Classification of VB on $X_{E,F}$ ($\mathcal{X}_{E,F}$)

$E/\mathbb{Q}_p$ finite, $\mathcal{O}_E$, $\pi \in \mathcal{O}_E$, $k = \mathcal{O}_E/\pi \cong \mathbb{F}_q$

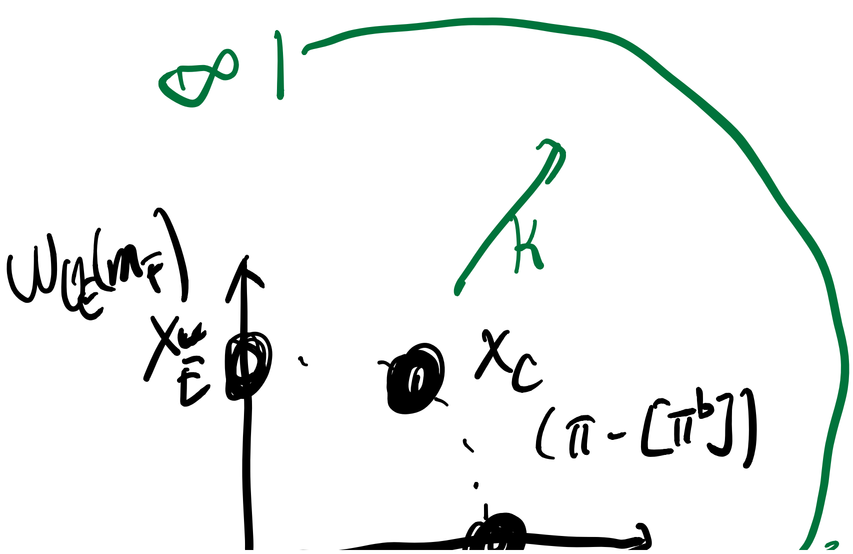
$F/\mathbb{F}_q$ alg closed, nonarchimedean complete.

$A_{inf} = W_{\mathcal{O}_E}(\mathcal{O}_F)$, C untilt of F of choice

$\theta : A_{inf} \longrightarrow \mathcal{O}_C$. k_C res. field of C

$$\breve{E} = W_{\mathcal{O}_E}(\underset{\overline{\mathbb{F}}_q}{\underset{\shortparallel}{k_C}})\left[\tfrac{1}{\pi}\right]$$

Spa(A_{inf}) (Spec(A_{inf}))



x_{k_C} $\qquad$ x_F $\qquad$ 0

$W_{\mathcal{O}_E}(m_F), (\pi)$ $\qquad$ (π)

① x_{k_C} unique nonanalytic point

$\mathcal{Y} := \mathrm{Spa}(A_{\inf}) \setminus \{x_{k_C}\}$ analytic adic space

② $\kappa : \mathcal{Y} \longrightarrow [0,\infty]$ radius map

$|Y_{[0,\infty)}| = \{ I \subseteq A_{\inf} \mid I = (d) \}$

" $\mathcal{Y}_{[0,\infty)} = \mathcal{Y} \setminus \{x_{\breve{E}}\}$ "

③ $|Y_{E,F}| := |Y_{[0,\infty)}| \setminus \{(\pi)\}$

" $\mathcal{Y}_{E,F} = \mathcal{Y}_{[0,\infty)} \setminus \{x_F\}$

$X_{E,F} = \mathcal{Y}_{E,F}/\phi^{\mathbb{Z}}$ adic FF curve

$$X_{E,F} = \mathrm{Proj}\left(\bigoplus_{d\geq 0} B^{\varphi=\pi^d}\right)$$

$$= \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0(X_{E,F}, \mathcal{O}_{X_{E,F}}(d))\right)$$

④ Recall

<u>Thm</u> (FF) $X_{E,F}\setminus\{x_C\}$ is an affine scheme $\mathrm{Spec}(B_e)$ with B_e PID.

<u>Rmk</u> • notation : $\infty = x_C$ if there's no ambiguity

• $X = X_{E,F}$ . . — -

• $B_e = \Gamma(X\setminus\{\infty\}, \mathcal{O}_X)$

now we have

$$\begin{array}{ccccc} \mathrm{Spec}(C) & \xhookrightarrow{i_\infty} & X & \hookleftarrow & X\setminus\{\infty\} \\ \shortparallel & & & & \\ x_C & & & & \end{array}$$

Rmk · $B_{dR}^{+} = \widehat{\mathcal{O}_{X,\infty}}$, $t \in B_{dR}^{+}$ uniformizer

· $B_{dR} = B_{dR}^{+}[\frac{1}{t}]$

Def an SES of VB on $X = X_{E,F}$

$$0 \to \mathcal{E}' \to \mathcal{E} \to i_{\infty *} W \to 0$$

where W is a fin dim C-vector space is a (miniscule) modification of $\mathcal{E}$ at ∞.

Example/Fact last time:

$(D, \varphi) \in \varphi\text{-Mod}_{\breve{E}}$, $\mathcal{E}(D, \varphi)$ a VB on X

Let $\Lambda \subseteq D \otimes_{\breve{E}} B_{dR}^{+}$ be a lattice.

$$\| $$

$$\widehat{\mathcal{E}(D, \varphi)_{\infty}}$$

$\exists$ a modification for $\mathcal{E}(D, \varphi)$

$$0 \to \mathcal{E}(D, \varphi, \Lambda) \to \mathcal{E}(D, \varphi) \to i_{\infty *}\left(D \underset{E}{\otimes} B_{dR}^{+} / \Lambda\right) \to 0$$

Furthermore, the following sets are in bijection

$$\left\{ \Lambda \subseteq D \underset{E}{\otimes} B_{dR}^{+} \text{ lattice} \;\middle|\; t D \underset{E}{\otimes} B_{dR}^{+} \subseteq \Lambda \subseteq D \underset{E}{\otimes} B_{dR}^{+} \right\}$$

$$\overset{1:1}{\longleftrightarrow} \left\{ \mathrm{Fil}\, D_C \subseteq D \underset{E}{\otimes} C \text{ sub } C\text{-vector space} \right\}$$

Via this bijection, can construct for any $\mathrm{Fil}\, D_C \subseteq D \underset{E}{\otimes} C$, sub C-VS, a modification of $\mathcal{E}(D, \varphi)$

$$0 \to \mathcal{E}(D, \varphi, \mathrm{Fil}\, D_C) \to \mathcal{E}(D, \varphi) \to i_{\infty *}\left(D \underset{E}{\otimes} C / \mathrm{Fil}\, D_C\right) \to 0$$

<u>Thm</u> (Fargues, Scholze-Weinstein)

pair

$\left\{ (T, \Xi) \;\middle|\; \begin{array}{l} \cdot\ T \text{ finite free } \mathcal{O}_E\text{-module} \\ \cdot\ \Xi \subseteq T \underset{\mathcal{O}_E}{\otimes} B_{dR},\ \text{a } B_{dR}^{+}\text{-lattice} \end{array} \right\}$

$$\updownarrow$$

$$\left\{ \begin{array}{l} (\mathcal{F}', \mathcal{F}, \beta, T) \\ \cdot\ \mathcal{F}', \mathcal{F} \text{ VB on } X,\ \mathcal{F}' \text{ trivial} \\ \cdot\ \beta : \mathcal{F}'|_{X \setminus \{\infty\}} \xrightarrow{\sim} \mathcal{F}|_{X \setminus \{\infty\}} \\ \cdot\ T \subseteq H^0(X, \mathcal{F}') \text{ an } \mathcal{O}_E\text{-lattice} \end{array} \right\}$$

In particular, the subcat corres. to the condition

$$T \otimes_{\mathcal{O}_E} B_{dR}^+ \subseteq \Xi \subseteq t^{-1}(T \otimes_{\mathcal{O}_E} B_{dR}^+)$$

is equivalent to π-divisible $\mathcal{O}_E$-modules/$\mathcal{O}_C$.

Def p-divisible group /R ring of height h is

$G = (G_n, i_n)_{n \in \mathbb{N}}$ with

- G_n are ffgs of p^{nh}
- $i_n : G_n \to G_{n+1}$ closed immersions

such that $\forall$ all $n \in \mathbb{N}$

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{[p]} G_{n+1}$$

is left exact.

<u>Examples</u> • $\mu_{p^\infty} = (\mu_{p^n}, i_n)$

• $E[p^\infty] = (E[p^n](\bar{\mathbb{Q}}_p), i_n)$

<u>Rmk</u> • Cartier dual $G^\vee = (G_n^\vee, i_n^\vee)$

$G_n^\vee = \mathrm{Hom}(G_n, \mathbb{G}_m)$

• Tate module $T_pG = \varprojlim G[p^n](\bar{\mathbb{Q}}_p)$

free $\mathbb{Z}_p$-module of finite rank.

• R noeth complete local ring $\exists$ SES

$$0 \to G^0 \to G \to G^{et} \to 0$$

connected part — étale part

If R is perfect field then the SES split.

Def Let $S \to \mathrm{Spec}(\mathcal{O}_E)$ $\mathcal{O}_E$-scheme

(resp. $S \to \mathrm{Spf}(\mathcal{O}_E)$ $\mathcal{O}_E$-formal scheme)

H is a <u>π-divisible (formal) $\mathcal{O}_E$-module</u> over S if H is a p-divisible gp $/ \mathcal{O}_E$ w/ an $\mathcal{O}_E$-action s.t. the induced action on $\mathrm{Lie}\, H$ is compatible w/ $S \to \mathrm{Spec}(\mathcal{O}_E)$.

Rmk $\cdot\, ht_{\mathcal{O}}(H) = \dfrac{ht(H)}{[E:\mathbb{Q}_p]}$ "$\mathcal{O}_E$-height"

- H_0 π-divisible $\mathcal{O}_E$-module over $\overline{\mathbb{F}_q}$, φ

can define Dieudonné module $\mathbb{D}_{\mathcal{O}}(H)$

free $\mathcal{O}_{\breve{E}}$-module of rank $ht_{\mathcal{O}}(H_0)$

$$(\mathbb{D}_{\mathcal{O}}(H)[\tfrac{1}{\pi}], \pi^{-1}\varphi) \in \pi^{-1}\varphi\text{-}Mod_{\breve{E}}$$

Construction 1 Take such $H_0 / \overline{\mathbb{F}_q}$ we have

$\hat{\mathcal{M}}$ a deformation space parameterizing the pairs (H, ρ) (up to iso) where

$\downarrow$

$\mathrm{Spf}(\mathcal{O}_{\breve{E}})$

formal $\mathcal{O}_{\breve{E}}$-scheme

- H π-divisible $\mathcal{O}_E$-module $/\mathcal{O}_C$
- $\rho : H_0 \otimes_{\overline{\mathbb{F}}_q} \mathcal{O}_C/p\mathcal{O}_C \longrightarrow H \otimes_{\mathcal{O}_C} \mathcal{O}_C/p\mathcal{O}_C$ quasi-isogeny

- $\mathcal{M}$ generic fibre of $\hat{\mathcal{M}}$. Fact: $\mathcal{M}$ is a Berkovic analytic space $/\breve{E}$

- The Grassmannian

$$\mathbb{P}^{dR} := \{ V \subseteq \mathbb{D}_{\mathcal{O}}(H_0)[\tfrac{1}{\pi}] \mid \mathrm{codim}_{\breve{E}} V = \dim H_0 \}$$

$$\pi^{-1}\varphi\text{-}\mathrm{Mod}_{\breve{E}}$$

as an $\breve{E}$-analytic space.

- $\pi^{dR} : \mathcal{M} \longrightarrow \mathbb{P}^{dR}$

Thm (Gross-Hopkins, Rapoport-Zink)

π^{dR} is an étale map

Rmk • $[(H,\rho)] \longmapsto \mathrm{Fil}(\mathbb{D}(H_0)_C \subseteq \mathbb{D}_{\mathcal{O}}(H_0)[\frac{1}{\pi}] \otimes C$, where $\mathbb{D}_{\mathcal{O}}(H_0)_C = \mathbb{D}(H_0)_C$

"Hodge-de Rham filtration"

• $z \in \mathbb{P}^{dR}(C)$, have a modification

$$0 \to \underbrace{\mathcal{E}(z)}_{\text{UB corres to } (H,\rho)} \to \underbrace{\mathcal{E}(\mathbb{D}_{\mathcal{O}}(H_0)[\frac{1}{\pi}], \pi^{-1}\varphi)}_{\text{UB assoc } H_0} \to i_{\infty *}\left(\mathbb{D}(H_0)_C / \mathrm{Fil}(\mathbb{D}_{\mathcal{O}}(H_0))\right) \to 0$$

Thm if $z \in \overbrace{\mathrm{Im}(\pi^{dR})}^{\text{"admissible locus"}}$, then $\mathcal{E}(z)$ is a trivial vector bundle.

Rmk $\mathrm{Im}(\pi^{dR}) \subseteq \mathbb{P}^{dR}$ open.

Thm (Lafaille, Gross-Hopkins) Let H_0 be 1-dim formal π-divisible $\mathcal{O}_E$-module of

$\mathcal{O}_E$ height n over $\overline{\mathbb{F}_q}$, and here $\mathbb{P}^{dR} = \mathbb{P}^{n-1}$ associated Grassmannian. Then π^{dR} surjective.

<u>Rmk</u> • H_0 in the assumption are unique up iso

• this thm gives

<u>Thm A</u> Let $0 \to \xi \to \mathcal{O}_X(\frac{1}{n}) \to i_{\infty *} k(\infty) \to 0$ be a modification of $\mathcal{O}_X(\frac{1}{n})$, then $\xi \simeq \mathcal{O}_X^n$.

<u>Construction 2</u> $K \subseteq GL_n(\mathcal{O}_E)$ open subgroup

• can define $\mathcal{M}_K \to \mathcal{M}$ finite étale cover (via level structures)

• have "$\mathcal{M}_\infty := \varprojlim_{\substack{K \subseteq GL_n(\mathcal{O}_E) \\ \text{open}}} \mathcal{M}_K$" as a generalized rigid analytic space (Lubin–Tate tower)

- $\mathbb{C}$-points

$$\mathcal{M}_\infty(\mathbb{C}) = \{(H, \rho, \eta)\}/\sim$$

- $(H, \rho) \in \mathcal{M}(\mathbb{C})$
- $\eta : \mathcal{O}_E^n \xrightarrow{\sim} T_\pi(H) \otimes \mathcal{O}_E$

- Grassmannian

$$\mathbb{P}^{HT} = \{U \subseteq E^n \mid \dim_E U = \dim H_0\}$$

as an $\breve{E}$-analytic space.

- $\pi^{HT} : \mathcal{M}_\infty \longrightarrow \mathbb{P}^{HT}$

$$(H, \rho, \eta) \longmapsto \text{datum of HT SES of } H$$

- $z \in \mathbb{P}^{HT}(\mathbb{C})$, have $\mathcal{H}(z)$ VB on X

Thm If $z \in \mathbb{P}^{HT}(C) \cap \mathrm{Im}(\pi^{HT})$, then

$$\mathcal{H}(z) \simeq \mathcal{E}(\mathbb{D}_{\mathcal{O}}(H_0)[\tfrac{1}{\pi}], \pi^{-1}\varphi)$$

Construction 3 Let H_0 be π-divisible $\mathcal{O}_E$-module $/\overline{\mathbb{F}}_q$ with $\dim H_0^\vee = 1$. Then $\mathbb{P}^{HT} = \mathbb{P}^{n-1}$

as $\breve{E}$-analytic space.

Define $\mathbb{P}_i^{n-1} = \{x \in \mathbb{P}^{n-1} \mid \dim_E(E^n \cap \mathrm{Fil}_x k(x)^n) = i\}$

$\rightsquigarrow$ get a stratification of $\mathbb{P}^{n-1}$

$$\mathbb{P}^{n-1} = \bigsqcup_i \mathbb{P}_i^{n-1}$$

open stratum is $\mathbb{P}_0^{n-1} \simeq \Omega^{n-1}$ Drinfeld half space of dimension $n-1$.

For each i, have fibration

$$\begin{array}{ccc} \mathbb{P}_i^{n-1} & & \mathcal{X} \\ \downarrow \tau_i & & \downarrow \\ Gr^i = \{ V \subseteq E^n \mid \dim_E V = i \} & & E^n \cap Fil_x k(x)^n \end{array}$$

fibers of $\tau_i \simeq \Omega^{n-1-i}$ DHS of dim $n-1-i$

__Rmk__ π-divisible $\mathcal{O}_E$-module $H' / \overline{\mathbb{F}_q}$ of height n dim $n-1$ are classified by $ht_{\mathcal{O}}(H'^{et}) \in \{0, \dots, n-1\}$ up to iso.

__Notation__ write $H_0^{(i)}$ for π-div $\mathcal{O}_E$-module $/ \overline{\mathbb{F}_q}$ with $ht_{\mathcal{O}}(H_0^{(i) et}) = i$.

Let $\mathcal{M}^{(i)}$ be the deformation space of $H_0^{(i)}$ by quasi-isogenies

· period morphisms

$$\pi_i^{HT} : \mathcal{M}_\infty^{(i)} \longrightarrow \mathbb{P}_i^{n-1} \quad \text{for each } i$$

<u>Thm</u> Hodge-Tate period map

$$\pi^{HT} : \coprod_{i=0,\dots,n-1} \mathcal{M}_\infty^{(i)} \longrightarrow \mathbb{P}^{n-1} \quad \text{is surj.}$$

<u>Rmk</u> pf has to show $\pi^{HT} : \mathcal{M}_\infty^{(0)} \to \Omega^{n-1}$ surj.

for each $z \in \mathrm{Im}(\pi^{HT})$

The datum $(T_\pi H^\vee \otimes_{\mathcal{O}_E} \mathcal{O}_X, \mathcal{H}(z), \rho_{H^\vee}, \mathrm{Lie}\, H[\frac{1}{\pi}])$

<u>Thm B</u> Let

$$0 \to \mathcal{O}_X^n \to \mathcal{E} \to i_{\infty *} k(\infty) \to 0 \text{ be}$$

a modification of $\mathcal{E}$ on X at ∞. Then

$\exists\ i = 0, \dots, n-1$ s.t.

$$\mathcal{E} \simeq \mathcal{O}_X^i \oplus \mathcal{O}_X\left(\frac{1}{n-i}\right)$$

Pf of A+B $\square$

Thm A+B $\Leftrightarrow$ Main Thm

Main Thm (Fargues-Fontaine, Hartl-Pink, Kedlaya)

$$\{\lambda_1 \geq \cdots \geq \lambda_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\simeq} \mathrm{Bun}_X/\sim$$

Idea of pf of A+B $\Leftrightarrow$ Main (+ thm B)

- Harder-Narasimhan filtration of $\mathcal{E}$ on X (?)
  $\rightsquigarrow$ reduce to $\mathcal{E}$ semistable
- passing on to (pro-) Galois cover of X

recall $X_{E_h} = X \times_E E_h$, E_h/E finite unram of deg h.

with $\pi_h : X_{E_h} \longrightarrow X_E$

$\lambda = \frac{d}{\cdot}$

(e.g. $\pi_{n*}\mathcal{O}_{X_{E_n}}(d) \cong \underbrace{\mathcal{O}_X\left(\frac{d}{n}\right)}$)

$\rightsquigarrow$ reduce to $\mathcal{E}$ of "slope 0"

Left to show: $\mathcal{E}$ semistable of slope 0

$\Rightarrow \mathcal{E} = \mathcal{O}_X^n$ for some n.

$$0 \to \mathcal{O}_X(-d) \to \mathcal{E} \to \underset{\text{deg 1 modifications}}{\mathcal{E}'} \to 0$$

induction on rank $\mathcal{E}$ $\square$

<u>Cor of thm</u>

- HN-filtration split on Bun_X
- $\mathcal{E}$ semistable of slope λ on X then $\mathcal{E} \simeq \mathcal{O}_X(\lambda)^n$

<u>Thm</u> (GAGA)

$$\mathcal{X}_{E,F} \to X_{E,F} \quad \text{induce}$$

$$\mathrm{Bun}_{\mathcal{X}} \cong \mathrm{Bun}_X$$