Oberseminar
The Fargues-Fontaine Curve

WS 20/21

Introduction

It is a well-known phenomenon, that if there are geometric - and number theoretic (or arithmetic) versions of a statement, then the geometric ones are more accessible, as it is for example the case for the Riemann hypothesis or the Langlands program. This is why the discovery of the Fargues-Fontaine curve (see [FF18]) is so spectacular, since it allows one to transfer purely arithmetic or algebraic objects into geometric ones, which then can be used to find new results and new proofs of old results concerning these objects. For example, let X be the Fargues-Fontaine curve attached to $\mathbb{Q}_p$ and $F := \widehat{\overline{\mathbb{F}_p((t))}}$ the t-adic completion of the algebraic closure of the Laurent power series with coefficients in $\mathbb{F}_p$. Then the $\mathbb{Q}_p$-linear continuous finite dimensional representations of the absolute Galois group $G := G(\mathbb{Q}_p)$ can be seen as certain G-equivariant vector bundles on X. (Note that G acts on X.) Also the isomorphism classes of isocrystals over $\overline{\mathbb{F}}_p$ can be canonically identified with the isomorphism classes of vector bundles on X. (Here *isocrystal over* $\overline{\mathbb{F}}_p$ means a finite dimensional vector space M over $K_0 = \mathrm{Frac}W(\overline{\mathbb{F}_p})[1/p]$ together with an isomorphism $\varphi^*M \xrightarrow{\simeq} M$, where $\varphi : K_0 \to K_0$ is the Frobenius lift.) The above can be used to give a geometric proof of one of the most important results in p-adic Hodge theory (originally due to Colmez and Fontaine), namely that the category of crystalline Galois representations of G is equivalent to the category of weakly admissible filtered φ-modules.

Let us list some properties of the Fargues-Fontaine curve X attached to $\mathbb{Q}_p$ and $F = \widehat{\overline{\mathbb{F}_p((t))}}$:

(i) X is a noetherian regular $\mathbb{Q}_p$-scheme of Krull dimension 1;

(ii) the closed points of X are (up to Frobenius twists) in bijection to untilts of $\mathcal{O}_F$ (i.e. algebraically closed non-archimedean extensions of $C/\mathbb{Q}_p$ with an isomorphism $\varprojlim_{x\mapsto x^p} \mathcal{O}_C \cong \mathcal{O}_F$);

(iii) $\mathrm{Pic}(X) \cong \mathbb{Z}$

(iv) let $\infty \in X$ be a closed point, then $\widehat{\mathcal{O}_{X,\infty}} = B_{\mathrm{dR}}^+$ and $X = \mathrm{Spec}\, B_{\mathrm{cris}}^{\varphi=\mathrm{id}} \sqcup_{\mathrm{Spec}\, B_{\mathrm{dR}}} \mathrm{Spec}\, B_{\mathrm{dR}}^+$, where B_{cris} and B_{dR} are Fontaine's period rings;

(v) $\pi_1^{\text{ét}}(X) \cong \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, where $\pi_1^{\text{ét}}$ denotes the étale fundamental group.

Note that X shares many properties with $\mathbb{P}^1_{\mathbb{Q}_p}$, but the residue fields of its closed points are given by the fields C as in (ii) above, whence it is not of finite type over $\mathbb{Q}_p$.

In this seminar we want to define the Fargues-Fontaine curve starting from scratch, prove some of its properties and also discuss the applications listed above.

The Talks

We will closely follow the lecture notes [Ans]. If there is no reference written, the numbering refers to [Ans].

1. Introduction and discussion (04.11.20)

The idea and main features of the Fargues-Fontaine curve will be shortly explained. After that follows a discussion on the seminar and the following talks.

Kay

2. Ramified Witt vectors (11.11.20)

Introduce and explain the notations p, $E/\mathbb{Q}_p$, $\mathcal{O}_E \subset E$, π, $\mathbb{F}_q$ from [Ans, 1.]. Then define the ramified Witt vectors $W_{\mathcal{O}_E,\pi}$ as in Lemma 3.4 (cf. [Hes, Prop 2]) and explain the proof of this lemma. Define the classical Witt vectors of a $\mathbb{Z}_p$-algebra A by $W(A) := W_{\mathbb{Z}_p,p}(A)$ (i.e. we take $E = \mathbb{Q}_p$ and $\pi = p$). Then explain Example 3.6, Proposition 3.7, Lemma 3.8. Introduce the notions *perfect prism*, *distinguished element* and *perfectoid* $\mathcal{O}_E$*-algebra* as in Definition 3.12 and discuss the first three items of Remark 3.13. Finally state Proposition 3.17 an deduce that the p-adic completion $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ of $\cup_{n\geq 0}\mathbb{Z}_p[p^{1/p^n}]$ is perfectoid.

Kay

3. Tilt and Untilt (18.11.20)

Introduce the *tilt* $A^\flat$ of a π-complete $\mathcal{O}_E$-algebra A as in Definition 3.2 and explain its ring structure as below Proposition 3.3 (see also [Lur, Lecture 2, Prop 6, Cor 7, Rem 8]). Explain Proposition 3.3 and its proof. Explain that the tilt of the perfectoid $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ from the previous lecture is the t-adic completion of $\mathbb{F}_p[t^{1/p^\infty}]$ (where $t = (p, p^{1/p}, p^{1/p^2}, \ldots)$), see [Sch13, Ex 2.2]. Recall the ramified Witt vectors from the first lecture and state and prove Proposition 3.9, also mention Remark 3.10 and Lemma 3.11. Define the *untilt* as in Definition 3.14 and state Exercise 3.15 (it would be nice to see an idea for the proof). State and proof Proposition 3.16 as detailed as possible. Then state Lemma 3.18.

Grétar

4. The ring $\mathbb{A}_{\text{inf}}$ (25.11.20)

Introduce the ring $\mathbb{A}_{\text{inf}}$ and go carefully through everything from 4.1 to 4.6. Stress also the following two points which are hidden in the text:

(i) instead of the second sentence after Def 4.4 state Theorem 5.4 and refer to the next lecture for its proof;

(ii) $\mathcal{O}_C \cong \mathbb{A}_{\text{inf}}/(\xi) \cong B_{\text{dR}}^+/(\xi)$ (see before Def 4.5, see also the proof of Prop 3.16 and the third item in Remark 3.13 discussed in the previous lecture).

If time remains you can also discuss 4.7, 4.8.

Christoph

5. The set $|Y|$ and untilts of $\mathcal{O}_F$ (02.12.20)

The aim of the talk is to prove Theorem 5.4. For this go through section 5 without 5.5.

Georg

6. Newton Polygons (09.12.20)

Go through 6.1 to 6.19 in [Ans] to discuss the Newton polygon and its properties of a polynomial or power series with coefficients in a non-archimedean local field. This talk is independent of the talks before.

Pablo

7. The Newton Polygon of elements in $\mathbb{A}_{\inf}$ (16.12.20)

Give Definition 6.20, 6.21, explain Lemma 6.22. Then state Theorem 7.1 and explain as much of the proof as possible. In particular also give the notations in Definition 5.5. You can skip Remark 7.3.

Raju

8. The (schematic) Fargues-Fontaine curve (06.01.21)

Go through section 8 in [Ans]. In particular introduce the ring B and the Fargues-Fontaine curve. Also give [FF18, Example 1.6.3]

Dennis

9. The multiplicative structure of the graded ring defining the FF curve(13.01.21)

Go through section 9 to explain the proof of Thm 9.2 and its more precise version Thm 9.14, which is used in the next talk to show that the Fargues-Fontaine curve is regular of Krull dimension 1.

Raju

10. It's a curve! (20.01.21)

Go carefully through 10.1 - 10.9 this comprises in particular the basic fundamental geometric properties of the Fargues-Fontaine curve (e.g. it is a curve). In the context of Lemma 10.7 also recall Thm 5.4, which yields that the closed points of X are given by untilts of $\mathcal{O}_F$. If time remains discuss Prop 10.10.

Fei

11. Vector bundles on the Fargues-Fontaine curve (27.01.21)

Go through section 11. In particular explain everything carefully needed to state Theorem 11.14 and to get the corollary stated below Thm 11.14, that any vector bumdle on X is a sum of $\mathcal{O}_X(\lambda)$'s.

Sascha

12. Sketch of the proof (03.02.21)

Go through sections 12 and 13 and explain a sketch of a proof of Thm 11.14 and the ingredients entering in the proof.

Yinying

13. Weakly admissible implies admissible (10.02.21)

Explain the statement of Thm 14.1 and sketch a proof using the Fargues-Fontaine curve as in section 14.

NN

References

[Ans] Johannes Anschütz. Lectures on the Fargues-Fontaine Curve. Lecture notes available at `http://www.math.uni-bonn.de/people/ja/thecurve/vorlesung_the_curve.pdf`.

[FF18] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Astérisque*, (406):xiii+382, 2018. With a preface by Pierre Colmez.

[Hes] Lars Hesselholt. Lecture notes on witt vectors. Available at `http://web.math.ku.dk/~larsh/papers/s03/wittsurvey.pdf`.

[Lur] Jacob Lurie. The Fargues-Fontaine curve. Lecture notes available at `https://www.math.ias.edu/~lurie/205.html`.

[Sch13] Peter Scholze. Perfectoid spaces: a survey. In *Current developments in mathematics 2012*, pages 193–227. Int. Press, Somerville, MA, 2013. `https://arxiv.org/abs/1303.5948`.