Ramified Witt vectors

Aim (Talk 2 - Talk 4) construct B_{dR}

Talk 2: W_{O_E}

Talk 3: $A \longrightarrow A^b$

Talk 4: $\mathbb{A}_{inf} = W_{O_E}(O_C^b)$, $B_{dR} = \mathrm{Frac}\left(A_{inf}[1/\pi]^{\wedge}_{\xi} \right)$
$(\xi = \pi - \pi^b)$

Setup (today and rest of seminar)

- p = prime
- $E/\mathbb{Q}_p$ finite extension

$\longrightarrow$ E cdvf with ring of integers O_E, $m = (\pi)$

$$O_E/m = \mathbb{F}_q$$

Have $p = \pi^e u$, $u \in O_E^\times$
$e \geq 1$

Def: A $\mathcal{O}_E$-alg

$a = (a_i)_{i \geq 0} \in A^{\mathbb{N}}$

$$gh_{n,\pi}(a) := \sum_{i=0}^{n} \pi^i a_i^{q^{n-i}} = a_0^{q^n} + \pi a_1^{q^{n-1}} + \dots + \pi^n a_n$$

$$gh_\pi(a) := \left(gh_{n,\pi}(a)\right)_{n \geq 0} \in A^{\mathbb{N}}$$

("ghost map")

① Lemma: A = torsion-free $\mathcal{O}_E$-alg

$\Rightarrow gh_\pi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is inj

Assume further $\exists\ \varphi : A \to A$ q-Frob lift

(i.e. $\varphi(a) \equiv a^q \mod \pi A$)

Then

$b = (b_n)_{n \geq 0} \in \operatorname{Im}(gh_\pi) \Longleftrightarrow b_{n+1} \equiv \varphi(b_n) \mod \pi^{n+1}$

<u>Pf:</u> <u>inj:</u> $gh_0(a) = a_0$, $gh_1(a) = a_0^q + \pi a_1, \dots$

<u>$\Longrightarrow$:</u>

$$\varphi(gh_n(a)) = \sum_{i=0}^{n} \pi^i \underbrace{\varphi(a_i)^{q^{n-i}}}$$

$$\equiv a_i^{q^{n+1-i}} \mod \pi^{n+1-i}$$

(*) $\qquad [\varphi(a_i) \equiv a_i^q \mod \pi]$

$$\equiv \sum_{i=0}^{n+1} \pi^i a_i^{q^{n+1-i}} \qquad \mod \pi^{n+1}$$

<u>$\Longleftarrow$:</u> Assume $b_{n+1} \equiv \varphi(b_n) \mod \pi^{n+1} \quad \forall n$

Set $a_0 := b_0 \quad \Rightarrow \quad gh_0((a_0, \dots)) = b_0$

Assume $a_0, \dots, a_n$ are constr. with $gh_i(a_0, \dots, a_n) = b_i$
$i \leq n$

Look for $a_{n+1} \in A$ with

$$gh_{n+1}(a_0, \dots, a_n, a_{n+1}, \dots) = b_{n+1}$$

$$\Longleftrightarrow \quad \pi^{n+1} a_{n+1} = b_{n+1} - \sum_{i=0}^{n} \pi^i a_i^{q^{n+1-i}}$$

$$\underset{(*)}{\equiv} \varphi(b_n) - \varphi(gh_n(a)) \mod \pi^{n+1}$$

$$= 0$$

$$\underset{A\ \pi\text{-torsion free}}{\Longrightarrow} \quad \exists!\ a_{n+1} \ \text{ s.t. } \ gh_{n+1}(a_0, \dots, a_{n+1}, \dots) = b_{n+1}$$

$\square$

Def - Thm:

$\exists !$ functor

$$W_{\mathcal{O}_E,\pi} : (\mathcal{O}_E\text{-alg}) \longrightarrow (\mathcal{O}_E\text{-alg})$$

s.t.:

- $W_{\mathcal{O}_E,\pi}(A) = \{ (a_0, a_1, a_2, \ldots) \mid a_i \in A \}$ (as a set)

- $f : A \to B$: $W_{\mathcal{O}_E,\pi}(f) : W_{\mathcal{O}_E,\pi}(A) \to W_{\mathcal{O}_E,\pi}(B)$
  $(a_0, a_1, \ldots) \mapsto (f(a_0), f(a_1), \ldots)$

- $gh : W_{\mathcal{O}_E,\pi}(A) \longrightarrow A^{\mathbb{N}}$
  $a \longmapsto gh(a)$

is a natural transformation of functors of $\mathcal{O}_E$-algebras.

Pf:

$$\mathcal{U} = \mathcal{O}_E[x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots]$$

$$\varphi : \mathcal{U} \longrightarrow \mathcal{U} \qquad x_i \longmapsto x_i^q, \quad y_i \longmapsto y_i^q$$

q-Frob lift. $\quad \mathcal{O}_E$-lin

$$x = (x_0, x_1, \dots), \quad y = (y_0, y_1, \dots) \in W_{\mathcal{O}_E,\pi}(\mathcal{U})$$

$$\Rightarrow \quad \varphi(gh_n(x) + gh_n(y)) \equiv gh_{n+1}(x) + gh_{n+1}(y) \mod \pi^{n+1}$$

① + φ map of $\mathcal{O}_E$-alg

$$\varphi(gh_n(x) \cdot gh_n(y)) \equiv gh_{n+1}(x)\, gh_{n+1}(y) \quad ''$$

$$\varphi(\lambda\, gh_n(x)) \equiv \lambda\, gh_{n+1}(x) \quad ''$$

$\lambda \in \mathcal{O}_E$

$\overset{①}{\Longrightarrow}$ $\exists !$ $\quad S = (S_n(x,y))$, $\quad P = (P_n(x,y))$, $\quad \mu_\lambda(x) = (\mu_{\lambda,n}(x))$ with

$$gh(S) = gh(x) + gh(y)$$
$$gh(P) = gh(x)\, gh(y)$$
$$gh(\mu_\lambda(x)) = \lambda\, gh(x)$$

let A : $\mathcal{O}_E$ - alg

for $a, a' \in W_{\mathcal{O}_E, \pi}(A)$, $\lambda \in \mathcal{O}_E$

define :

$$a +_w a' := S(a, a')$$

$$a \cdot_w a' := P(a, a')$$

$$\lambda \cdot_w a := \mu_\lambda(a)$$

$\rightarrow$ defines a ring structure on $W_{\mathcal{O}_E, \pi}(A)$

with $0_w = (0, 0, \ldots)$

$1_w = (1, 0, \ldots)$

s.t. $gh : W_{\mathcal{O}_E, \pi}(A) \to A^{\mathbb{N}}$

is a $\mathcal{O}_E$ - alg hom $\forall A$ $\square$

<u>Bsp:</u> $s_0(x,y) = gh_0(s(x,y)) = gh_0(x) + gh_0(y) = x_0 + y_0$

$$s_0^q + \pi s_1 = gh_1(s) = gh_1(x) + gh_1(y)$$
$$= (x_0^q + \pi x_1) + (y_0^q + \pi y_1)$$

$$\Rightarrow s_1 = \frac{x_0^q + y_0^q - (x_0+y_0)^q}{\pi} + x_1 + y_1$$

Similar

$$P_0(x,y) = x_0 y_0$$
$$P_1(x,y) = x_0^q y_1 + y_0^q x_1 + \pi x_1 y_1$$

$$M_{\lambda,0}(x) = \lambda x_0 \quad , \quad M_{\lambda,1}(x) = \frac{\lambda - \lambda^q}{\pi} x_0^q + x_1 , \ldots$$

Thus $a, b \in W_{O_E,\pi}(A)$

$$\Rightarrow \quad a+b = \left(a_0+b_0 , \frac{a_0^q + b_0^q - (a_0+b_0)^q}{\pi} , \ldots\ldots \right)$$
$$ab = \left(a_0 b_0 , a_0^q b_1 + a_0^q b_1 + \pi a_1 b_1 , \ldots\ldots \right)$$
$$\lambda \cdot a = \left(\lambda a_0 , \frac{\lambda - \lambda^q}{\pi} a_0^q + a_1 , \ldots\ldots \right)$$

<u>Def:</u> The construction above also works for

$$(\mathcal{O}_E, q, \pi) \longleftrightarrow (\mathbb{Z}, p, p)$$

$\rightsquigarrow$ $W(A)$ <u>ring of p-typical Witt vectors</u> defined for any ring A

If A = $\mathbb{Z}_p$-alg

$\Rightarrow W(A) = W_{\mathbb{Z}_p, p}(A)$

<u>Note</u>: $\mathcal{U}$, φ as above.

$\mathfrak{m} = (\pi) = (\pi') \subset \mathcal{O}_E$

(1) $\Rightarrow$

$$\begin{tikzcd} W_{\mathcal{O}_E,\pi}(\mathcal{U}) \arrow[r, "gh_\pi", "\cong"'] \arrow[d, dotted, "\mu_{\pi,\pi'}", "\cong"'] & \{(b_n)_{n\geq 0} \in \mathcal{U}^{\mathbb{N}} \mid b_{n+1} \equiv \varphi(b_n) \bmod \mathfrak{m}\} \\ W_{\mathcal{O}_E,\pi'}(\mathcal{U}) \arrow[ur, "gh_{\pi'}", "\cong"'] & \end{tikzcd}$$

In part. $s_{\pi'} = \mu_{\pi,\pi'}(s_\pi)$ etc. ...

$\rightsquigarrow$ obtain

$$\begin{tikzcd} W_{\mathcal{O}_E,\pi} \arrow[r, "\mu_{\pi,\pi'}"] \arrow[dr, "gh_\pi"'] & W_{\mathcal{O}_E,\pi'} \arrow[d, "gh_{\pi'}"] \\ & (-)^{\mathbb{N}} \end{tikzcd}$$

$\exists$ son of $\mathcal{O}_E$-alg

<u>Def</u>: $W_{\mathcal{O}_E}(A) := \varprojlim_{\pi} W_{\mathcal{O}_E,\pi}(A)$

$gh : W_{\mathcal{O}_E}(A) \longrightarrow W_{\mathcal{O}_E,\pi}(A) \xrightarrow{gh_\pi} A^{\mathbb{N}}$.

② **Prop.**

(1) $[-] : A \longrightarrow W_{\mathcal{O}_E}(A)$, $a \longmapsto [a] = (a, 0, 0, \dots)$

multipl.

(not additive) $\quad (u_{\pi,\pi'}([a]) = [a])$

(2) $\exists\ F : W_{\mathcal{O}_E}(A) \longrightarrow W_{\mathcal{O}_E}(A)$

nat $\mathcal{O}_E$-alg hom s.t. $gh_n(F(a)) = gh_{n+1}(a)$

$\rightsquigarrow$ · $F([a]) = [a^q]$

· $F : W_{\mathcal{O}_E}(A) / \pi W_{\mathcal{O}_E}(A) \longrightarrow W_{\mathcal{O}_E}(A) / \pi$

$\bar{a} \longmapsto \bar{a}^q$

· if $\pi A = 0 \Rightarrow$

$F(a_0, a_1, a_2, \dots) = (a_0^q, a_1^q, a_2^q, \dots)$

(also denoted φ in this case)

(3) $\exists\ V_\pi : W_{\mathcal{O}_E}(A) \longrightarrow W_{\mathcal{O}_E}(A)$ $\quad \mathcal{O}_E$-lin.

$(a_0, a_1, \dots)_\pi \longmapsto (0, a_0, a_1, \dots)_\pi$ $\quad$ (dep on π)

s.t. $FV_\pi = \pi$, $V_\pi(F(x)\cdot y) = x\, V_\pi(y)$, $V_{\pi'} = \frac{\pi'}{\pi} V_\pi$

(4) $$W_{\mathcal{O}_E}(A) \cong \varprojlim_n \underbrace{W_{\mathcal{O}_E}(A) / V_\pi^n W_{\mathcal{O}_E}(A)}_{\text{indep of } \pi \text{ by (3)}}$$

and any Elt can be uniquely written as

$$a = \sum_{n \geq 0} V_\pi^n([a_n]) \quad , a_n \in A$$

(5) If $\pi A = 0 \Rightarrow V_\pi F = \pi$

(6) If $\pi A = 0$ and A is perfect $\left(\text{i.e. } \begin{array}{l} A \to A \\ a \mapsto a^p \end{array} \text{ bij.}\right)$

$\Rightarrow$
- $V_\pi^n W_{\mathcal{O}_E}(A) = \pi^n W_{\mathcal{O}_E}(A)$
- $W_{\mathcal{O}_E}(A)$ is π-adic complete
- any Elt $a \in W_{\mathcal{O}_E}(A)$ can be uniquely written as

$$a = \sum [a_n'] \pi^n \quad , a_n' \in A$$

Pf: for (1) - (4) it suff to consider

$A = \mathfrak{A}$ and check after applying gr

(except in (2) the $\pi A = 0$ case, which follows from $V_\pi \mathcal{F} = \mathcal{F} V_\pi$ by (5))

(5) show $V_\pi([1]) = \pi[1]$ if $\pi A = 0$

$$\Rightarrow \quad V_\pi(\mathcal{F}(a)) \underset{(3)}{=} a \, V_\pi([1]) = \pi a$$

(6) $\Leftarrow$ (5), (3) since

$$V_\pi^{-1}([a]) \underset{\substack{\uparrow \\ \text{A perf.}}}{=} V_\pi^{-1}([a' {}^{q}]) = V_\pi^{-1}(\mathcal{F}^{-1}[a']) = \pi^{-1}[a'] \qquad \square$$

Rmk.

$q = p^f$, $a \in W(A)$

$\Rightarrow \left(b_n := gh_{nf,p}(a) \right)_{n \geq 0}$ satisfies $b_{n+1} = \varphi(b_n) \bmod \pi^{n+1}$

$\underset{(I)}{\Rightarrow}$ $\exists!\ \alpha \in W_{\mathcal{O}_E}(A) : gh(\alpha) = (b_n)$

$\rightsquigarrow$
$$\begin{tikzcd} W(A) \arrow[r] \arrow[dr, "(gh_{nf})_{n\geq 0}"'] & W_{\mathcal{O}_E}(A) \arrow[d, "gh"] \\ & A^{\mathbb{N}} \end{tikzcd}$$

Assume A = perfect $\mathbb{F}_q$-alg

$\Rightarrow$ $W(A) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E \xrightarrow{\cong} W_{\mathcal{O}_E}(A)$

Have: • $W(\mathbb{F}_q)$ = unramified ext of $\mathbb{Z}_p$ with res field $\mathbb{F}_q$

• E'/E fin $\mathcal{O}_{E'}/m' = \mathbb{F}_{q'}$

$\Rightarrow$ $W_{\mathcal{O}_E}(\mathbb{F}_{q'})$ = max'l unramified subextension of $\mathcal{O}_{E'}/\mathcal{O}_E$

Lemma: R perfect $\mathbb{F}_q$-alg.

$\alpha \in W_{O_E}(R)$

Then $\exists!\ r_i \in R$, $i \geq 0$: $\alpha = \sum_{i=0}^{\infty} [r_i]\pi^i$ (by ②, (6))

and

$$\frac{F(\alpha)-\alpha^q}{\pi} \in W_{O_E}(R)^\times \iff r_1 \in R^\times$$

Pf:

$x \in W_{O_E}(R)$

Then x unit $\iff \bar{x} \in \left(\frac{W_{O_E}(R)}{(\pi)}\right)^\times = R^\times$

And

$$\frac{F(\alpha)-\alpha^q}{\pi} = \frac{\sum_{i=0}^{\infty}[r_i^q]\pi^i - \left(\sum_{i=0}^{\infty}[r_i]\pi^i\right)^q}{\pi}$$

$$= [r_1^q] - \underbrace{q\,[r_1][r_0]^{q-1}}_{\in \pi(\dots)} + \pi(\dots)$$

$$\equiv r_1^q \mod \pi \qquad \square$$

<u>Def:</u> R perfect $\mathbb{F}_q$-alg

$d \in W_{\mathcal{O}_E}(R)$ is called <u>distinguished</u>

if $\frac{F(d) - d^q}{\pi} \in W_{\mathcal{O}_E}(R)$

$\left(\underset{\text{above}}{\Longleftrightarrow} \quad d = \sum_{i \geq 0} [r_i] \pi^i \quad \text{with } r_1 \in R^\times \right)$

<u>Def:</u>

(1) A <u>perfect prism</u> over $\mathcal{O}_E$ is a pair

$(W_{\mathcal{O}_E}(R), I)$ with R perfect $\mathbb{F}_q$-alg, $I = (d)$, d disting.

(2) A $\mathcal{O}_E$-alg

We say A is <u>perfectoid</u> $\Longleftrightarrow$

$\exists$ <u>perfect prism</u> $(W_{\mathcal{O}_E}(R), I)$ s.t.

$$A \cong W_{\mathcal{O}_E}(R) / I$$

Rmk: • A perfect $\mathbb{F}_q$-alg

$\Rightarrow$ A perfectoid

(since $A \cong W_{O_E}(A)/(\pi)$)

Prop: S Ring

$\alpha \in S$ non-zero div s.t.

$\alpha^p \mid p$, S is α-adically complete.

and $\varphi : S/\alpha \longrightarrow S/\alpha^p$ is an Isom.

$\bar{a} \longmapsto \bar{a}^p$

$\Rightarrow$ S perfectoid.

Example

$$S_0 := \mathbb{Z}_p[p^{1/p^\infty}] := \bigcup_{n \geq 0} \mathbb{Z}_p[p^{1/p^n}] = \frac{\mathbb{Z}_p[X_1, X_2, X_3, \dots]}{(X_1^p - p,\ X_2^p - X_1,\ X_3^p - X_2, \dots)}$$

$$S = \varprojlim_N S_0/p^N$$

$$\alpha := p^{1/p} = \overline{X}_1$$

$\Rightarrow$ $(p) \subset (\alpha)$ and $(\alpha^p) = (p)$

$\Rightarrow$ S is α-adically complete

$$S/\alpha = \frac{\mathbb{F}_p[X_2, X_3, \dots]}{(X_2^p,\ X_3^p - X_2 \quad)} \xrightarrow{\varphi} S/\alpha^p = \frac{\mathbb{F}_p[X_1, X_2, X_3, \dots]}{(X_1^p,\ X_2^p - X_1, \dots)}$$

$$\begin{array}{cccc} i \geq 2 & \overline{X}_i & \longmapsto & \overline{X}_{i-1} \quad \text{isom} \\ & \uparrow & & \uparrow \\ S_0 \ni & \overline{X}_i & \longmapsto & \overline{X}_i^p = X_{i-1} \end{array}$$

$\Rightarrow$ S perfectoid.