Dennis Part 1

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The Fargues - Fontaine - Curve

topic: The ring B

Setting:

- $q = p^r \in \mathbb{N}$ prime power
- $E/\mathbb{Q}_p$ finite ex.
- $(\mathcal{O}_E, (\pi))$ ring of integers of E with uniformizer $\pi \in \mathcal{O}_E$
- $\mathbb{F}_q = \mathcal{O}_E / (\pi)$ res. field
- $(F/\mathbb{F}_q, |\cdot|)$ a non-archimedian al. closed ex.
  with a norm $|\cdot|: F \to \mathbb{R}_{\geq 0}$
  $a \mapsto q^{-v(a)}$
  correspon valuation $v: F \longrightarrow \mathbb{R}_{\geq 0}$
- $(\mathcal{O}_F, m_F)$ ring of integers of F with
  $\mathcal{O}_F = \{x \in F \mid |x| \leq 1\}$ and $m_F = \{x \in F \mid |x| < 1\}$
- C/E alg. closed, non-archimed. ex.
  with a valuation $v_C: C \to \mathbb{R} \cup \{+\infty\}$, s.th.
  $(\mathcal{O}_C, v_C)$ valuation ring.
  $\forall r \geq 0$ there are valuations

$v_r : \mathbb{A}_{\inf} \longrightarrow \mathbb{R} \cup \{\infty\}$

$$f = \sum_{i=0}^{\infty} [x_i] \cdot \pi^i \longmapsto \inf_{i \in \mathbb{Z}} \{ v(x_i) + ir \}$$

and $\mathrm{New}(f)$ is the convex, decreasing, piecewise linear function with Legendre-transform.

$$\mathcal{L}(\mathrm{New}(f)) := \begin{cases} v_r(f), & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

The following commutative diagram

$$\begin{tikzcd} \mathbb{A}_{\inf} \arrow[rr, "v_r"] \arrow[dr, hook] & & \mathbb{R} \cup \{\infty\} \\ & B^b \arrow[ur, "\tilde{v}_r"'] & \end{tikzcd}$$

with $B^b := \mathbb{A}_{\inf}\left[\frac{1}{\pi}, \frac{1}{\varpi}\right] = \left\{ \sum_{n >> -\infty}^{\infty} [x_n] \pi^n \;\middle|\; \inf_{i \in \mathbb{Z}} \{ v(x_i) \} > -\infty \right\}$

$\tilde{v}_r : B^b \longrightarrow \mathbb{R} \cup \{\infty\}$

$$f = \sum_{n >> -\infty}^{\infty} [x_n] \pi^n \longmapsto \inf_{i \in \mathbb{Z}} \{ v(x_i) + ir \}$$

defines the ex. $\tilde{v}_r$ of v_r and $\varpi \in \mathfrak{m}_F \setminus \{0\}$.

Furthermore

$$\mathcal{L}(\mathrm{New}(f)) = \begin{cases} \tilde{v}_r(f), & r \geq 0 \end{cases}$$

Furthermore

$$h\,(\mathcal{M}_{\text{cont}}(f)) := \begin{cases} \tilde{v}_r(f), & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

is the correspon. ex. from $\mathcal{M}_{\text{cont}}(f)$ to B^b.

__Def__:

Let $I \subset (0,\infty)$ be an interval. The completion of B^b for the family of valuations $(v_r)_{r\in I}$ is given by

$$B_I = \varprojlim_{\mathcal{U}\in F} B^b/\mathcal{U}$$

with the fundamental system

$$F_I := \left\{ \bigcap_{i=1}^{n} v_{r_i}^{-1}[m,\infty) \;\middle|\; n,m \in \mathbb{N},\, r_i \in I \right\} \subseteq B^b$$

are open?

__Def__:

We set $\underline{\underline{B}} := B_{(0,\infty)}$.

$\forall I \subset (0,\infty)$ there is an interval $q\,I := \{ q\cdot y \mid y \in I\} \subset (0,\infty)$

and the bijective map

$$I \longrightarrow q\,I$$

$$x \mapsto qx$$

$$I \longrightarrow qI$$
$$y \longmapsto q \cdot x$$

There is a commutative diagram

$$\begin{tikzcd} B^b \arrow[r, "\varphi", "\sim"'] \arrow[d] & B^b \arrow[d] \\ B_I \arrow[r, "\bar{\varphi}", "\sim"'] & B_{qI} \end{tikzcd}$$

with the maps

$$\varphi : B^b \longrightarrow B^b$$
$$\sum_{n \gg -\infty}^{\infty} [x_n] \pi^n \longmapsto \sum_{n \gg -\infty}^{\infty} [x_n^q] \pi^n$$

$$B^b \longrightarrow B_I$$
$$x \longmapsto (x + \mathcal{U})_{\mathcal{U} \in \mathcal{F}_I}$$

$$\bar{\varphi} : B_I \xrightarrow{\sim} B_{qI}$$
$$\left(\sum_{n \gg -\infty}^{\infty} [x_{n,\mathcal{U}}] \pi^n + \mathcal{U} \right)_{\mathcal{U} \in \mathcal{F}_I} \longmapsto \left(\sum_{n \gg -\infty}^{\infty} [x_{n,\mathcal{U}}^q] \cdot \pi^n + \mathcal{U} \right)_{\mathcal{U} \in \mathcal{F}_{qI}}$$

and the inverse maps

$$\psi : B^b \longrightarrow B^b$$

$$\psi : B^b \longrightarrow B^b$$
$$\sum_{n \gg -\infty}^{\infty} [y_n] \pi^n \mapsto \sum_{n \gg -\infty}^{\infty} [(y_n)^{\frac{1}{q}}] \pi^n$$

$$\tilde{\psi} : B_{qI} \longrightarrow B_I$$
$$\left(\sum_{n \gg -\infty}^{\infty} [x_{n,\mathcal{U}}] \pi^n + \mathcal{U} \right)_{\mathcal{U} \in \mathcal{F}_{qI}}$$
$$\downarrow$$
$$\left(\sum_{n \gg -\infty}^{\infty} [x_{n,\mathcal{U}}^{\frac{1}{q}}] \pi^n + \mathcal{U} \right)_{\mathcal{U} \in \mathcal{F}_I}$$

If we look at $I = (0, +\infty)$

we have
$$\begin{tikzcd} B^b \arrow[r, "\varphi"] \arrow[d, hook] & B^b \arrow[d, hook] \\ B \arrow[r, "\tilde{\varphi}"] & B \end{tikzcd}$$

[<u>Alternative Def of B :</u>

$I \subset (0,1)$ be an interval.

The compl. of B^b for the family of norms $(|\cdot|_\rho)_{\rho \in I}$ is given by

$$B_I = \varprojlim B^b / \mathcal{U}$$

$$B_I = \varprojlim_{U \in \mathcal{F}} B^b / U$$

and the open neighborhoods U of 0 are given by the topology of the complete norm

$$\|\cdot\|_I := \sup_{\rho \in I} |\cdot|_\rho : B_I \to [0, +\infty]$$

Here is

$$\underline{\underline{|\cdot|_\rho}} : B^b \to \mathbb{R} \cup \{+\infty\}$$

$$\sum_{n \gg -\infty}^{\infty} [x_n] \pi^n \mapsto \sup_{n \in \mathbb{Z}} |x_n| \rho^n$$

$\forall \rho \in (0,1)$. $\forall \rho \in (0,1)$ $\exists!\, r \in (0, +\infty)$ s. th. $\rho = q^r$ and $|\cdot|_\rho = \rho^{-v_r(\cdot)}$,

where $v_r : B^b \to \mathbb{R} \cup \{+\infty\}$

$$B := B_{(0,1)}$$

<u>Ex</u> :

Let $I = [\rho_1, \rho_2] \subset (0,1)$ with $a, b \in F$, s. th. $|a| = \rho_1$ and $|b| = \rho_2$.

Further $B := \underline{B(0,1)}$ and it follows

$$B(0,1) = \{ x \in B^b \mid \|x\|_I \leq 1 \}$$

$$= A_{inf}\left[\frac{[a]}{\pi}, \frac{\pi}{[b]} \right]$$

with $B^b = A_{inf}\left[\frac{1}{\pi}, \frac{1}{[\varpi]} \right]$, the $\|\cdot\|_I$ norm in the altern. def.

and $B_I = \widehat{A_{inf}\left[\frac{[a]}{\pi}, \frac{\pi}{[b]} \right]}\left[\frac{1}{\pi} \right]$

in the π-adic topology.

<u>Def:</u> (The schematic Fargues-Fontaine curve)

$$X := X_{E,F} := \mathrm{Proj}\left(\bigoplus_{d \geq 0} B^{\varphi = \pi^d} \right)$$

with $B^{\varphi=\pi^d} = \{ x \in B \mid \varphi(x) = \pi^d \cdot x \}$

$\forall\, d \geq 0$.

<u>Ex:</u>

simplest setting:

$E = \mathbb{Q}_p \subset F = \mathrm{Quot}(\mathcal{O}_C^\flat)$ with $C = \widehat{\overline{\mathbb{Q}}_p}$

and $B = B(0,1)$ to look at the graded ring structr. on $\bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^n}$

Let $f \in B^{\varphi = p^m}$, $g \in B^{\varphi = p^n}$

we know $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$

$= (p^m f) \cdot (p^n \cdot g)$

$= p^{m+n} fg \in B^{\varphi = p^{m+n}}$

For $f = \sum_{n \in \mathbb{Z}} [c_n] \cdot p^n \in \bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^n}$ converging,

i.e. $c_n \in C^\flat$ $\forall n \in \mathbb{Z}$ with

- $\limsup_{n > 0} |c_n|^{\frac{1}{n}} \leq 1$
- $\lim_{n \to \infty} |c_{-n}|^{\frac{1}{n}} = 0$

Recall. $\sum [c_n^p] \cdot p^n = \varphi(f) = p^k \cdot f$

Koeff. compar.

$$\sum_{n\in\mathbb{Z}}{}' \,[c_n^p]\cdot p^n = \varphi(f) = p^k\cdot f$$

$$= \sum_{n\in\mathbb{Z}} [c_n]\, p^{n+k}$$

$$= \sum_{n\in\mathbb{Z}} [c_{n-k}]\cdot p^n$$

Hence $c_{n-k} = c_n^p$
$\forall n, k \in \mathbb{Z}$.

<u>1 case:</u> Let $k = 0$

Hence $c_n = c_n^p \ \forall n \in \mathbb{Z}$, i.e. $c_n \in \mathbb{F}_p \subset C^\flat$.

This induces a map

$$\begin{array}{ccccc}
\mathbb{F}_p & \hookrightarrow & W_{\mathcal{O}_E}(\mathbb{F}_p) & \hookrightarrow & W_{\mathcal{O}_E}(\mathbb{F}_p)[\tfrac{1}{p}] \\
c_n & \mapsto & [c_n] & \longmapsto & [c_n] \qquad \| \\
 & & & & W(\mathbb{F}_p)[\tfrac{1}{p}] \\
 & & & & \| \\
 & & & & \mathbb{Q}_p \\
 & & & & \downarrow \\
 & & & [c_n]\cdot p & B^{\varphi=p}
\end{array}$$

Case : $k>0$

Then the condition $c_{n-k} = c_n^p$ implies

$$c_k^p = c_0 \Rightarrow c_k = c_0^{\frac{1}{p}}$$

$$c_{k+1}^p = c_1 \Rightarrow c_{k+1} = c_1^{\frac{1}{p}}$$

$$c_{k+2}^p = c_2 \Rightarrow c_{k+2} = c_2^{\frac{1}{p}}$$

$$c_{k+3}^p = c_3 \Rightarrow c_{k+3} = c_3^{\frac{1}{p}}$$

$$\vdots$$

$$c_{2k}^p = c_k \Rightarrow c_{2k} = c_k^{\frac{1}{p}} = c_0^{\frac{1}{p^2}}$$

$$c_{2k+1}^p = c_{k+1} \Rightarrow c_{2k+1} = c_{k+1}^{\frac{1}{p}} = c_1^{\frac{1}{p^2}}$$

$$c_{2k+2}^p = c_{k+2} \Rightarrow c_{2k+2} = c_{k+2}^{\frac{1}{p}} = c_2^{\frac{1}{p^2}}$$

$$\vdots$$

Therefore $c_0, \dots, c_{k-1}$ determines the sequence $(c_n)_{n \in \mathbb{Z}_{\dots}}$

sequence $(c_n)_{n\in\mathbb{Z}}$.

Claim:

If f converge in B^b,

$c_0,\dots,c_{k-1}\in\mathfrak{m}^b_C\subset\mathcal{O}^b_C$.

proof:

Analog to Ref. [10] in Proposition 4.2.1. part (2)

1:1 – correspondence

$$\left\{\begin{aligned}(\mathfrak{m}^b_C)^k &\longrightarrow B^{\varphi=p^k}\\ (c_0,\dots,c_{k-1}) &\longmapsto \sum_{i=0}^{k-1}\sum_{n\in\mathbb{Z}}[x_i^{p^{-n}}]\,p^{nk+i}.\end{aligned}\right.$$

where $f\in B^{\varphi=p^k}$

$$\left(\;\cdot\quad\begin{tikzcd} \mathcal{O}^b_C \arrow[r, "\mathcal{F}"] & B^{\varphi=p^k} \\ \mathfrak{m}^b \arrow[u, hook] \arrow[ur] & \end{tikzcd}\right)$$

m_c^b

Lemma:

Let $(x_n)_{n\in\mathbb{Z}} \subset F^{\mathbb{Z}}$ a seq. s.th.

$$\lim_{|n|\to\infty} v(x_n)+rn = \infty \quad \forall r\in(0,\infty).$$

Then $\sum_{n\in\mathbb{Z}} [x_n]\pi^n$ converge in B.

proof:

It sufficies to show that

$$v_r([x_n]\pi^n) \to \infty \quad \forall |n|\to\infty \quad \forall r\in(0,\infty).$$

$$\left(\Leftarrow \quad |[x_n]\cdot\pi^n|_\rho = 0 \quad \forall |n|\to\infty, \ \forall \rho\in(0,1) \right)$$

$$v_r([x_n]\pi^n) = v(x_n)+rn \to \infty$$

$\forall |n|\to\infty \quad \forall r\in(0,\infty)$. □

Remark:

The construction of the bijection

$$\mathfrak{m}_F \longrightarrow B^{\varphi=\pi}.$$

Let $a \in \mathfrak{m}_F$, i.e. $v(a) > 0$.

Define $$f_a := \sum_{i \in \mathbb{Z}} [a^{q^{-i}}]\pi^i$$

converges in B, because

$$v(a^{q^{-i}}) + ir = q^{-i}\, v(a) + ir \longrightarrow \infty$$

$\forall$ $|i| \to \infty$ and $v(a) > 0$.

Moreover $$\varphi(f_a) = \sum_{i \in \mathbb{Z}} [a^{q^{-(i-1)}}]\pi^i = \pi \cdot f_a$$

$$\rightsquigarrow \quad \mathfrak{m}_F \longrightarrow B^{\varphi=\pi}$$
$$a \longmapsto f_a$$