $\mathfrak{m}^{n+1}$. Notice that we don't check that the maps $\tau_{i_s,\pi_s,n}$ are independent of the factorization $f = \pi \circ i$, which is not needed for the injectivity result of Theorem 6.4.

7. Reciprocity Functors

Recall that the key tool to prove the main results in the Sections 3 and 4 was the action of certain correspondences described in section 5. Also one of the main ingredients in proving Theorem 6.2 was the construction of a kind of pushforward map $Rf_{s*}W_n\mathcal{O}_{Y_s} \to W_n\mathcal{O}_{X_s}$ (with the notation from Section 6), which naturally raises the question whether there is a cycle action lurking in the background. (For example the map $\tau_{i_s,\pi_s,n}$ from Section 6 should then be given by the action of the graph of f, $\Gamma_f \subset Y \times_Y X$.) Therefore one would like to view the cohomology theories appearing in the Sections 3–6 as something *motivic*. This is very vague and in fact cannot be stated more precisely at the moment, since the motivic theory developed so far aims to explain only $\mathbb{A}^1$-homotopy invariant theories, which none of the above cohomology theories is. In Chapter 5 we want to go a little step in the direction of non-homotopy invariant motives. More precisely we are seeking for a replacement of the homotopy invariant Nisnevich sheaves with transfers which play a crucial role in the theory of motives as developed by Voevodsky et al. (see e.g. [Voe00b]). Let us explain this in a little bit more detail.

Let k be a perfect ground field of characteristic $p \geq 0$. Denote by Sm_k the category of smooth separated k-schemes and denote by $SmCor_k$ the category with the same objects but with morphisms given by

$$\mathrm{Hom}_{SmCor_k}(X, Y) = Cor(X, Y).$$

Here $Cor(X, Y)$ denotes the group of finite correspondences from X to Y, i.e. the free abelian group generated by integral closed subschemes $V \subset X \times_k Y$ which are finite and surjective over a connected component of X. The composition is defined in a similar way to the one defined for $\mathcal{C}_S$ at the beginning of Section 5, but instead of Fulton's refined Gysin homomorphism one can use the physical intersection of cycles with multiplicities given by Serre's *Tor*-formula. (Notice that there is a natural *contravariant* functor $SmCor_k^o \to \mathcal{C}_k$.) Now Voevodsky defines a *homotopy invariant Nisnevich sheaf with transfers* to be a contravariant additive functor $F : SmCor_k^o \to$ (abelian groups), which is a sheaf in the Nisnevich topology on Sm_k with the property that the pullback along the projection $p_1^* : F(X) \to F(X \times_k \mathbb{A}^1)$ is an isomorphism for all $X \in Sm_k$ (see [Voe00b, 3.1]). We denote by $\mathbf{HI}_{\mathrm{Nis}}$ the category of homotopy invariant Nisnevich sheaves with transfers.

In the following a point over k, or for short a *k-point*, is a point $x = \operatorname{Spec} K$, where K is a finitely generated field extension of k. We denote by pt_k the category of k-points, with the obvious morphisms. For a k-point x we define

$$\hat{F}(x) = \varinjlim_{U \ni x} F(U), \tag{7.1.4}$$

where the limit is over all smooth models of x. Then it follows from [Voe00b, Prop. 3.1.11] and [Voe00a, Cor. 4.19] that for any smooth k-scheme X with generic point η and any $F \in \mathbf{HI}_{\mathrm{Nis}}$ the natural map $F(X) \to \hat{F}(\eta)$ is injective. Furthermore, for any map of k-points $f : x \to y$ we naturally obtain a pullback $f^* : \hat{F}(y) \to \hat{F}(x)$ and if f is finite also a pushforward or trace

$$f_* = \mathrm{Tr}_f = \mathrm{Tr}_{x/y} : \hat{F}(x) \to \hat{F}(y).$$

(Since the transpose of the graph of f spreads out to a finite correspondence from a model of y to a model of x.) There are some natural compatibilities, which the pushforward and pullback satisfy. A functor $\mathrm{pt}_k^o \to$ (abelian groups) with these properties is called a Mackey functor and we denote by $\mathbf{MF}$ the category of Mackey functors. Thus we obtain a functor

$$\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{MF}, \quad F \mapsto \hat{F}$$

which is conservative by [Voe00a, Prop. 4.20]. On the other hand only considering points makes it harder to express the homotopy-invariance, which involves the 1-dimensional scheme $\mathbb{A}^1_k$. (In fact it is possible via the theory of Rost cycle modules, but there one needs some extra data for each geometric discrete valuation defined on finitely generated field extensions of k, see [Dég03].) Therefore it is natural to consider the category $\mathrm{Reg}_k^{\leq 1}$ of regular at most 1 dimensional k-schemes, which are of finite type over some k-point and we can define the category $\mathrm{Reg}^{\leq 1}Cor_k$ in an analog way to $SmCor_k$. We make the following definition:

Definition 7.2. A *Mackey functor with specialization map* is a contravariant functor $\mathcal{M} : \mathrm{Reg}^{\leq 1}Cor_k^o \to$ (abelian groups), which satisfies the following conditions:

(Nis) $\mathcal{M}$ is a sheaf in the Nisnevich topology on $\mathrm{Reg}_k^{\leq 1}$.

(Inj) For all open immersions $j : U \hookrightarrow V$ between connected schemes in $\mathrm{Reg}_k^{\leq 1}$ the restriction map
$$j^* : \mathcal{M}(V) \hookrightarrow \mathcal{M}(U)$$
is injective.

(FP) For all connected $X \in \mathrm{Reg}_k^{\leq 1}$ with generic point η the natural map
$$\varinjlim_{U \subset X} \mathcal{M}(U) \xrightarrow{\simeq} \mathcal{M}(\eta)$$
is an isomorphism, where the limit is over all non-empty open subsets $U \subset X$.

We denote the category of Mackey functors with specialization map by **MFsp**.

In particular if C is a regular curve, which is of finite type over some k-point and $P \in C$ is a closed point, the graph of the inclusion $P \hookrightarrow C$ defines a finite correspondence from P to C and hence induces a specialization map
$$s_P : \mathcal{M}(C) \to M(P)$$
for any $\mathcal{M} \in \mathbf{MFsp}$. This explains the name. We obtain a conservative functor
$$\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{MFsp}, \quad F \mapsto \hat{F}$$
and the $\mathbb{A}^1$-homotopy invariance translates in
$$s_1 = s_0 : \hat{F}(\mathbb{A}^1_x) \to \hat{F}(x)$$
for all k-points x. A key example for an object in $\mathbf{HI}_{\mathrm{Nis}}$ is given by the algebraic group $\mathbb{G}_m$ and more general by any semi-abelian variety. Now we want to relax the $\mathbb{A}^1$-homotopy invariance condition in such a way that also $\mathbb{G}_a$ and more general any connected smooth commutative algebraic group satisfies this new condition. It was suggested by B. Kahn that the modulus condition of Rosenlicht-Serre might be a good replacement. This motivates the following definition:

Definition 7.3. A *reciprocity functor* is a Mackey functor with specialization map $\mathcal{M}$, such that for any regular connected curve, which is projective over some k-point, any non-empty open subset $U \subset C$ and any section $a \in \mathcal{M}(U)$, there exists an effective divisor $\mathfrak{m}$ with support equal to $C \setminus U$ satisfying the following condition
$$(MC) \quad \sum_{P \in U} v_P(f)\mathrm{Tr}_{P/x_C} s_P(a) = 0, \quad \text{for all } f \in k(C)^\times \text{ with } f \equiv 1 \bmod \mathfrak{m}$$
where the sum is over all closed points $P \in U$, v_P is the associated normalized discrete valuation, $x_C = \operatorname{Spec} H^0(C, \mathcal{O}_C)$ and the condition $f \equiv 1 \bmod \mathfrak{m}$ means $\sum_{P \in C \setminus U} v_P(f-1) \cdot [P] \geq \mathfrak{m}$.

The condition (MC) is exactly the modulus condition of Rosenlicht-Serre for smooth, connected and commutative algebraic groups in [Ser84, III, §1] (there only for curves over an algebraically closed ground field). Also we learned that in the 90's Bruno Kahn was working on a similar but more global approach of Mackey functors with reciprocity, see [Kah]. We denote by **RF** the category of reciprocity functors. The name "reciprocity functor" comes from the following fact, which is proven as in [Ser84, III]: Let $\mathcal{M}$ be a reciprocity functor. Then for any regular connected curve C which is projective over some k-point and with generic point η, there exists a family of biadditive pairings
$$(-,-)_P : \mathcal{M}(\eta) \times \mathbb{G}_m(\eta) \to \mathcal{M}(x_C)$$
indexed by the closed points $P \in C$ satisfying the following conditions:

(i) For all $P \in C$ and all $a \in \mathcal{M}(\eta)$, there exists an integer $n_0 \geq 1$ such that $(a, f)_P = 0$ for all $f \in k(C)^\times$ with $v_P(f-1) \geq n_0$.

(ii) For all $P \in C$, all open neighborhoods U of P and all $a \in \mathcal{M}(U)$ we have $(a, f)_P = v_P(f)\mathrm{Tr}_{P/x_C} s_P(a)$ for all $f \in \mathbb{G}_m(\eta)$.

(iii) For all $a \in \mathcal{M}(\eta)$ and all $f \in \mathbb{G}_m(\eta)$ we have $\sum_{P \in C}(a, f)_P = 0$. (Notice that the sum is finite by (ii).)

Furthermore the family $\{(-,-)_P\}_{P \in C}$ is uniquely determined by the above properties and is called the family of local symbols attached to $\mathcal{M}$. For $P \in C$ as above we can use the symbol to define the following groups

$$\mathrm{Fil}^0_P \mathcal{M}(\eta) = \mathcal{M}_{C,P} = \varinjlim_{U \ni P} \mathcal{M}(U),$$

and for $n \geq 1$

$$\mathrm{Fil}^n_P \mathcal{M}(\eta) = \{a \in \mathcal{M}(\eta) \,|\, (a, u)_P = 0 \text{ for all } u \in 1 + \mathfrak{m}_P^n\},$$

where $\mathfrak{m}_P$ denotes the maximal ideal in $\mathcal{O}_{C,P}$. Thus the $\mathrm{Fil}^\bullet_P \mathcal{M}(\eta)$ form an increasing and exhaustive filtration of $\mathcal{M}(\eta)$. We denote by $\mathbf{RF}_n$, $n \geq 0$ the full subcategory of $\mathbf{RF}$ whose objects satisfy $\mathrm{Fil}^n_P \mathcal{M}(\eta) = \mathcal{M}(\eta)$ for all curves C and closed points $P \in C$ as above. We have the following examples (see [IR, Sec 2]):

- We have a conservative functor $\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{RF}_1$, $F \mapsto \hat{F}$.
- Any smooth, connected and commutative algebraic group G over k defines a reciprocity functor, more precisely we have:

$$G \in \begin{cases} \mathbf{RF}_0 & \text{if } G \text{ is an Abelian variety,} \\ \mathbf{RF}_1 & \text{if } G \text{ is an semi-Abelian variety,} \\ \mathbf{RF} \setminus \bigcup_{n \geq 0} \mathbf{RF}_n & \text{if } G \text{ is unipotent.} \end{cases}$$

  In case $G = \mathbb{G}_m$ the symbol equals the tame symbol composed with the norm,

$$(a, f)_P = (-1)^{v_P(a) v_P(f)} \mathrm{Nm}_{P/x_C}(\frac{a^{v_P(f)}}{f^{v_P(a)}});$$

  in case $G = \mathbb{G}_a$ the symbol equals the residue map,

$$(a, f)_P = \mathrm{Res}_P(a \frac{df}{f}).$$

- For $n \geq 1$ the functor $\mathrm{pt}_k \mapsto$ (abelian groups), $x \mapsto K^M_n(x) = n$-th Milnor K-theory of $k(x)$, can be extended to a reciprocity functor, which for $n = 1$ coincides with $\mathbb{G}_m$. (This can be checked directly via the theory of Rost cycle modules from [Ros96], but is also a special case of the first example by [Dég03].)
- For all $n \geq 1$, the absolute Kähler differentials $\mathrm{Reg}^{\leq 1}_k \ni X \mapsto \Omega^n_{X/\mathbb{Z}}$ have the structure of a reciprocity functor. For $n = 0$ it coincides with $\mathbb{G}_a$.

7.3.1. *K-groups of reciprocity functors.* Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be a finite family of reciprocity functors in $\mathbf{RF}_1$ which admit *fine symbols*, i.e. for each regular, connected curve C, which is projective over some k-point and with generic point η and each closed point $P \in C$ there is a biadditive pairing

$$\partial_P : \mathcal{M}_i(\eta) \times \mathbb{G}_m(\eta) \to \mathcal{M}_i(P),$$

such that $(-,-)_P = \mathrm{Tr}_{P/x_C} \circ \partial_P$. Then one can define define for each k-point x the Somekawa K-group $K(x; \mathcal{M}_1, \ldots, \mathcal{M}_n)$. It is defined as a quotient of the tensor product of Mackey functors $(\mathcal{M}_1 \otimes^M \ldots \otimes^M \mathcal{M}_n)(x)$ by the relations

$$(SK) \quad \sum_{P \in C} \mathrm{Tr}_{P/x}(s_P(a_1) \otimes \ldots \partial_P(a_{i(P)}, f) \ldots \otimes s_P(a_n)) = 0,$$

where the a_i's are in $\mathcal{M}_i(\eta)$ and we assume that for all $P \in C$ there exists an index $i(P)$ such that $a_i \in \mathcal{M}_{i,C,P}$ for all $i \neq i(P)$ and $f \in k(C)^\times$ is any non-zero function. This definition was first introduced for semi-Abelian varieties by Somekawa following an idea of Kato in [Som90]. It was modified to work in various other situations by several authors, see [RS00], [Akh04], [KY11]. There are two problems with this definition: First it does not generalize to reciprocity functors which do not have a fine symbol (such as $\mathbb{G}_a$ in the case k has positive characteristic) and second it has a priori only the structure of a Mackey functor and not of a reciprocity functor. The K-groups of reciprocity functors constructed in [IR, Sec 4] solve both this problems.

More precisely, let $\mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{N}$ be reciprocity functors. Then we say that a collection Φ of maps $\Phi(X) : \mathcal{M}_1(X) \times \ldots \times \mathcal{M}_n(X) \to \mathcal{N}(X)$, for integral $X \in \mathrm{Reg}_k^{\leq 1}$, is an *n-linear map of reciprocity functors* if all the $\Phi(X)$ are n-linear maps of $\mathbb{Z}$-modules, which are compatible with pullbacks and satisfy a certain projection formula and for all integers $r_1, \ldots, r_n \geq 1$ and any regular curve which is projective over some k-point and has generic point η and all closed points $P \in C$ we have

$$(L3) \quad \Phi(\mathrm{Fil}_P^{r_1}\mathcal{M}_1(\eta) \times \ldots \times \mathrm{Fil}_P^{r_n}\mathcal{M}_n(\eta)) \subset \mathrm{Fil}_P^{\max\{r_1,\ldots,r_n\}}\mathcal{N}(\eta).$$

Then in the above above situation we have the following Theorem:

Theorem 7.4 ([IR, Thm 4.2.4])**.** *The functor from* **RF** *to the category of abelian groups, which sends a reciprocity functor $\mathcal{N}$ to the abelian group of n-linear maps from $\mathcal{M}_1 \times \ldots \times \mathcal{M}_n$ to $\mathcal{N}$ is representable by a reciprocity functor $T(\mathcal{M}_1, \ldots, \mathcal{M}_n)$.*

Roughly the construction of T also starts from the tensor product of Mackey functors as in the Somekawa case but the relation (SK) is replaced by the following relation:

$$\sum_{P \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_P(f)\mathrm{Tr}_{P/x}(s_P(a_1) \otimes \ldots \otimes s_P(a_n)),$$

where C is a regular connected curve which is projective over some k-point x, the $\mathfrak{m}_i$ are effective divisors on C, such that $a_i \in \mathcal{M}(C \setminus |\mathfrak{m}_i|)$ has modulus $\mathfrak{m}_i$, for all $i = 1, \ldots, n$, and $f \in k(C)^\times$ is a function satisfying $f \equiv 1 \mod \max_i\{\mathfrak{m}_i\}$. And then one needs to divide out some more relations to guarantee the property (Inj). Actually one would like to call T a tensor functor and it has also all the properties (commutativity, compatibility with direct sums, unit) but the associativity. In general the universal property only gives a surjection

$$T(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \to T(T(\mathcal{M}_1, \mathcal{M}_2), \mathcal{M}_3).$$

Here again one problem which is in the way of associativity is the unnatural property (Inj) (but also (L3) causes problems). Let us point out that it is a priori not clear at all whether the groups $T(\mathcal{M}_1, \ldots, \mathcal{M}_n)(x)$ coincide with the Somekawa K-groups, in case the latter are defined. But in fact they do in all the examples below, where it is shown by computing both sides directly. We have the following computations:

Theorem 7.5 ([IR, Thm 5.1.8])**.** *Let $F_1, \ldots, F_n \in \mathbf{HI}_{\mathrm{Nis}}$ be homotopy invariant Nisnevich sheaves with transfers. There exists a canonical and functorial isomorphism of reciprocity functors*

$$T(\hat{F}_1, \ldots, \hat{F}_n) \xrightarrow{\sim} (F_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} F_n)\widehat{\ }.$$

This result was suggested by the main Theorem in [KY11], which proves the corresponding statement on étale k-points with the left hand side replaced by the Somekawa type K-groups. In particular we obtain an isomorphism

$$K(k; \hat{F}_1, \ldots, \hat{F}_n) \cong T(\hat{F}_1, \ldots, \hat{F}_n)(k).$$

But notice that we don't know the relation between the two groups on function fields in positive characteristic. (Except in the case where the F_i are split tori, see below.) In the same way as in [KY11] one can deduce form the above theorem:

Corollary 7.6 ([IR, Cor 5.2.5])**.** *Let $X_1, \dots, X_n$ be smooth projective k-schemes and $r \geq 0$ an integer. Then for all S-points x we have an isomorphism*

$$T(\mathrm{CH}_0(X_1), \dots, \mathrm{CH}_0(X_n), \mathbb{G}_m^{\times r})(x) \cong \mathrm{CH}_0(X_{1,x} \times_x \dots \times_x X_{n,x}, r).$$

The version of this corollary using Somekawa K-groups was first proved in [RS00] (see also [Akh04]).

Theorem 7.7 ([IR, Thm 5.3.3])**.** *For all $n \geq 1$ and all k-points x, there is a canonical isomorphism*

$$T(\mathbb{G}_m^{\times n})(x) \xrightarrow{\simeq} K_n^M(x),$$

where $K_n^M(x)$ denotes the n-th Milnor K-group of $k(x)$.

The version of this theorem using Somekawa K-groups is proved in [Som90]. In particular we have an isomorphism of Mackey functors $K(-; T_1, \dots, T_n) \cong T(T_1, \dots, T_n)$, where the T_i are split tori.

A form of the following theorem was suggested by B. Kahn.

Theorem 7.8 ([IR, Thm 5.4.7])**.** *There is a natural morphism of reciprocity functors*

$$\theta : \Omega^n_{-/\mathbb{Z}} \to T(\mathbb{G}_a, \mathbb{G}_m^{\times n}),$$

which is an isomorphism if the characteristic of k is zero.

The map θ is constructed using a presentation of $\Omega^n_{x/\mathbb{Z}}$ as a quotient of $k(x) \otimes_{\mathbb{Z}} (k(x)^\times)^{\otimes n}$ from [BE03b]. In characteristic zero it is easy to see that the filtration $\mathrm{Fil}^\bullet_P \mathbb{G}_a(\eta)$, for $C \in \mathrm{Reg}_k^{\leq 1}$ a curve with generic point η and $P \in C$ a closed point, is the pole order filtration. It is therefore straightforward to check, that the collection of maps

$$\mathbb{G}_a(X) \otimes_{\mathbb{Z}} \mathbb{G}_m(X)^{\otimes n} \to \Omega^n_{X/\mathbb{Z}}(X), \quad (a, b_1, \dots, b_n) \mapsto a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_n}{b_n},$$

for integral $X \in \mathrm{Reg}_k^{\leq 1}$, satisfies the condition (L3) above and hence induces an $(n+1)$-linear map of reciprocity functors. The universal property of T thus gives a map in the other direction, which is inverse to θ.

Combining the above theorem with the theorem of Bloch and Esnault in [BE03a] which identifies $\Omega^{n-1}_{x/\mathbb{Z}}$ with the additive higher Chow groups of x of level n and with modulus 2, we obtain

$$T(\mathbb{G}_a, \mathbb{G}_m^{\times n-1})(x) \cong \mathrm{TCH}^n(x, n, 2), \quad \text{if } \mathrm{char}(k) = 0.$$

In positive characteristic it is not true any more that the map θ above is an isomorphism. The reason is essentially that in positive characteristic the algebraic group (or the reciprocity functor) $\mathbb{G}_a$ has more endomorphisms than in characteristic zero, namely the absolute Frobenius comes into the game. This forces the following:

Corollary 7.9 ([IR, Cor 5.4.12])**.** *Assume the characteristic of k is $p > 0$. Then for all k-points x we have a surjective morphism*

$$\Omega^n_{x/\mathbb{Z}}/B_\infty \twoheadrightarrow T(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$$

and the following commutative diagram

$$\begin{tikzcd} \Omega^n_{x/\mathbb{Z}}/B_\infty \arrow[r, "C^{-1}"] \arrow[d, two heads] & \Omega^n_{x/\mathbb{Z}}/B_\infty \arrow[d, two heads] \\ T(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x) \arrow[r, "F\otimes \mathrm{id}"] & T(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x), \end{tikzcd}$$

where $F : \mathbb{G}_a \to \mathbb{G}_a$ is the absolute Frobenius, $C^{-1} : \Omega^n_{x/\mathbb{Z}} \to \Omega^n_{x/\mathbb{Z}}/d\Omega^{n-1}_{x/\mathbb{Z}}$ is the inverse Cartier operator and B_∞ is the union over B_n, where $B_1 = d\Omega^{n-1}_{x/\mathbb{Z}}$ and B_n is the preimage of $C^{-1}(B_{n-1})$ in $\Omega^n_{x/\mathbb{Z}}$ for $n \geqslant 2$.

One would like to define an inverse of the map $\Omega^n_{x/\mathbb{Z}}/B_\infty \twoheadrightarrow T(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$ above by sending an element $(a, b_1, \ldots, b_n) \in \mathbb{G}_a(x) \times \mathbb{G}_m(x)^{\times n}$ to $a\frac{db_1}{b_1} \wedge \ldots \wedge \frac{db_n}{b_n}$ in $\Omega^n_{x/\mathbb{Z}}/B_\infty$ and use the universal property of T. But we don't know whether the condition (L3) above is satisfied (and thus whether it is an $(n+1)$-linear map of reciprocity functors). The problem is that we do not know how to control the filtration $\mathrm{Fil}^\bullet_P\mathbb{G}_a(\eta)$ in the case where C is a regular projective curve over some k-point x and $P \in C$ is a closed point which is purely inseparable over x.

The following result was suggested by B. Kahn and is proved by an easy computation:

Theorem 7.10 ([IR, Thm 5.5.1])**.** *Assume the characteristic of k is not 2. Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be reciprocity functors. Then*

$$T(\mathbb{G}_a, \mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_n) = 0.$$

Let us end with the following comment: The category of reciprocity functors defined above is only the first attempt to get in the direction of a nice theory of non-homotopy invariant Nisnevich sheaves with transfers. One wish for a more general theory surely is to get something more global; the other wish is to replace the very artificial condition (Inj) (which causes lots of problems) by some more natural conditions, which force the condition (Inj) to hold and still are satisfied by the above examples. There is work in progress by B. Kahn, S. Saito and T. Yamazaki, which solves these problems (and goes beyond).

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