Geometric properties of the FF-curve

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List of symbols

- $p$ a fixed prime
- $E/\mathbb{Q}_p$ finite extension (E.g. $E = \mathbb{Q}_p$)
- $\mathcal{O}_E \subset E$ ring of integers (E.g. $\mathcal{O}_E = \mathbb{Z}_p$)
- $\pi \in \mathcal{O}_E$ uniformizer (E.g. $\pi = p$)
- $\mathbf{F}_q = \mathcal{O}_E/(\pi)$ the residue field of $\mathcal{O}_E$ (E.g. $\mathbf{F}_p$)
- $F$ = an algebraically closed cvf extension of $\mathbf{F}_q$ with non-trivial non-archimedean rank-1-valuation $v$ (E.g. $F = \widehat{\overline{\mathbf{F}_p((T))}}$, value group $v(F^*) = \mathbb{Q}$)
- $\mathcal{O}_F$ the ring of integers of $F$ [3] (E.g. $\mathcal{O}_F = \widehat{\overline{\mathbf{F}_p[[T]]}}^F$)
- $\mathfrak{m}_F$ the maximal ideal of $\mathcal{O}_F$
- $\varpi \in \mathfrak{m}_F - \{0\}$ pseudo-uniformizer (E.g. $\varpi = T \in \mathfrak{m}_F$)
- $k = \mathcal{O}_F/\mathfrak{m}_F$ (E.g. $k = \overline{\mathbf{F}}_p$)

Further list of symbols, towards the schematic Fargues-Fontaine curve:

- $\mathbf{A}_{\text{inf}} = \mathbf{A}_{\text{inf},E,F} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \{\sum_{i\geq 0}[x_i]\pi^i \mid x_i \in \mathcal{O}_F\} \subset W_{\mathcal{O}_E}(F)$
- $B^b = \mathbf{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}] = \{\sum_{i\gg-\infty}^{\infty}[x_i]\pi^i \mid x_i \in F, \sup_i |x_i| < \infty\} \subset W_{\mathcal{O}_E}(F)[\frac{1}{\pi}]$ [5]
- $v_r$ the valuation associated to $r \in \mathbb{R}$ [6]
- $|\cdot|_\rho$ the Gauss norm [7]
- $B_I$ the completion of $B^b$ with respect to valuations $v_r$ for all $r \in I$ [8]
- $B = B_{(0,\infty)}$ [9]
- $P = P_{E,F,\pi} = \bigoplus_{d\in\mathbb{N}} B^{\varphi=\pi^d}$
- $X = X_{E,F,\pi} = \text{Proj}(P_{E,F,\pi})$, the schematic Fargues-Fontaine curve

Further list of symbols, towards the adic Fargues-Fontaine curve:

- $\text{Prim}_d = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\text{inf}} \mid x_i \in \mathcal{O}_F;\ x_0 \in \mathfrak{m}_F \setminus \{0\},\ x_1, \dots, x_{d-1} \in \mathfrak{m}_F,\ x_d \in \mathcal{O}_F^*\}$. In particular,
  $\text{Prim}_0 = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\text{inf}} \mid x_0 \in \mathcal{O}_F^*\} = \mathbf{A}_{\text{inf}}^*$
  $\text{Prim}_1 = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\text{inf}} \mid x_0 \in \mathfrak{m}_F \setminus \{0\}, x_1 \in \mathcal{O}_F^*\}$
  $= \{[x_0] + u\pi \in \mathbf{A}_{\text{inf}} \mid x_0 \in \mathfrak{m}_F \setminus \{0\},\ u \in \mathbf{A}_{\text{inf}}^*\}$
  $= \{\text{distinguished elements}\} \setminus \{u\pi \in \mathbf{A}_{\text{inf}} \mid u \in \mathbf{A}_{\text{inf}}^*\}$
- $|Y|_{[0,\infty)} = \{(d) \subset \mathbf{A}_{\text{inf}} \mid d \text{ is a distinguished element}\}$
  $= \{([x_0] + u\pi) \subset \mathbf{A}_{\text{inf}} \mid x_0 \in \mathfrak{m}_F, u \in \mathbf{A}_{\text{inf}}^*\}$
- $|Y| = |Y|_{[0,\infty)} \setminus \{(\pi)\}$
  $= \text{Prim}_1 / \mathbf{A}_{\text{inf}}^*$
  $\overset{!}{=} \{(C, \iota) \mid C/E \text{ non-arch, complete, alg closed}, \iota : \mathcal{O}_C^\flat \simeq \mathcal{O}_F\}$

# Geometric properties of the FF-curve

1. Fundamental exact sequence of p-adic Hodge
2. It's regular curve
3. Cohomology of line bundles
4. The Picard
5. The étale fundamental group.

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Convention : $E = \mathbb{Q}_p$

Recall last time $\quad X = \text{Proj}(P) = \text{Proj}\left(\bigoplus_{d\geq 0} B^{\varphi = p^d}\right)$

$\boxed{\text{Goal}}$ 1. $P$ is gen as a $P_0$-alg by $P_1$

2. Stronger: Every element $s \in P_d = B^{\varphi=p^d}$ can be written as a product of irr elements from $P_1 = B^{\varphi=p}$

$$s = s_1 \, s_2 \cdots s_d$$

and this expression is unique up to $\mathbb{Q}_p^\times$ the factors.

3. This expression essentially takes the form

$$\lambda \log([\varepsilon_1]) \cdots \log([\varepsilon_n])$$

with $\lambda \in B^{\varphi = id} \overset{\text{Rajn}}{=} \mathbb{Q}_p$, $\varepsilon_1, \ldots, \varepsilon_n \in 1 + m_F$.

<u>Cor</u> For any $\varepsilon \in 1 + m_F \setminus \{1\}$,

$$(\log([\varepsilon]))$$

is a prime homogeneous ideal of deg 1 in $P$.

$\boxed{\text{Better than Goal}}$ $\exists$ divisor map which is an iso of gded monoids:

$$\mathrm{div} : \bigcup_{d \geq 0} \left( B^{\varphi = p^d} \setminus \{0\} \right) / \mathbb{Q}_p^\times \xrightarrow{\sim} \underbrace{\mathrm{Div}^+(|Y| / \varphi^{\mathbb{Z}})}$$

$$f \longmapsto \underbrace{\mathrm{div}(f)} \quad := \mathrm{Div}^+(|Y|)^\varphi$$

set of $\varphi$-inv "loc. finite" Weil divisors.

$$:= \sum_{y \in |Y|} \mathrm{ord}_y(f) \, y$$

<u>Recall how Rajn showed surjectivity</u>:

Given $y \in |Y|$.

<u>Claim</u> $\exists \, \varepsilon_y \in 1 + m_F$

$t := t_y := \log([\varepsilon_y]) \in B^{\varphi = p}$

Simply take

$$\varepsilon_y := (1, \zeta_p, \zeta_{p^2}, \ldots) \in \lim_{x \mapsto x^p} \mathcal{O}_{C_y}/p = \mathcal{O}_{C_y}^\flat \overset{c_y}{\simeq} \mathcal{O}_F.$$

Intuition / critical observation : $\forall a \in (0,1)$

$$D^+(|Y|/\varphi^{\mathbb{Z}}) \xleftrightarrow{1-1} D^+(|Y_{[a^p,a)}|)$$

$$\sum_{n\in\mathbb{Z}} [\varphi^n(y)] \longleftarrow y$$

symbol for Gauss norms.

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$E = \mathbb{Q}_p$. $F$ CNA rank-1-v.f.

$y = (C_y, \iota_y) \in |Y|$

$C := C_y$ for some $y$.

Starting point of the story :

Fontaine theta map.

(1) Fontaine theta map

$$\mathbb{A}_{\inf} := W(\mathcal{O}_C^\flat) \xrightarrow{\theta} \mathcal{O}_C$$

$$\sum_{i\geq 0} [a_i] p^i \longmapsto \sum_{i\geq 0} a_i^\# p^i$$

* Surjectivity : OK bc $(-)^\#$ is surj bc $C$ is alg closed

kedlaya

(again bc $F$ is alg closed)

(2) The extended Fontaine theta map.

$\because C$ is rank-1

$$\theta : B^b \longrightarrow \mathcal{O}_C[\tfrac{1}{p}] = C$$

$$\sum_{i\gg -\infty} [a_i] p^i \longmapsto \sum_{i\gg -\infty} a_i^\# p^i$$

$\rightsquigarrow$ extends to

$$B := \varprojlim_{U\in\mathcal{J}} B^b/U \xrightarrow{\theta} C \qquad \text{well-def'd}$$

where $\mathcal{J} := \left\{ \bigcap_{i=1}^{n} v_{r_i}^{-1}([m,\infty)) \;\middle|\; \begin{array}{l} m, n\in\mathbb{N} \\ r_i \in (0,\infty) \end{array} \right\}$

notation wrt valuations

(i.e. $\rho \in (0,1)$ for [illegible])

notation wrt valuations
(i.e. $\rho \in (0,1)$ for Gauss norms)

* surjectivity  suffices to show

$$1+\mathfrak{m}_F \xrightarrow{\log([-])} B^{\varphi=p} \xrightarrow{\theta} C$$

is surj. Direct from the formula for $\theta$, this is the same as

$$1+\mathfrak{m}_F \xrightarrow{(-)^{\#}} 1+\mathfrak{m}_C \xrightarrow{\log} C$$

Now – log is surj : observation

$$1+\mathfrak{m}_C \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{m}_C \quad \text{mutual inverses}$$

$\Rightarrow$ $\mathfrak{m}_C$ is mapped onto

$\Rightarrow$ $C$ is mapped onto

[ $\forall x \in C$, $\exists N$ s.t. $p^N x \in \mathfrak{m}_C$

$\to \exists y \in C$ s.t. $\log(y) = p^N x$

$\to \log\left(y^{\frac{1}{p^N}}\right) = \frac{1}{p^N}\log(y) = x$ ]

[ Recall: $(-)^{\#} : \mathcal{O}_C^{\flat} \to \mathcal{O}_C$ ]

– $(-)^{\#} : 1+\mathfrak{m}_F \to 1+\mathfrak{m}_C$ is surj : direct from the formula.

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Tiny rmks • In fact, $\log([-]) : 1+\mathfrak{m}_F \to B^{\varphi=p}$ is an iso.

lands here : shown by Rajn

• All surjectivities depends on $F = \overline{F}$. (also on Thm of Kedlaya)

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Next : what is the kernel of theta ?

Recall Grétar, talk 3 $\ker(A_{\inf} = W(\mathcal{O}_C^{\flat}) \to \mathcal{O}_C) = (\xi)$

[illegible] [illegible] $\ker(A_{\inf} = W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C) = (\xi)$

where $\xi = p - [p^\flat]$

Raju : $\ker(B \to \mathcal{O}_C[\tfrac{1}{p}] = C) = (\xi)$

Q Restrict the map

$$B^{\varphi = p} \to C$$

what is the kerne?

$\rightsquigarrow$ Raju : $t = \log([\varepsilon]) \in B^{\varphi = p}$

with $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_C^\flat$

Lemma $t \in \ker\left(B^{\varphi = p} \xrightarrow{\theta} C\right)$

Pf Super easy.

$$\log([\varepsilon]) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon] - 1)^n}{n} \quad \in B^{\varphi = p}$$

$\rightsquigarrow$ $\theta([\varepsilon]) = 1$

$\theta([\varepsilon] - 1) = \theta([\varepsilon - 1]) = 0$ $\square$

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Next$^2$ want to generalize

(3) The lifted Fontaine theta map

The map $\theta : B \to C$ can be lifted to

$$\boxed{\theta : B \to B_{dR}^{+}}$$

**Rmk** This is the map that Raju has constructed to compute "$\mathrm{ord}_y(-)$".

Here's how: $B^b := A_{\inf}\left[\frac{1}{p}, \frac{1}{[\omega]}\right] \xrightarrow{\theta} C$

$\xrightarrow{\xi \in \ker(\theta)}$ $B^b/\xi^n \xrightarrow{\theta} C$

$\rightsquigarrow$ $\underbrace{\varprojlim B^b/\xi}_{=: B_{dR}^+} \xrightarrow{\theta} C$

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On the other hand: by construction of $B$. $\exists$ can map

$$B^b \to B$$

$\rightsquigarrow$ $B_{dR}^+ := \varprojlim_n B^b/\xi^n \xrightarrow{\sim} \varprojlim(B/\xi^n) = \widehat{B}^{\xi}$

Raju

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All together

$$\theta : B \to \widehat{B}^{\xi} \xleftarrow{\sim} B_{dR}^+ \xrightarrow{\theta} C$$

"hol fun on $D^\times$"

Now we have $\theta : B \to B_{dR}^+$.

$\rightsquigarrow$ $\theta : B^{\varphi = p^d} \to B_{dR}^+/\xi^d$

**Q** Is this surj? What is the kernel?

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## §1. FES of p-adic Hodge

$\varepsilon_y := (1, \zeta_p, \zeta_{p^2}, -) \in 1 + m_F$

$t_y := \log([\varepsilon_y]) \in B^{\varphi = p}$

## §1. FES of p-adic Hodge

$\varepsilon_y := (1, \zeta_p, \zeta_{p^2}, \ldots) \in 1+\mathfrak{m}_F$

$t_y := \log([\varepsilon_y]) \in B^{\varphi=p}$.

<u>Thm</u> Let $y \in |Y|$. This is exact:

$$0 \to \mathbb{Q}_p \xrightarrow{t_y^d} B^{\varphi=p^d} \xrightarrow{\theta_y} B_{dR,y}^+/\xi_y^d \to 0$$

<u>pf.</u> Show only $d=0$

- <u>Injectivity</u> Clear bc $\mathbb{Q}_p$ is a field.
- <u>Exactness in the middle</u>

– $\mathrm{Im}(t_y) \subset \ker(\theta_y)$. $\longrightarrow$ Done in lemma.

– $\ker(\theta_y) \subset \mathrm{Im}(t_y)$

Let $f \in \ker(B^{\varphi=p} \xrightarrow{\theta_y} B_{dR,y}^+/\xi_y)$

$\Rightarrow \mathrm{ord}_y(f) \geq 1$

$\Rightarrow \mathrm{div}(f) \geq [y]$ $\qquad\qquad \mathrm{Div}^+(|Y|)$

$\overset{\mathrm{div}(f)\ \varphi\text{-inv}}{\Longrightarrow} \mathrm{div}(f) \geq \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

Better than Goal

$$\mathrm{div}: \bigcup_{d \geq 0} (B^{\varphi=p^d} \setminus 0) \setminus \mathbb{Q}_p^\times \xrightarrow{\sim} \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$$
$$t_y \longmapsto \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$$

$\Rightarrow f \in \mathbb{Q}_p \cdot t_y'$ for some $t_y' \in B^{\varphi=p}$

$\Rightarrow t_y'$ & $t_y$ both map to $\sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

$\rightarrow$ they differ by $\mathbb{Q}_p^\times$

$\Rightarrow f \in \mathbb{Q}_p \cdot t_y$

$\Rightarrow f \in \mathrm{Im}(t_y)$.

• Surjectivity: $B^{\varphi=p} \xrightarrow{\theta_y} B^+_{dR,y}/\xi_y \cong C$

have done in intro. $\boxed{\checkmark}$.

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## §2. Regular curve

Recall: $X = X_{\mathbb{Q}_p, F, \pi} := \mathrm{Proj}(P)$

where $P = \bigoplus_{d \geq 0} B^{\varphi=p^d}$

Q What are the pts on $X$?

First observations

- $(0)$ is a homogeous prime ideal of $P$ ↙ int domain

  ⇝ generic pt $\eta$

- " Generate from the untilts "

Take $y = (C_y, \iota_y) \in |Y|$

$\boxed{\text{Better than Goal}}$ $\Rightarrow \exists\, \varepsilon \in 1 + m_F$ s.t. $t = t_\varepsilon = \log([\varepsilon]) \in B^{\varphi=p}$ (unique up to $\mathbb{Q}_p^\times$)

vanishes at $y \in |Y|$ i.e. $t_\varepsilon(y) := \theta_y(t_\varepsilon) = 0$ in $C_y$.

$\boxed{\text{Goal 3 Cor}}$ $\Rightarrow (t_\varepsilon)$ is a homogeneous prime of deg 1 in $P$.

⇝ corresp to a pt $x_\varepsilon \in X$

well-def'd (does not depend on the choice of $\varepsilon$)

. while

depend on the choice of $\xi$)

write
$$x_y := x_\xi$$

Also we will use the symbol
$$t_y := t_\xi$$

bc we usually don't care this non-uniqueness up to $\mathbb{Q}_p^\times$.

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$|Y|$ is equipped w/ Frob $\varphi$-action: $\varphi((\xi_y)) := (\varphi^*(\xi_y))$

**Claim** whenever $y, y'$ come from the same $\varphi$-orbit
$$\Rightarrow x_y = x_{y'}$$

$\boxed{\text{Better than Goal}}$

$$\mathrm{div} : \bigcup_{d \geq 0} \left(B^{\varphi = p^d} \setminus \{0\}\right) / \mathbb{Q}_p^\times \xrightarrow{\sim} \mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$$

$\exists$ ($!$ up to $\mathbb{Q}_p^\times$) $t_y, t_{y'} \in B^{\varphi = p}$ s.t.

$$\mathrm{div}(t_y) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)] \qquad \mathrm{div}(t_{y'}) = \sum_{n \in \mathbb{Z}} [\varphi^n(y')]$$

bc $y, y' \in$ same $\varphi$-orbit

$\Rightarrow$ $t_y$ differ $t_{y'}$ by $\mathbb{Q}_p^\times$

$\Rightarrow$ $(t_y) = (t_{y'})$

$\Rightarrow$ $x_y = x_{y'}$

i.e. $\varphi$ induces trivial action on $X$.

**Q** Are
$$\{\eta\} \cup \{x_y \in X \mid y \in |Y|/\varphi^{\mathbb{Z}}\}$$
all the pts of $X$?

**A** Yes! (but we postpone)

**Thm** (Closed pts) $\exists$ a canonical bijections
$$|X| \overset{(1)}{=} |Y|/\varphi^{\mathbb{Z}} \overset{(2)}{=} (P_1 \setminus \{0\})/\mathbb{Q}_p^{\times}.$$

**Rmk** ($\mathbb{P}^1_{\mathbb{Q}_p}$ – analog)
$$|\mathbb{P}^1_{\mathbb{Q}_p}| = (k[x_0, x_1]_{(1)} \setminus \{0\})/\mathbb{Q}_p^{\times}$$

Pf. (2) precisely deg 1 part of $\boxed{\text{Better than Goal}}$.

(1)
$$|Y|/\varphi^{\mathbb{Z}} \overset{1-1}{\longleftrightarrow} \{x_y \in X \mid y \in |Y|/\varphi^{\mathbb{Z}}\}. \quad \square$$

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**Thm** (regular curve) Let $f \in P_1 \setminus \{0\} = B^{\varphi = p} \setminus \{0\}$

Then
$$D^+(f) = \operatorname{Spec}(P_{(f)}) = \operatorname{Spec}(B[\tfrac{1}{f}]^{\varphi = id})$$
is the Spec of a PID. (i.e. $B[\frac{1}{f}]^{\varphi=id}$ is a PID).

In other word:

(1) $X := \operatorname{Proj}(P) \overset{!}{=} \operatorname{Spec}(B[\frac{1}{f}]^{\varphi = id}) \cup V^+(f)$

and $V^+(f)$ is a one pt set ← direct from Better than Goal

(2) $X$ is noetherian regular scheme |

(2) $X$ is noetherian regular scheme of krull dim 1. i.e. it's Dedekind.

direct from Better than Goal (bc. div preserves deg)

Pf. Goal 1 Take $f \in P_1 = B^{\varphi = p}$ suffices to give a cover for $X$.

Let $x \in D^+(f) \subset X$. $\mathfrak{p} \in \mathrm{Spec}\, B[\frac{1}{f}]^{\varphi = id}$ the corresp. prime ideal.

Suppose $x$ is non-generic $\Rightarrow \mathfrak{p} \neq 0$

Choose an element $\neq 0$ from $\mathfrak{p}$, $\frac{g}{f^n}$ with $g \in B^{\varphi = p^n}$.

$\boxed{\text{Goal 3}}$ $\Rightarrow$ can decompose

$$\frac{g}{f^n} = \lambda \frac{\log([\varepsilon_1])}{f} \cdots \frac{\log([\varepsilon_n])}{f} \in \mathfrak{p}$$

for $\lambda \in B^{\varphi = id}$ $(= \mathbb{Q}_p)$, $\varepsilon_1, \dots, \varepsilon_n \in 1 + m_F$.

$\mathfrak{p}$ is prime $\Rightarrow \exists$ some $\frac{\log([\varepsilon_i])}{f} \in \mathfrak{p}$

[bc $\mathfrak{p}$ can not only contain $\lambda$ otherwise $\mathfrak{p}$ = whole ring.]

Claim $\mathfrak{p} \subset B[\frac{1}{f}]^{\varphi = id}$ is gen by $\frac{\log([\varepsilon_i])}{f} \in B[\frac{1}{f}]^{\varphi = id}$

Sketch: $B[\frac{1}{f}]^{\varphi = id} \subset B[\frac{1}{f}] \longrightarrow C_y$

$\ldots] \subset B[\tfrac{1}{f}] \longrightarrow C_y \}$

is surj & $\ker = \left( \dfrac{\log([\varepsilon])}{f} \right)$

<u>Consequence</u> $B[\tfrac{1}{f}]^{\varphi = id}$ is a PID.

(so it's also Dedekind.) $\square$

Look back at the "closed pts" thm:

$$|X| \overset{1-1}{\longleftrightarrow} \{ x_y \in X \mid y \in |Y|/\varphi^{\mathbb{Z}} \}.$$

set of closed pts

<u>Continuation of the pf of "closed pts thm":</u>

Let $x \in X$ non-generic.

$\Rightarrow x \in D^+(f)$ for some $f \in P_1$ &

corresp to $\mathfrak{p} \neq 0$ with $\mathfrak{p} \in \Gamma(D^+(f), \mathcal{O})$

$\mathfrak{p}$ is gen by $\dfrac{\log([\varepsilon])}{f}$ for some $\varepsilon \in 1 + \mathfrak{m}_F$

Then $t_\varepsilon := \log([\varepsilon])$

$\boxed{\text{Better than Goal}}$ $\Rightarrow$ gives a $y \in |Y|$ s.t. $t_\varepsilon$ vanishes

i.e. $t_\varepsilon(y) := \theta_y(t_\varepsilon) = 0$ in $C_y$.

$\Rightarrow x = x_y \in X$ $\boxtimes$

Cor of "thm of reg curve"

<u>Thm</u> (Residue fields) The residue field of $x = x_y$ for

Thm (Residue fields) The residue field of $x = x_y$ for $y = (C_y, \iota_y) \in |Y|$ is $C_y$.

Warning • Residue fields of closed pts of $X$ are definitely ext of $\mathbb{Q}_p$ of inf. degrees

This is different from $\mathbb{P}^1_{\mathbb{Q}_p}$, where all res fields of closed pts are finite extensions of $\mathbb{Q}_p$

• This shows that $X$ is not of fin type over $\mathbb{Q}_p$.

$\mathbb{C}_p / \mathbb{Q}_p$ transcendence extension $\quad \overline{\mathbb{Q}}_p \underset{\neq}{\hookrightarrow} \mathbb{C}_p$

Thm (Local rings) For $y \in |Y|$ with $x = x_y \in |X|$ there is a can iso

$$\widehat{\mathcal{O}}_{X,x} = B^+_{dR,y}$$

So we will also write $B^+_{dR,x}$.

Pf. Skip

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## §3. Cohomology of line bundles

Recall what we mean by $\mathcal{O}(1)$?

The coh sheaf $\widetilde{M}$ on $X$ corresp to the graded module

$$M = \bigoplus_{n \in \mathbb{Z}} M_n = \bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^{n+1}}$$

By construction of "proj tilde" $\to$ for $t = \log([\varepsilon]) \in P_1$

$$\widetilde{m}(D^+(t)) = M[\tfrac{1}{t}]_0 = P[\tfrac{1}{t}]_1 = B[\tfrac{1}{t}]^{\varphi = p}.$$

Thm (Cohomology)

(1) $H^0(X, \mathcal{O}(n)) = B^{\varphi = p^n}$. In part,

- $= \mathbb{Q}_p$ when $n = 0$ (Raján)
- $= m_F$ when $n = 1$ (Lubin-Tate)
- $= 0$ when $n < 0$.

(2) $H^i(X, \mathcal{O}(n)) = 0$ for $i > 0$, $n \geq 0$

Rmk This is the same as $\mathbb{P}^1_{\mathbb{Q}_p}$.

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Consequences

Cor (Geometric explaination of FES of p-adic Hodge)

i.e. Thm of cohomology $\Rightarrow$ Thm of FES.

Pf. $0 \to \mathcal{O} \xrightarrow{t^d} \mathcal{O}(d) \to \underbrace{\mathcal{O}_{X,\infty_t}}_{\widehat{\ } \ \mathcal{O}^+(t)} / \mathfrak{z}^d_{\infty_t} \to 0$

LES $\rightsquigarrow$ $0 \to H^0(X, \mathcal{O}) \xrightarrow{t^d} \underline{H^0(X, \mathcal{O}(d))} \to H^0(X, B^+_{dR,\infty_t}/\mathfrak{z}^d_{\infty_t}) \to 0$

This is precisely FES

$0 \to \mathbb{Q}_p \xrightarrow{t^d} \underline{B^{\varphi = p^d}} \to B^+_{dR,\infty_t}/\mathfrak{z}^d_t \to 0$

have used

$\underbrace{\mathcal{O}_{X,\infty_t}}_{DVR} / \mathfrak{z}^d_{\infty_t} \cong \widehat{\mathcal{O}}_{X,\infty_t} / \mathfrak{z}^d_{\infty}$

$0 \to \mathbb{Q}_p \to \underline{B} \to B_{dR,\infty_t}^+/\xi_t^d \to 0$ $\qquad$ $\underbrace{\phantom{xx}}_{\text{DVR}} / \xi_{\infty_t} \cong \mathcal{O}_{X,\infty_t}/\xi_{\infty_t}^d$

$\| \text{ Thm of local rings}$

$B_{dR,\infty_t}^+/\xi_{\infty_t}^d$.

$\boxed{\lightning}$

([Ans] : This is a fake proof.)

[Kedlaya 18 : Simple connectivity of FF-curves]