$\mathbb{A}_{inf}$

„The ring to rule them all"

— P. Colmez

Setup · p prime

$E/\mathbb{Q}_p$ finite

$\cup$

$\mathcal{O}_E$ ring of integers

$\vee$

(π) max. ideal gen by unif. π

· $\mathcal{O}_E/(\pi) = \mathbb{F}_q$, $q = p^n$

· $F/\mathbb{F}_q$ non-arch. alg. closed

$v: F \to \mathbb{R} \cup \{\infty\}$

non-noeth

+ $\mathcal{O}_F = \{x \in F \mid v(x) \geq 0\}$

$\cup$

$\mathfrak{m}_F = \{x \in F \mid v(x) > 0\}$

Def:

$$\mathbb{A}_{inf} := W_{O_E}(O_{F^\flat})$$

ring of ram. Witt vectors of the perf. $\mathbb{F}_q$-alg. $O_{F^\flat}$

$$\mathbb{A}_{inf} = \left\{ \sum_{n=0}^{\infty} [x_n] \pi^n \;\middle|\; x_n \in O_{F^\flat} \right\}$$

ring of „formal power series in π"

$\varphi : \mathbb{A}_{inf} \to \mathbb{A}_{inf}$ „Frobenius"

$$\sum_{n=0}^{\infty} [x_n] \pi^n \mapsto \sum_{n=0}^{\infty} [x_n^q] \pi^n$$

Compare with $\mathcal{O}_F[|u|]$:

- $\varphi\left(\sum_{n\geq 0}^{\infty} x_n u^n\right) = \sum_{n=0}^{\infty} x_n^q u^n$

- $\mathcal{O}_F[|u|]$ non-noeth., local int. domain

<u>Geometric Interpretation:</u>

Is the ring of bounded functions on

$$\mathbb{D}_F = \{x \mid v(x) > 0\}, \ \bar{F}$$

• Arnold (1975): $\dim \mathcal{O}_{\overline{F}}[[V]] = \infty$

Concretize:

For $a \in m_{\overline{F}}$ have

$$ev_a : \mathcal{O}_{\overline{F}}[[V]] \to \overline{F}$$
$$f \mapsto f(a)$$

with $\ker ev = (v - a)$

Theorem

$$\operatorname{Spec} \mathcal{O}_F[U] = \{0, (m_F, U)\}$$
$$\cup \{(U - a) \mid a \in m_F\}$$
$$\cup \operatorname{Spec}(\mathcal{O}_F[U]_{m_F[U]})$$

Proof: Let $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F[U])$ s.t. $\mathfrak{p} \not\subseteq m_F[U]$

$\rightsquigarrow$ exist $f \in \mathfrak{p}$ $f = \sum_{i=0}^{d} x_i U^i$ with some $x_i \in \mathcal{O}_F^\times$

Weierstraß-fact.

$\leadsto$ $f = g\,t$ with $g \in \mathcal{O}_F[U]$,

$\deg(g) = \min\{i \mid x_i \in \mathcal{O}_F^\times\}$,

F alg. cl. + $\mathcal{O}_F[[U]]^\times$

$\leadsto$ $g = \prod (U - a_i)$ with $a_i \in \mathcal{O}_F$

+ $g \in \mathfrak{p}$

$\mathfrak{p}$ prime for some i

$\leadsto$ $(U - a_i) \subseteq \mathfrak{p} \subseteq (\mathfrak{m}_F, U)$

$\square$

$\mathbb{A}_{inf}$

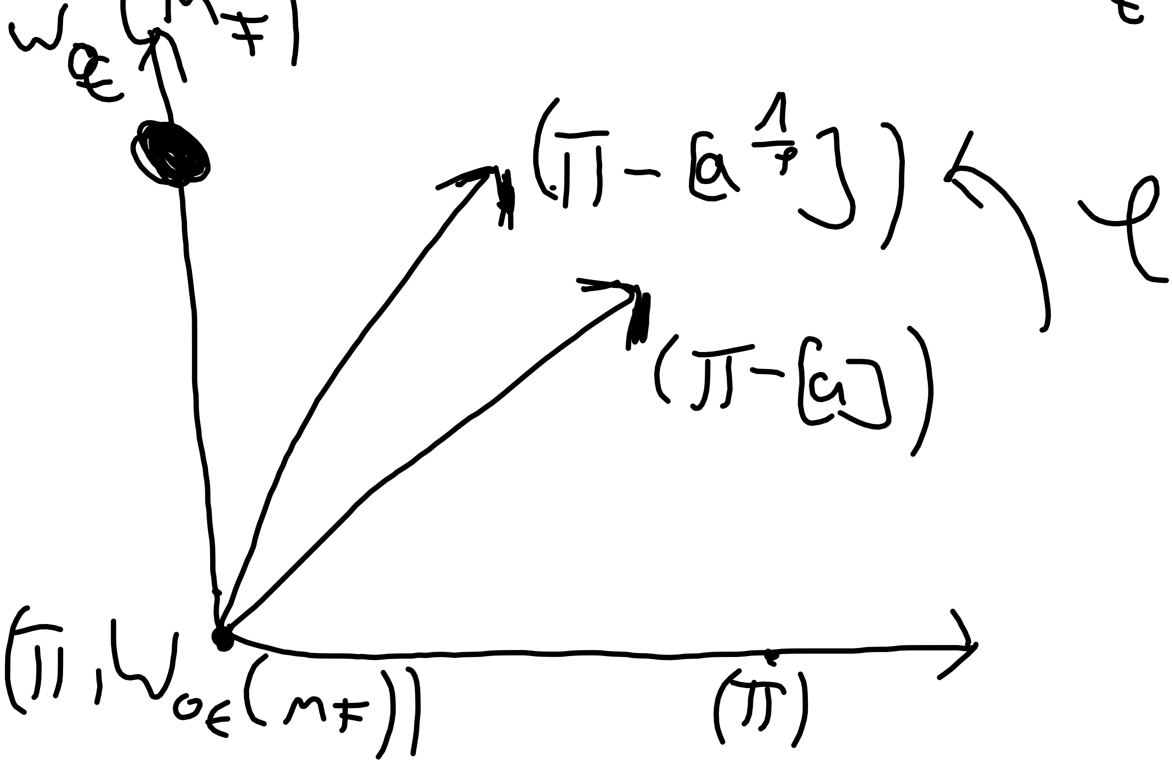
- $\mathbb{A}_{inf}$ non-no[illegible]th[illegible] local int. dom.
- $+ \ (\pi, [\bar{\omega}])$ - [illegible]ically complete for all $\bar{\omega} \in m_{\mathcal{O}} \setminus \{0\}$

Theorem (Lang, [illegible]ig 2019)

- $\dim \mathbb{A}_{inf} = \infty$

+

+ „exotic" primes lie in $W_{O_E}(m_F)$



- harder to give geom. interpr.
  $\rightsquigarrow$ use adic spaces

Def.:

- $|Y|_{[0,\infty)} = \{ I \subset A_{\inf} \mid I = (d),\ d \text{ distinguished} \}$

  $= \{ (u\pi - [a]) \subset A_{\inf},\ u \in A_{\inf}^{\times},\ a \in m_F \}$

- $|Y| = |Y|_{[0,\infty]} \setminus \{ (\pi) \}$

Remark:

- for $O_F[[u]]$ would get m_F and $\pi_F^{-1}\{0\}$

Next talk:

$\{ (C,i)_{E} \text{ non-arch., alg. closed},\ i : O_C^{b} \cong O_F \} \xrightarrow{\cong} |Y|$

$(C,i) \mapsto \ker(A_{\inf} \xrightarrow{i^{-1}} W_{O_E}(O_C^{b}) \xrightarrow{\theta} O_C)$

From last time:

for C/E alg. closed, non-arch.

$+ \ \mathcal{O}_C^{\flat} \cong \mathcal{O}_F$ that

$$A_{\inf}/(\xi) \cong \mathcal{O}_C \qquad \mathcal{O}_F$$

$$\pi - [\pi^{\flat}] \qquad \left[\theta : A_{\inf}(\mathcal{O}_C^{\flat}) \to \mathcal{O}_C\right.$$

<u>Definition:</u>

$$B_{dR}^+ \cong B_{dR,C}^+ = \mathbb{A}_{inf}\left[1/\pi\right]^\wedge_\xi$$

$$\xi = \pi - [\pi]$$

is ξ-adic compl of $\mathbb{A}_{inf}[\frac{1}{\pi}]$

- $B_{dR} = \mathrm{Frac}(B_{dR}^+)$

„Fontaines field of p-adic periods"

- $B^+_{dR} = \varprojlim A_{inf}[1/\pi]/\xi^n$
- $B^+_{dR}/\xi \cong A_{inf}[1/\pi]/\xi = C$

<u>Lemma</u> The nat. morph.

$$\mathbb{A}^{\mathrm{inf}} \longrightarrow B_{dR}^{+}$$

is inj.

+ B_{dR}^{+}, $\mathbb{A}_{\mathrm{inf}}/(\xi)$ are DVR

Proof: 1) As $\xi \in A_{inf}$ non-zero div, O_C π-torsion free, we have

A_{inf}/ξ^n

is π torsion free for each $n \geq 0$

$\rightsquigarrow A_{inf}/\xi^n \hookrightarrow A_{inf}/\xi^n[1/\pi]$

for all $n \geq 0$

$\varprojlim$ + (M J-adically comp! + $I \subset J$

i.e. f.g $\Rightarrow$ M I-adically comp)

$\rightsquigarrow A_{inf} \subset \varprojlim A_{inf}/\xi^n \hookrightarrow B_{dR}^+$

2) B^+_{dR} is DVR:

- use ($R/_I$ noeth., I f.g $\Rightarrow$ $R^\wedge$ is noeth.) to see that B^+_{dR} is noeth.
- B^+_{dR} is local integ. domain with principal maximal ideal $\neq (0)$

$\Rightarrow$ B^+_{dR} DVR

$\underline{A_{\inf}(\xi)\ \text{DVR}:}$ $\mathfrak{p} \in \operatorname{Spec}(A_{\inf}(\xi))$

s.t. $\xi \notin \mathfrak{p}$, $A_s(\mathfrak{p}) \subset (\xi)$

we have for $a \in \mathfrak{p}$ that

$$a = \underset{A_{\inf}(\xi)}{\underset{\cap}{b}} \cdot \xi \overset{\mathfrak{p}\ \text{prime}}{\Rightarrow} b = \frac{a}{\xi} \in \mathfrak{p}$$

$$\Rightarrow \xi \mathfrak{p} = \mathfrak{p}$$

For $\mathfrak{q} = \mathfrak{p} B_{dR}^{+}$ we get

$$\xi \mathfrak{q} = \mathfrak{q}$$

$$\underset{B_{dR}^{+}\ \text{DVR}}{\Rightarrow} \mathfrak{q} = 0$$

$$\Rightarrow \mathfrak{p} = 0$$

Hence $\operatorname{Spec} A_{\inf}(\xi) = \{(0), (\xi)\}$

$\Rightarrow A_{\inf}(\xi)$ is DVR

Explanation for none A_{inf}.

<u>Def:</u> • R π-complete O_E-alg

• $f: D \to R$ surj of O_E-algebras with kernel I

• D is $I+(\pi)$-adically compl.

Then D is called

<u>π-adic pro-infinitesimal thickening of R</u>

Example: $R = \mathcal{O}_C$, $\bar{R} = \mathcal{O}_C/\pi$

then

$$\theta : A_{\inf} \to R$$

is a pro-inf. th.

<u>Lemma</u> $R \in \{\mathcal{O}_C, \mathcal{O}_C/\pi\}$.

Then $\mathbb{A}_{inf}$ is the universal π-adic pro-inf. thick i.e. for π-adic pro-inf. th. $D \to R$ we have unique

$$\begin{tikzcd} \mathbb{A}_{inf} \arrow[r] \arrow[dr] \arrow[dr, phantom, "\mathbin{/\!/}"] & D \arrow[d] \\ & R \end{tikzcd}$$

.

Proof: By last time we have

$$D^\flat \simeq R^\flat \left(\underset{x \mapsto x^p}{\overset{\text{use}}{\lim}} A \simeq (A/I)^\flat \right)$$

By the adjunction $(\ )^\flat$ with $W_{\mathcal{O}_E}(-)$ we get a map

$$A_{\inf} \to D \quad \text{red. to } D^\flat \simeq R^\flat$$

· checking uniqueness follows as usual. □