Vector bundles on the FF-curve

Goals: i) Construct and describe vb's on the FF-curve X.

ii) Consider the Harder-Narasimhan machinery for vb's on X.

We start with ii).

Motivation: Let X be a smooth conn. proj. curve $/ K = \overline{K}$, $\mathcal{E}$ a vb on X.

Set

$$\deg(\mathcal{E}) := \deg(\Lambda^{rk(\mathcal{E})} \mathcal{E}) \quad \text{"degree"}$$

$$\mu(\mathcal{E}) := \deg(\mathcal{E}) / rk(\mathcal{E}), \quad \mathcal{E} \neq 0, \quad \text{"slope"}$$

A vb $\mathcal{E}/X$ is called semistable

$:\Leftrightarrow \forall$ subbundles $\mathcal{F} \subset \mathcal{E}$ =

$\mathcal{A}$

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

Rk: Concept is useful for studying moduli spaces (GIT).

Thm (Harder-Narasimhan). For every vb $\mathcal{E}/X$ $\exists!$ filtration:

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subset \dots \subsetneq \mathcal{E}_r = \mathcal{E}$$

by subbundles s.t. $\mathcal{E}_i/\mathcal{E}_{i-1}$ is ss $\forall i$ and

$$\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1) \geq \dots > \mu(\mathcal{E}/\mathcal{E}_{r-1})$$

Can generalize this theory as follows.

Let
- $\mathcal{E}$ be an exact category (Quillen).

together with 2 additive functions:

$\deg : \mathrm{ob}(\mathcal{E}) \to \mathbb{Z}$ exact seq.

$\mathrm{rk} : \mathrm{ob}(\mathcal{E}) \to \mathbb{N}$

- $\mathcal{A}$ be an abelian category
- $F : \mathcal{E} \to \mathcal{A}$ exact faithful functor: s.t. $\forall\, E \in \mathcal{E}$

$$\{\underbrace{\text{strict subobjects of } E}\} \xrightarrow{\sim} \{\text{subobjects of } F(E)\}.$$

$\exists$ exact seq. $0 \to U \to E \to V \to 0$.

s.t. - rk factorises over $\mathcal{A}$.

- $\mathrm{rk}\, E = 0 \iff E \simeq 0$.
- Let $f \in \mathrm{Mor}(E, E')$ s.t. $F(f)$ is an isom.
  $\Rightarrow \deg E' \leq \deg E$ with "=" $\iff$ f is an isom.

<u>Examples</u> a) $\mathcal{E} = \mathrm{Bun}_X$, X as above.

$\mathcal{A} = \{\text{f.-d. } K(X)\text{-VS}\}$.

F = taking generic fibre.

b) $\mathcal{C}$ = category of filtered vector spaces, Higgs bundles, quiver representations etc.

c) Let $A = W_{O_E}(\mathbb{F}_q)\left[\frac{1}{\pi}\right] = \breve{E} \circlearrowleft \varphi$ Frob.

$\varphi-\mathrm{Mod}_A$ = category of φ-module $/\breve{E}$

fin.-dim $\breve{E}$-vs M together with

an φ-linear iso $\varphi_M : M \xrightarrow{\sim} M$.

$rk(M, \varphi_M) = \dim M$

$\deg(M, \varphi_M) = \deg(\Lambda^{\dim M}(M, \varphi_M)) = val_\pi(\det \varphi_M)$

<u>Thm</u> (Dieudonné - Manin - classification)

$\varphi-\mathrm{Mod}_{\breve{E}}$ is semi-simple and the simple objects are param. by $\mathbb{Q}$.

$$\mathbb{Q} \longrightarrow \varphi-\mathrm{Mod}_{\breve{E}}$$

$$\underset{r>0}{(d,r)=1} \quad \frac{d}{r} = \lambda \longmapsto D(\lambda) := \left(\breve{E}^r, \begin{pmatrix} 0 & & \pi^d \\ 1 & \ddots & \\ 0 & \ddots 1 & 0 \end{pmatrix} \varphi . id\right)$$

SS-objects = isoclinic objects = iso. to $D(\lambda)^n$ for some $\lambda \in \mathbb{Q}, n \in \mathbb{N}$.

Proof of HN-Thm. (Sketch).

1) Let $\mathcal{E} \in ob(\mathcal{C})$. If $\mathcal{E}$ is ss. ✓.

If not show that $S := \{\mu(\mathcal{E}') \mid \mathcal{E}' \text{ strict subobj. of } \mathcal{E}\}$

is finite / bounded.

2) Show that $\exists^!_{\circ}$ strict subobject $\mathcal{E}'$ of $\mathcal{E}$

with $\mu(\mathcal{E}') = \max S = \lambda_0$ of max. rk.

[Suppose there are two such objects: $\mathcal{F}, \mathcal{G}$.

Then consider $\mathcal{F} \oplus \mathcal{G} \to \mathcal{E}$

ss. of slope λ_0. If $\mathcal{F} \neq \mathcal{G}$ then

the image is again of slope λ_0]

3) Set $\mathcal{E}_1 := \mathcal{E}'$ and apply induction to

$\mathcal{E}/\mathcal{E}'$

D.

Now we come to c) $\overset{\text{P}}{\shortparallel}$ Rayal's talk

Recall that $X_E = X = \mathrm{Proj}\Big(\underbrace{\bigoplus_{d\geq 0}^{\infty} B^{\varphi = \pi^d}}_{P_d}\Big)$

Let $D = (D, \varphi_D) \in \varphi\text{-Mod}_A$ Set

$$\mathcal{E}(D) := \overbrace{\bigoplus_{d\geq 0} \left(B \otimes_{\breve{E}} D\right)^{\varphi\otimes\varphi_D = \pi^d}}$$

graded P-module

Get a functor.

$$\varphi\text{-Mod}_A \to \mathrm{Coh}_X$$

__Example:__ $\mathcal{E}(D(-d)) \simeq \mathcal{O}_X(d)$ for $d \in \mathbb{Z}$.

$\Big(D(-d) = \breve{E}$ with $\varphi_D = \pi^{-d} \cdot \varphi$. $\Rightarrow$

$(B \otimes_{\breve{E}} \breve{E}(-d))^{\varphi\otimes\varphi_D = \pi^e} = B^{\varphi = \pi^{d+e}}\Big)$.

__Prop:__ $\mathcal{E}(D)$ is a vb $\forall D \in \varphi\text{-Mod}_A$.

Lemma: Let E_n/E be an unramified extension of degree $n \in \mathbb{N}$ and let $D \in \varphi$-Mod.

a) There are isom. $\forall d$: Frob.on

$$E_n \otimes_E (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d} \cong (B \otimes_{\breve{E}} D)^{\varphi^n \otimes \varphi_D^n = \pi^{n \cdot d}}$$

b) $X_{E_n} \cong X_E \times_E E_n$

c) The following diagram is commutative,

$$\begin{tikzcd} (D, \varphi_D) \arrow[d, mapsto] & \varphi\text{-Mod}_{\breve{E}} \arrow[r, "\mathcal{E}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Coh}_{X_E} \arrow[d, "- \otimes_E E_n"] \\ (D, \varphi_D^n) & \varphi^n\text{-Mod}_{\breve{E}} \arrow[r, "\mathcal{E}"] & \mathrm{Coh}_{X_{E_n}} \end{tikzcd}$$

Pf: a) Have an action of $G := \mathrm{Gal}(E_n/E) \cong \mathbb{Z}/n\mathbb{Z}$ on RHS via $1 \mapsto \pi^{d} \varphi \otimes \varphi_D$

One checks that.

□

$$RHS^{\check{G}} = (B \otimes_{\check{E}} D)^{\varphi \otimes \varphi_D = \pi^d}.$$

Thus the claim follows from Galois descent.

b) Follows from a) and

$$X_{E_n} = \mathrm{Proj}\left(\bigoplus_{d \geq 0} B^{\varphi^h = \pi^d}\right) = \mathrm{Proj}\left(\bigoplus_{d \geq 0} B^{\varphi^h = \pi^{hd}}\right),$$

c) similar to b) $\square$.

Pf of Prop.: May assume that $D = D(\lambda)$;

$\lambda = \frac{d}{r}$. If $r = 1$ then ✓ by above example.

Otherwise let $h := r$. Suffices to check

$\mathcal{E} := \mathcal{E}(D) \otimes_E E_n$ is a v.b. But

$$\mathcal{E}(D) \otimes_E E_n = \mathcal{E}(D(\lambda), \varphi_{D(\lambda)}^r) \quad \text{by Lemma c)},$$

and $(D(\lambda), \varphi^r_{D(\lambda)}) = D(d)^r \Rightarrow \mathcal{E} \cong \mathcal{O}_X(-d)^r$

$$\left(\begin{pmatrix} 0 & & \pi^d \\ 1 & \ddots & \\ & \ddots & \\ & 1 & 0 \end{pmatrix}^r = \pi^d \cdot E_r \right)$$

<u>Def</u> $\mathcal{O}_X(\lambda) := \mathcal{E}(D(\lambda))$ for $\lambda \in G$.

<u>Thm</u> (F.-F.) Have a bijection

$$\varphi\text{-Mod}_{\breve{E}}/\cong \xrightarrow{\mathcal{E}} \mathrm{Ban}_X/\cong$$

<u>Rmk.</u> i) $\mathcal{E}$ is not an equivalence of categories; Ban_X is not abelian.
ii) But $\mathcal{E}$ is compatible with HN-filtrations. (and HN-filtr. splits)