# The set $|Y|$ and the untilts of $\mathcal{O}_F$

**Setting**

- $E/\mathbb{Q}_p$ fin., $\mathcal{O}_E \supset \mathfrak{m}_E = (\pi)$
  $\mathcal{O}_E/\mathfrak{m}_E = \mathbb{F}_q$, $q = p^f$
- $F/\mathbb{F}_q$ non-Arch., alg closed, $\mathcal{O}_F \supset \mathfrak{m}_F$
  $k_F = \mathcal{O}_F/\mathfrak{m}_F$

**Recall:**

- $\mathbb{A}_{\inf} := W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{n \geq 0} [x_n] \pi^n \,\middle|\, x_n \in \mathcal{O}_F \right\}$
- $|Y|_{[0,\infty)} := \{ I \subset \mathbb{A}_{\inf} \mid I = (d),\ d \text{ distinguished} \}$

  $|Y| := |Y|_{[0,\infty)} \setminus \{(\pi)\}$

**Goal:** Start "attaching geom. meaning $|Y|$", relate $|Y|$ to the untilts of $\mathcal{O}_F$

Def 1:

(i) $x = \sum_{n \geq 0} [x_n] \pi^n \in A_{inf}$ is <u>primitive</u> if $x_0 \neq 0$ and $\exists d \geq 0 : x_d \in \mathcal{O}_F^\times$.

(ii) For $x \in A_{inf}$ primitive, let

$$\deg(x) := \min \{ d \mid x_d \in \mathcal{O}_F^\times \}$$

and

$$\mathrm{Prim}_d := \{ \text{prim. elts of deg } d \} \subset A_{inf}$$

Remarks 2:

(i) Def does not depend on $\pi$: For $(\pi) = (\pi')$:

$$\sum_{n \geq 0} [x_n] \pi^n = \sum \Big[ \underbrace{\left(\tfrac{\pi}{\pi'}\right)^n}_{\in \mathcal{O}_F^\times} x_n \Big] \pi'^n$$

(ii) Consider $A_{inf} \longrightarrow W_{\mathcal{O}_E}(k_F)$

$$x = \sum [x_n] \pi^n \mapsto \tilde{x} = \sum [\tilde{x}_n] \pi^n$$

Then

$x \in A_{inf}$ primitive of deg $d$

$\Longleftrightarrow x \notin \pi \cdot A_{inf}$ and $v_\pi(\tilde{x}) = d$

($W_{\mathcal{O}_E}(k_F)$ is a DVR)

(iii) Have seen: $x \in \mathbb{A}_{\inf}^{\times} \iff \bar{x} \in \left(\mathbb{A}_{\inf}/\pi\right)^{\times} = \mathcal{O}_F^{\times}$

Hence $\mathrm{Prism}_0 = \mathbb{A}_{\inf}^{\times}$.

(iv) Had lemma:

$x$ distinguished (i.e. $\frac{F(x) - x^q}{\pi} \in \mathbb{A}_{\inf}^{\times}$)

$\iff x_1 \in \mathcal{O}_F^{\times}$

Hence

$x \in \mathrm{Prism}_1 \iff x$ dist. and $\pi \nmid x$

Moreover

$$\mathrm{Prism}_1/\mathbb{A}_{\inf}^{\times} \xleftrightarrow{1:1} |Y|$$

$$x \longmapsto (x)$$

Have equivalence [Ans, after 3.7.4]

$$\left\{\begin{array}{l}\text{untilts } (A,\iota)\\ \text{of } \mathcal{O}_F \text{ over } \mathcal{O}_E\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l} I \subset A_{\inf} \text{ s.t.}\\ (A_{\inf}, I) \text{ perfect prism}\end{array}\right\} \overset{\text{Def}}{=} |Y|_{[0,\infty)}$$

$$(A,\iota) \longmapsto \operatorname{Ker}\big(A_{\inf} \xrightarrow{\iota^{-1}} W_{\mathcal{O}_E}(A^\flat) \xrightarrow{\theta} A\big)$$

$\iota : A^\flat \xrightarrow{\sim} \mathcal{O}_F$ $\qquad$ $\iota^{-1}$: by adjunction $\qquad$ $\theta$: Fontaine's map

<u>Prop</u> [talk 3]

Let $C/E$ non-Arch, alg. closed

Then $\mathcal{O}_C$ is a perfectoid $\mathcal{O}_E$-alg.

Hence get

$$\begin{tikzcd} \left\{(C,\iota) \,\middle|\, \begin{array}{l} C/E \text{ non-Arch, alg closed}\\ \iota : \mathcal{O}_C^\flat \xrightarrow{\sim} \mathcal{O}_F\end{array}\right\} \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] \arrow[rr, bend right, "\Psi"'] & \left\{\begin{array}{l}\text{untilts } (A,\iota)\\ \text{of } \mathcal{O}_F \text{ over } \mathcal{O}_E\end{array}\right\} \arrow[r, "\sim"] & {|Y|_{[0,\infty)}} \\ (*) & & {|Y|} \arrow[u] \end{tikzcd}$$

Note: We do not hit $I = (\pi) \in |\mathcal{Y}|_{[0,\infty)}$:

Assuming otherwise $(\pi) = \mathrm{Ker}(A_{\inf} \to \mathcal{O}_C)$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{char } p}}{\mathcal{O}_F} \cong A_{\inf}/(\pi) \hookrightarrow \underset{\substack{\uparrow \\ \text{char } 0}}{\mathcal{O}_C}$$

Theorem 3:

$$(*) \xrightarrow{\Psi} |\mathcal{Y}| \cong \mathrm{Prim}_1 / A_{\inf}^{\times} \quad \text{is a bijection.}$$

Thus, let $a \in \mathrm{Prim}_1$,

let $u \in A_{\inf}^{\times}$, $a_0 \in \mathfrak{m}_F \setminus \{0\}$ s.t. $a = u\pi - [a_0]$.

Let
$$D := A_{\inf}/(a), \quad \theta : A_{\inf} \to D \text{ proj.}$$

Prop 4:

(i) $D_b$ is $\pi$-cplt and $\pi$-torsion free

(ii) $D^b \cong O_F$

(iii) $D \to D$, $x \mapsto x^p$ is surjective

Proof:

One shows

$(\pi, a)$ regular seq in $A_{inf}$ $\Rightarrow (\pi^n, a)$ reg. seq. $\Rightarrow A_{inf}/\pi^n \xhookrightarrow{a} A_{inf}/\pi^n$

$\Rightarrow (a, \pi)$ reg. seq. $\Rightarrow$ (i)

and

$$D^b = \left(A_{inf}/(a)\right)^b \underset{!}{\cong} A_{inf}^b \cong O_F$$

Ad (iii):

Step 1: Reduce to $E = \mathbb{Q}_p$

Let $E/E_0/\mathbb{Q}_p$ be the max. unram. ext.

In Fall 2:

$$W(O_F) = W_{\mathbb{Z}_p}(O_F) \hookrightarrow W_{O_E}(O_F) \cong W(O_F) \otimes_{O_{E_0}} O_E$$

$\Rightarrow$ get norm homom.

$$\begin{tikzcd} W_{O_E}(O_F) \arrow[r, "Nm"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & W(O_F) \arrow[d] \\ W_{O_E}(k_F) \arrow[r, "\widetilde{Nm}"'] & W(k_F) \end{tikzcd}$$

As $W_{O_E}(k_F)/\pi \cong k_F \cong W(k_F)/p$,

have $v_p(\widetilde{Nm}(\tilde{x})) = v_\pi(\tilde{x})$, for $\tilde{x} \in W_{O_E}(k_F)$.

$\Rightarrow b := Nm(a) \in W(O_F)$ is primitive of deg 1.

also $b \cdot W_{O_E}(O_F) \subset (a)$

get $D' := W(O_F)/(b) \longrightarrow D$.

To reduce $O_p$, have to show that this is surj.

As $D, D'$ are $p$-cplt, suffices to show

$$W(O_F)/p \longrightarrow D'/p \longrightarrow D/p \qquad (**)$$

is surj.. Let $e = [E:E_0]$ i.e. $(p) = (\pi^e)$.

Then (**) becomes

$$\mathcal{O}_F \longrightarrow A_{\inf}/(p,a) \cong \mathcal{O}_F[T]\Big/\left(T^e, \sum_{i=0}^{e-1} a_i T^i\right)$$

As $a_1 \in \mathcal{O}_F^\times$, $T \in$ RHS has preimage.

Step 2: $E = \mathbb{Q}_p$, $\pi = p \neq 2$

- $[\,]$ multipl. + $\mathcal{O}_F$ perfect $\Rightarrow$ $\forall z \in \mathcal{O}_F : \theta([z])$ has $p$-th root
- $\theta([a_0]) = \theta(u)\cdot p$ in $D$ $\quad \boxed{a = u\pi - [a_0]}$

$\Rightarrow$ $D$ is $\theta([a_0])$-adically split and $D/\theta([a_0]) \cong \mathcal{O}_F/a_0$.

$\Rightarrow$ Can write each $x \in D$ in the form

$$x = \sum_{n\geq 0} \theta([x_n])\,\theta([a_0])^n$$

with $x_n = 0$ or $v_F(x_n) < v_F(a_0)$, $\forall n \geq 0$.

- Assume $x \neq 0$ and let $n_0 \geq 0$ minimal s.t. $x_{n_0} \neq 0$

$$\Rightarrow x = \underbrace{\theta([x_{n_0} a_0^{n_0}])}_{\uparrow\ p\text{-th root}}\cdot\left(1 + \sum_{n\geq 1}\dots\right)$$

$\Rightarrow$ May assume $x_0 \in \mathcal{O}_F^\times$

<u>Claim:</u> $\exists z \in \mathcal{O}_F^\times : x \equiv \theta([z]) \bmod p^2$

$\Rightarrow$ May assume $\exists w \in D$ s.t.

$x = 1 + p^2 w$

$$\overset{\text{Newton}}{\Rightarrow} (1+p^2 w)^{\frac{1}{p}} = \underbrace{\sum_{k \geq 0} \binom{\frac{1}{p}}{k} 1^{\frac{1}{p}-k} (p^2 w)^k}_{\text{converges } p\text{-adically}}$$

· For claim: Do explicit computation in $W_{\mathcal{O}_E,1}(\mathcal{O}_F)$ c.f. [Ans]. $\square$

Cor 5:

$D$ is a cplt. val ring with Frac$(D)$ alg. closed and val given by

$$v_D : D \longrightarrow \mathbb{R} \cup \{\infty\},\ d = \theta(\underset{\underset{\mathcal{O}_F}{\cap}}{[x]}) \longmapsto v_F(x)$$

(and this is well-def.)

Proof:

Have by Prop (iii) a mult. surjection

$$\mathcal{O}_F \xrightarrow[\sim]{4(ii)} D^b \xrightarrow{(\_)^\#} D \quad \text{projection}$$

$$x \longmapsto y := (\bar{x}, \bar{x}^{\frac{1}{q}}, \dots) \mapsto y^\# = \lim_{n\to\infty} \theta([x^{\frac{1}{q^n}}])^{q^n}$$

$$= \theta([x]) =: x^\natural$$

As $\theta([a_0]) = \theta(u) \underset{\in D^\times}{\pi}$, get a surj.

$$\mathcal{O}_F\left[\tfrac{1}{a_0}\right] \longrightarrow\!\!\!\!\to D\left[\tfrac{1}{\pi}\right].$$

Then

- $a_0 \in \mathfrak{m}_F \setminus \{0\}$ top. nilpotent

$\Rightarrow \mathcal{O}_F\left[\tfrac{1}{a_0}\right] = F$

$\Rightarrow D\left[\tfrac{1}{\pi}\right]$ is a field.

- $D$ $\pi$-torsion-free
  $\Rightarrow$ $D$ domain, $D[\frac{1}{\pi}] = \mathrm{Frac}(D)$.
- $\mathcal{O}_F$ val. ring
  $\Rightarrow$ $\forall d \in \mathrm{Frac}(D) \setminus \{0\}$: $d \in D$ or $d^{-1} \in D$
  $\overset{D \text{ domain}}{\Rightarrow}$ $D$ val ring

To show $\mathrm{Frac}(D)$ is alg closed, it suffices to show that every
$$P(T) = T^n + b_{n-1} T^{n-1} + \dots + b_0 \in D[T]$$
irred., $n > 0$

has a zero in $D$.

Let $Q(T) \in \mathcal{O}_F[T]$ s.t. $Q(T) \equiv P(T)$ in $D/_{\pi}[T] = \mathcal{O}_F/_{a_0}[T]$,
and $y \in \mathcal{O}_F$ s.t. $Q(y) = 0$.

Then $P(T + y^{\natural})$ has constant term $P(y^{\natural})$
and $P(y^{\natural}) \equiv 0$ in $D/_{\pi}$.

Let $c \in D$ s.t. $n \cdot v_D(c) = v_D(P(y^{\natural}))$ $(\geq v_D(\pi))$

and

$$P_1(T) := c^{-n} P(cT + y^{\natural}) \in \mathrm{Frac}(D)[T]$$

which is irred.

<u>Lemma</u> [Verbitsch, Alg ZT, Kor 4.7]

K non-Arch cplt field, $f = \sum_{i=0}^{n} a_i X^i \in K[X]$ irred.

$$\Rightarrow \max_{i=0}^{n} \{|a_i|\} = \max\{|a_0|, |a_n|\}$$

$\overset{\text{Lemma}}{\Longrightarrow}$ $P_1(T) \in D[T]$.

Like before, find $y_1 \in \mathcal{O}_F$ s.t. $v_D(P_1(y_1^{\natural})) \geq v_D(\pi)$

$$\Rightarrow v_D(\underbrace{P(cy_1^{\natural} + y^{\natural})}_{= c^n \cdot P_1(y_1^{\natural})}) = n\, v_D(c) + v_D(P_1(y_1^{\natural})) \geq 2\, v_D(\pi)$$

So on and get zero of $P(T)$ in the limit.

<u>Prop 6:</u> (Slight gen. from talk 3)

Let $A$ be a $\pi$-cplt $\mathcal{O}_E$-alg, $I \subset A$ ideal s.t. $A$ is $I+(\pi)$-adically cplt.

Then have multipl. bij.

$$\varprojlim_{x \mapsto x^q} A \xrightarrow{1:1} \left(A/I\right)^\flat, \quad (a_0, a_1, \dots) \mapsto (\underset{\substack{\cap \\ A/I+(\pi)}}{\bar{a}_0}, \bar{a}_1, \dots)$$

<u>Cor 7:</u>

Let $f: A \twoheadrightarrow B$ be a surj homom. of $\pi$-cplt $\mathcal{O}_E$-alg s.t. $A$ is $I+(\pi)$-adic. cplt., $I = \mathrm{Ker}(f)$. Then

$$A^\flat \longrightarrow \varprojlim_{x \mapsto x^q} A \longrightarrow \left(A/I\right)^\flat = B^\flat$$

is a ringisom.

<u>Proof:</u>

Use Prop 6. For additivity use

$$\begin{tikzcd} \varprojlim A \arrow[rr] \arrow[dr] & \cong & (A/I)^\flat \\ & A^\flat \arrow[ur] & \end{tikzcd}$$

$\square$

$a = u\pi - [a_0]$, $(\pi) + (a) = (\pi, [a_0])$

$a_0 \in \mathfrak{m}_F \setminus \{0\}$

$A_{\inf}$ $(\pi, [\varpi])$-adically cplt. for $\forall \varpi \in \mathfrak{m}_F$.