For me: $E = \mathbb{Q}_p$, $F / \mathbb{F}_p$ com. alg. closed non-arch field ($F = \widehat{\mathbb{F}_p((t))}$)

$$A_{inf} := W(\underline{\mathcal{O}_F}) \qquad \varpi = [t]$$

Perfectoid

$\{\text{Untilts } (K, \iota)\}/\sim$

real # $r(K,\iota) := |p|_K$

p

A_{inf}

$A_{inf}[\frac{1}{p}]$

$A_{inf}[\frac{1}{\varpi}]$

$B^b = A_{inf}[\frac{1}{p}, \frac{1}{\varpi}]$

Complex Analytic

$\mathbb{D}$ = open disk $\{z \in \mathbb{C} \mid |z| < 1\}$

$|z|$

coordinate fn z

Series $\sum_{n=0}^{\infty} c_n z^n$, $|c_n| \leq 1$

Series $\sum_{n \gg -\infty} c_n z^n$, $|c_n| \leq 1$

Series $\sum_{n=0}^{\infty} c_n z^n$, $|c_n|$ bounded

Series $\sum_{n \gg -\infty} c_n z^n$, $|c_n|$ bounded

all yield meromorphic fns on $\mathbb{D}$

Goal of introducing B: some very natural fns that we want to exist but don't in $A_{inf}[\frac{1}{p}, \frac{1}{\varpi}]$ b/c they have an "essential singularity at $p=0$"

B ...

Holomorphic fns on ...

$\mathcal{B}_{[a,b)}$ — Holomorphic fns on $a < |z| < b$

$\mathcal{B}$ — Holomorphic fns on $\mathbb{D}^\times$

=

Claim Let K be a complete, non-arch field of char 0. Let $x \in m_K$. Then

$$\log(1+x) := x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$ converges.

$(p > 2)$

$\rightsquigarrow$ **Claim** Let $e \in \mathcal{O}_F$ with $|e-1|_F < 1$

Then for <u>every</u> char 0 untilt (K, ι)

$$\rightsquigarrow e^\# \in K$$

$$\Rightarrow \log(e^\#) \in K$$

Question Does "$\log([e])$" exist (in A_{inf})

Answer: NO! "$\log([e])$" is not even in

$$A_{inf}\left[\frac{1}{p}, \frac{1}{\varpi}\right]$$

because it has an "essential singularity at $p=0$"

Note If $A / \mathbb{Q}_p$ is comm. algebra w/ a sub-mult. norm $|\cdot|_A$

$$|x \cdot y|_A < |x|_A |y_A|$$

Sub-mult. norm

$$|x\cdot y|_A \leq |x|_A |y_A|$$

- If $x \in A$ w/ $|x-1|_A < 1$
  $\Rightarrow \log(x) \in \hat{A}$

- If $y \in A$ w/ $|y-1|_A < 1$
  $\Rightarrow \log(xy) = \log(x) + \log(y)$ in $\hat{A}$

It follows that if $x \in F$ w/ $|x-1|_F < 1$
$\rightsquigarrow \log f([x])$ converges w.r.t all Gauss norms $|\cdot|_p$

$\rightsquigarrow \log([x]) \in B$

(also: If $y \in F$ w/ $|y-1|_F < 1$,
$\log([xy]) = \log([x]) + \log([y])$ in B.

$=$

Recall $\varphi : A_{inf} \to A_{inf}$ induces $\varphi : B \to B$

$B^{\varphi = p^d}$? P_d

Question $\log([x]) \in B^{\varphi = p^d}$ for some d?

Answer yes!

$\varphi(\log f[x])) = \log(\varphi([x]))$

$$\varphi(\log([x])) = \log(\varphi([x]))$$
$$= \log([x^p])$$
$$= p\log([x])$$

$$\Rightarrow \log([x]) \in B^{\varphi = p} \; !$$

Thm There exists a commutative diagram of sets

$$\begin{tikzcd}
 & 1+m_F \arrow[dr, "\log({[\cdot]})"] & \\
m_F \arrow[ur, dashed, "E"] \arrow[rr] & & B^{\varphi=p} \\
a \arrow[rr, mapsto] & & \displaystyle\sum_{n\in\mathbb{Z}} \frac{[a^{p^n}]}{p^n}
\end{tikzcd}$$

E is the "Artin-Hasse exponential"

Note: E is formally: $\exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots\right)$

=

Reminder $I \subseteq (0,1)$ (using norm notation)

B_I denotes the completion of $A_{inf}[\frac{1}{p}, \frac{1}{\varpi}]$ for $(|\cdot|_\rho)_{\rho \in I}$

$$P := \bigoplus_{d \geq 0} P^d := \bigoplus_{d\geq 0} B^{\varphi = p^d} \quad \text{a graded ring}$$

$$X = \operatorname{Proj}(P)$$

Thm (Good for today) ... w/ irreducible elements of

Thm (Good for today)
P is graded factorial w/ irreducible elements of deg 1: $f \in P_d$, $\exists\ t_1, \dots, t_d \in P_1$ w/

$$f = t_1 \cdot \dots \cdot t_d$$

Lemma let $I \subseteq (0,1)$, $y \in |Y_I|$

((K, ι) untilts w/ $|p|_K \in I$)

Then

$$\begin{tikzcd} \Theta_y : A_{\inf}\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] \arrow[r] \arrow[d, hook] & C_y \\ B_I \arrow[ur, dotted, "\text{extends}"'] & \end{tikzcd}$$

AND: $\ker(\Theta_y : B_I \to C_y) \subseteq B_I$
$= (\xi_y) B_I$, where
ξ_y is a gen of $\ker(A_{\inf} \to C_y)$

"Pf" Key fact: $I = [a,b]$, $\pi_a, \pi_b \in F$ w/ $|\pi_a| = a$, $|\pi_b| = b$

$$B_I = \widehat{A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]}\left[\frac{1}{p}\right]$$

p-adic completion

So the fact that Θ_y extends to B_I comes from:

$$\Theta_y\left(\frac{[\pi_a]}{p}\right) \in \mathcal{O}_{C_y}$$

$$\Theta_y\left(\frac{p}{[\pi_b]}\right) \in \mathcal{O}_{C_y}$$

ring of ints

$$\mathcal{O}_y\left(\frac{p}{[\pi_b]}\right) \in \mathcal{O}_{C_y}$$

and $\mathcal{O}_{C_y}$ is p-adically complete

Next ingredient.

Dennis explained there is a thy of NPs for rings B_I & B.

<u>Key Result from NPs</u>

Let $I \subseteq (0,1)$ be an interval, $f \in B_I$

If λ is a slope of $NP(f)$, then $\exists$ $a \in m_F$ $v(a) = -\lambda$ $(|a|_F = p^{\lambda})$

s.t. $f = \underbrace{(p - [a])}_{\text{distinguished element.}} g$, $g \in B_I$

<u>Corollary</u> If $I \subseteq (0,1)$ is compact, then B_I is a PID!!!

- B_I is a PID $\overset{\text{Exercise}}{\iff}$ factorial and each (non-invertible) element x generates a max ideal
- Let $f \in B_I$.. I compact $\Rightarrow \mathrm{Newt}_I(f)$ has finitely many slopes
- Key third result $\Rightarrow f = u \cdot \xi_{y_1} \cdot \ldots \cdot \xi_{y_n}$ $y_i \in |Y_I|$

each ξ_{y_i} generates a max ideal b/c $B_I/(\xi_{y_i}) \cong C_{y_i}$

Lemma $y \in |Y_I|$, $I \subseteq (0,1)$ any interval,

$\xi_y \in B_I \rightsquigarrow (B_I)^\wedge_{\xi_y} := \lim B_I/(\xi_y^n)$

Then $(B_I)^\wedge_{\xi_y} \simeq B^+_{dR,y}$

content comes from def of RHS

Reminder: $\xi_y \in A_{inf}[\frac{1}{p}, \frac{1}{\varpi}]$

$B^+_{dR,y} := \xi_y$-adic completion of $A_{inf}[\frac{1}{p}, \frac{1}{\varpi}]$

Output of this Lemma: $B^+_{dR,y}$ is a DVR w/ residue field C_y

"$\hat{\mathcal{O}}_{Y,y}$" $\simeq B^+_{dR,y}$

Now we start to talk about divisors.

Def $I \subseteq (0,1)$, interval,

$$\mathrm{Div}^+(|Y_I|) = \left\{ \text{monoid of sums } \sum n_y y,\ n_y \in \mathbb{N} \ \middle|\ \text{for every compact } J \subseteq I,\ \{y \in |Y_J| \text{ w/ } n_y \neq 0\} \text{ is finite} \right\}$$

for every $\{ y \in |Y_J| \text{ w/ } n_y \neq 0 \}$ is finite $\}$

Note: This is partially ordered.

Then φ acts on $\mathrm{Div}^+(|Y|)$

$$\varphi^*(y) = (\varphi^{-1}(\xi_y))$$

Def $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) := \mathrm{Div}^+(|Y|)^{\varphi^{\mathbb{Z}}} = \mathrm{Div}^+(|Y|)^{\varphi}$

Then $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) \xleftrightarrow[\text{bijective corr.}]{} \mathrm{Div}^+(|Y_{[a^p, a)}|)$

for any $a \in (0,1)$

(Note: for me, switch to valuation notation

$$(0,\infty) = \bigcup_{n \in \mathbb{Z}} [p^n a, p^{n+1} a))$$

We will prove something more precise than goal.

Notation/Def $f \in B_I$

$$\mathrm{div}(f) = \sum_{y \in |Y_I|} \mathrm{ord}_y(f) \cdot y \in \mathrm{Div}^+(|Y_I|)$$

?

where $\mathrm{ord}_y(f)$ is

$R \longrightarrow B^{\#}_{I,R}$ is the

$$B_I \longrightarrow B^+_{dR,y}$$ is the ξ_y-adic completion

$\mathrm{ord}_y(f) = \mathrm{val}(f)$ in $B^+_{dR,y}$

i.e., what power of $(\xi_y) B^+_{dR,y}$ is maximal power that contains the image of f.

<u>Thm</u> (Better than goal)

$$\exists \quad \mathrm{div}: \bigcup_{d \geq 0} (P_d \setminus \{0\})/\mathbb{Q}_p^\times \longrightarrow \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$$

that is an isomorphism of monoids.

<u>Check</u>: well-defined / makes sense.

$f \in P_d$, $\quad \varphi^*(\mathrm{div}(f)) = \mathrm{div}(\varphi(f)) = \mathrm{div}(p^d f) = \mathrm{div}(f)$

Injectivity is easier. Follows from:

- B_I is a PID if I compact
- $(B)^{\varphi = p^d} = \begin{cases} \mathbb{Q}_p & \text{if } d = 0 \\ 0 & d < 0 \end{cases}$

$=$

<u>Interesting</u>: surjectivity

$|Y|$ consider the

Interesting surjectivity

In other words, given $y \in |Y|$, can consider the divisor

$$\sum_{n \in \mathbb{Z}} \varphi^n(y) \in \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$$

What we need to do is show this is in the image of div.

Idea use the log!

In particular, find $\varepsilon \in 1+m_F$ w/

$$\mathrm{div}(\log([\varepsilon])) = \sum_{n \in \mathbb{Z}} \varphi^n(y)$$

(recall that we proved that $\log([\varepsilon]) \in B^{\varphi = p}$)

What is this ε?

(C_y, ι) the untilt associated to y.

It is an untilt of $F = \bar{F}$

$\Rightarrow$ C_y contains all p^{th} roots (in fact, $\Rightarrow C_y = \bar{C_y}$)

$\Rightarrow$ $\mathbb{Q}_p^{cyc} := \left(\bigcup_{n \geq 0} \mathbb{Q}_p[\zeta_{p^n}]\right)^\wedge \subseteq C_y$

Pick compatible p^{th} power roots

$$(1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \in C_y^\flat \xrightarrow[\sim]{\iota} F$$

$$\longmapsto \varepsilon$$

$$(1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \qquad \sim$$

$$\longmapsto \varepsilon$$

Claim: this works.