Fargues - Fontaine Seminar

Introduction

$K/\mathbb{Q}_p$ fin ext.

$G_K = \mathrm{Gal}(\overline{K}/K)$ abs Galois group.

We want to study

$\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ = p-adic representations

i.e. $G_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V)$ cts

V fin dim'l $\mathbb{Q}_p$ v.s.

[Top on $\mathrm{Aut}_{\mathbb{Q}_p}(V) \hookrightarrow \mathrm{End}_{\mathbb{Q}_p}(V) \times \mathrm{End}_{\mathbb{Q}_p}(V)$

$\varphi \longmapsto (\varphi, \varphi^{-1})$]

Look for smaller but still interesting subcats :

One way to produce interesting p-adic rep:

Take X proper$/K$

$$\rightsquigarrow \quad H^i_{et}(\underset{\substack{\| \\ X \otimes_K \bar{K}}}{X_{\bar{K}}}, \mathbb{Q}_p) \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$$

Depending on X this has extra properties

Digression: Assume $K \hookrightarrow \mathbb{C}$ and X sm proper$/K$

$$\Rightarrow \quad H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

and

$$H^i_{sing}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i(X, \Omega^{\bullet}_{X/K}) \otimes_K \mathbb{C}$$

extra structure ← has Hodge filt

But we lose G_K-action.

Have non-archimedean version with G_K-action:

<u>Fontaine</u> $\rightsquigarrow$

B_{dR} : cdvf with residue field $C = \widehat{\bar{K}}$

$\cup$

B_{dR}^+ : ring of integers

Have

- $G_K \curvearrowright B_{dR}$
- $\mathrm{Fil}^i B_{dR} = \mathfrak{m}^i_{B_{dR}}$
- $\bar{K} \hookrightarrow B_{dR}$ G_K-equiv., $B_{dR}^{G_K} = K$

(Note as a field $B_{dR} \cong C((t))$ but we cannot describe the G_K-action on the RHS)

$\boxed{\text{Talk } 4 \Leftarrow \text{Talk } 2, 3}$

<u>Thm</u> (Fa, Ts, Sch, Bei, ...) X sm proper$/K$

$\Rightarrow \exists$ filtered, G_K-equiv isom.

$$H^n_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\otimes} B_{dR} \cong H^n(X, \Omega^{\bullet}_{X/K}) \underset{K}{\otimes} B_{dR}$$

G_K $\quad$ Filtr $\quad\quad$ Filtr $\quad$ G_K

$\rightsquigarrow$ **Def:** We say $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is <u>de Rham</u>

$$:\Longleftrightarrow \quad \underbrace{(V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}}_{D_{dR}(V)} \otimes_K B_{dR} \cong V \otimes_{\mathbb{Q}_p} B_{dR}$$

$\nwarrow$ carries a filtr.

$\Rightarrow$ • $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ is de Rham

• Have $D_{dR} : \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{FilMod}_K$ exact, faithful

$\uparrow$

still too large

<u>Note</u> if X is sm proper $/K$

and has a smooth proper model

over $\mathcal{O}_K$ i.e. $\exists\ \mathcal{X} \longrightarrow \mathrm{Spec}\, \mathcal{O}_K$ sm proper

s.t. $\mathcal{X} \otimes_{\mathcal{O}_K} K = X$ (where $k = \mathcal{O}_K / \mathfrak{m}$, $X_0 = \mathcal{X} \otimes k$)

then (*) $H^i(X, \Omega^\bullet_{X/K}) \underset{B.O.}{\cong} H^i_{crys}(X_0 / K_0) \otimes_{K_0} K$

filtr. $\qquad$ $\circlearrowleft$ Frobenius action.

Don't have Frobenius on B_{dR}

Fontaine $\longrightarrow$ $B_{cris} \subset B_{dR}$

subring with • G_K - action

• $\varphi = \mathrm{Frob}$ - action

• $B_{cris}^{G_K} = K_0 := W(k)[1/p]$

• $K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$

and $\mathrm{Fil}^i(K \otimes_{K_0} B_{cris}) = \mathrm{Fil}^i B_{dR} \cap K \otimes_{K_0} B_{cris}$

<u>Thm</u> (Fa, Tsuji, Niziol, ...)

$\mathcal{X} \longrightarrow \mathrm{Spec}\, O_K$ sm proper

with $X := \mathcal{X} \otimes_{O_K} K$, $X_0 = \mathcal{X} \otimes_{O_K} k$

$$\Rightarrow \quad H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} = H^i_{crys}(X_0/K_0) \otimes_{K_0} B_{cris}$$

G_K

filtr (after $\otimes$) φ

filtr, φ (after $\otimes K$)

G_K

$\rightsquigarrow$ <u>Def:</u> We say $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$

is <u>crystalline</u> $:\Longleftrightarrow$

$$\underbrace{\left(V \otimes_{\mathbb{Q}_p} B_{cris}\right)^{G_K}}_{D_{crys}(V)} \otimes_{K_0} B_{cris} \cong V \otimes_{\mathbb{Q}_p} B_{cris}$$

$\Rightarrow$ • $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ is crystalline if X has a smooth proper model over $\mathcal{O}_K$

• $D_{crys}: \mathrm{Rep}^{crys}_{\mathbb{Q}_p}(G_K) \longrightarrow \varphi\text{-}\mathrm{FilMod}_{K/K_0}$

$\uparrow$

fin dim K_0-v.s. M

$+ \ \varphi: M \to M$ p-linear

$+$ filtr on $M \otimes_{K_0} K$

($\to$ more concrete)

Fontaine: D_{crys} is fully-faithful

$M \in \varphi\text{-FilMod}_{K/K_0}$ is

<u>weakly admissible</u>

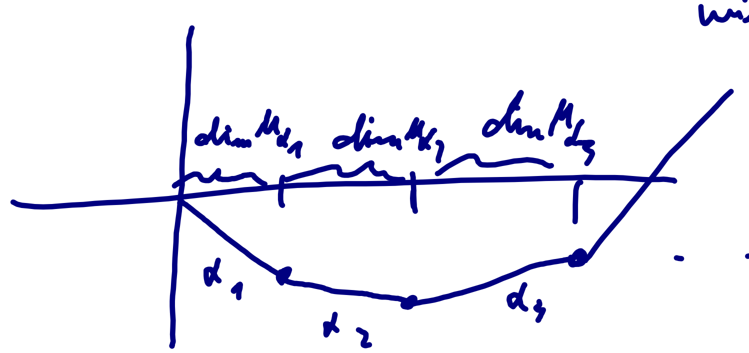
if the following holds

$$M = \bigoplus_\alpha M_\alpha$$

$\uparrow$ φ acts with EV of p-adic val $\alpha \in \mathbb{Q}$

Newton polygon: $\alpha_1 < \dots < \alpha_n \in \mathbb{Q}$ all α with $\dim_{K_0} M_\alpha \neq 0$

$P_N(M)$



<u>Hodge Polygon</u>: $i_1 < \dots < i_m \in \mathbb{N}$ all i with $gr^i(M \otimes_{K_0} K) \neq 0$, $h_j := \dim_K gr^{i_j}$

$P_H(M)$

M is weakly admissable

$\Longleftrightarrow$ (1) $\forall\ M' \subset M$ sub φ-Fil

$$P_H(M') \leq P_N(M')$$

(2) endpoints of

$P_H(M)$ and $P_N(M)$ are the same

<u>Thm</u> (Fontaine-Colmez)

$$D_{crys} : \mathrm{Rep}^{crys}_{\mathbb{Q}_p}(G_K) \xrightarrow{\cong} \varphi\text{-FilMod}^{w.a.}_{K/K_0}$$

equivalence of cat

$$\mathrm{Fil}^0(M \otimes_{K_0} B_{cris})^{\varphi = \mathrm{id}} \longleftarrow M$$

The Fontaine-Fargues curve

gives a way to understand this purely algebraic statement more geometrically

Fontaine–Fargues curve X (for $C^\flat$)

i) X regular $\mathbb{Q}_p$-scheme of dim 1
(not of fin. type)

ii) $\exists$ deg : $\mathrm{Pic}(X) \xrightarrow{\simeq} \mathbb{Z}$

(in part $\deg(\mathrm{div}(f)) = 0$ $\rightsquigarrow$ kind of complete)

iii) $\pi_1^{et}(X) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$
étale fundam. gp.

iv) $\exists$ $\mathcal{O}_X(\lambda)$ v.b. $\lambda = \frac{d}{r} \in \mathbb{Q}$ of rk r

s.t. any v.b on X is $\simeq \bigoplus_\lambda \mathcal{O}_X(\lambda)$

($\rightsquigarrow$ similar to $\mathbb{P}^1_{\mathbb{Q}_p}$)

- closed pts of X have residue field.
non-archimedean complete alg closed fields

- G_K acts on X
- $\exists\ \infty \in X$ fixed by G_K s.t.

$$\hat{\mathcal{O}}_{X,\infty} = \underbrace{B_{dR}^+}_{\text{DVR}}, \quad X \setminus \{\infty\} = \operatorname{Spec}\big(\underbrace{B_{crys}^{\varphi = id}}_{\text{PID}}\big)$$

and $X \setminus \{\infty\} \times_X \operatorname{Spec} \hat{\mathcal{O}}_{X,\infty} = \operatorname{Spec} B_{dR}$

$\rightsquigarrow$ For Y sm proper $/_K$

the isom

$$(*) \quad H^i_{et}(Y, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^i(Y, \Omega^\bullet_{Y/K}) \otimes_K B_{dR}$$

defines a v.b. $\mathcal{H}^i(Y)$ on X by

$$\mathcal{H}^i(Y)|_{X \setminus \{\infty\}} := H^i_{et}(Y, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{X \setminus \{\infty\}}$$

$$\mathcal{H}^i(Y)|_{\operatorname{Spec} \hat{\mathcal{O}}_{X,\infty}} := H^i(Y, \Omega^\bullet_{Y/K}) \otimes_K B_{dR}^+$$

glue along $(*)$ (fpqc descent)

$\boxed{\text{see talks 8 – 11}}$

Thm (FF) $\breve{\mathbb{Q}}_p = W(\bar{\mathbb{F}}_p)[1/p]$

$$\varphi\text{-}\mathrm{Mod}_{\breve{\mathbb{Q}}_p}/\cong \xrightarrow{1:1} \mathrm{Bun}_X/\cong$$

(= isom of v.b. on X)

(talk 12: sketch of pf ⇜ p-divisible gps)

We have

- $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \underbrace{\mathrm{Bun}_X^{G_K}}_{G_K\text{- equivariant v.b.}}$ fully faithful

$$V \longmapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

- $\varphi\text{-}\mathrm{Fil}\ \mathrm{Mod}_{K/K_0} \longrightarrow \mathrm{Bun}_X^{G_K}$

This can be used to proof the Thm of Colmes-Fontaine. (⇝ sketch in talk 13)