The Fargues-Fontaine-Curve
Chapter 8: The ring B

Setting:

- $q = p^r \in \mathbb{N}$ with $p \in \mathbb{P}$ fixed
- $E/\mathbb{Q}_p$ finite extension
- $(\mathscr{O}_E, (\pi))$ ring of integers of E with uniformizer $\pi \in \mathscr{O}_E$
- $\mathbb{F}_q = \mathscr{O}_E/(\pi)$ residue field
- $(F/\mathbb{F}_q, |-|)$ a non-archimedian algebraically closed extension with a norm $|-| : F \to \mathbb{R}_{\geq 0}$, $a \mapsto q^{-v(a)}$ corresponding to a valuation $v : F \to \mathbb{R} \cup \{+\infty\}$.
- $(\mathscr{O}_F, \mathfrak{m}_F)$ ring of integers of F with $\mathscr{O}_F := \{x \in F \,|\, |x| \leq 1\}$ and $\mathfrak{m}_F := \{x \in F \,|\, |x| < 1\}$
- C/E algebraic closed, non-archimedian extension with a valuation $v_C : C \to \mathbb{R} \cup \{+\infty\}$, such that $(\mathscr{O}_C, v_C)$ as valuation ring

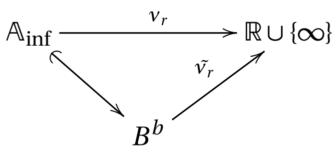
In a talk before we have seen that for all $r \geq 0$ there are valuations

$$v_r : \mathbb{A}_{\inf} \to \mathbb{R} \cup \{\infty\}$$
$$f = \sum_{i=0}^{\infty} [x_i] \cdot \pi^i \mapsto \inf_{i \in \mathbb{Z}} \{v(x_i) + ir\}$$

Newt(f) is the convex, decreasing, piecewise linear function with Legendre transform

$$\mathscr{L}(\text{Newt}(f)) := \begin{cases} v_r(f) & , r \geq 0 \\ -\infty & , r < 0 \end{cases}.$$

The following commutative diagram



with

$$B^b := \mathbb{A}_{\inf}[\frac{1}{\pi}, \frac{1}{[\omega]}] = \{ \sum_{n >> -\infty}^{\infty} [x_n]\pi^n \,|\, \inf_{i \in \mathbb{Z}} \{v(x_i)\} > -\infty\}$$

$$\tilde{v_r} : B^b \to \mathbb{R} \cup \{\infty\}$$
$$f = \sum_{n >> -\infty}^{\infty} [x_n]\pi^n \mapsto \inf_{i \in \mathbb{Z}} \{v(x_i) + ir\}$$

defines an extension $\tilde{v_r}$ of v_r to B^b and $\overline{\omega} \in \mathfrak{m}_F \setminus \{0\}$. Futhermore

$$\mathscr{L}(\mathrm{Newt}(f)) := \begin{cases} \tilde{v}_r(f) & , r \geq 0 \\ -\infty & , r < 0 \end{cases}.$$

is the corresponding extension from $\mathrm{Newt}(f)$ to B^b.

Definition 1. *Let $I \subset (0,\infty)$ be an interval. The completion of B^b for the family of valuations $(v_r)_{r\in I}$ is given by*

$$B_I = \varprojlim_{\mathscr{U}\in\mathscr{F}} B^b/\mathscr{U}$$

with the fundamental system $\mathscr{F} := \{\bigcap_{i=1}^{n} v_{r_i}^{-1}[m,\infty) | n, m \in \mathbb{N}, r_i \in I\} \subset B^b$.

Bemerkung 2. *The completion comes from the following idea in commutative algebra: Let R be a topological ring s.th. 0 has a fundamental system $\mathscr{F} = \{\mathscr{U}$ neighborhood of $0 | \mathscr{U}$ subgroups$\}$. Then the ring*

$$\hat{R} = \varprojlim_{\mathscr{U}\in\mathscr{F}} R/\mathscr{U}$$

is the completion of R.

Definition 3. *We set $B := B_{(0,\infty)}$.*

For any $I \subset (0,\infty)$ there is the interval $qI := \{q \cdot y | y \in I\} \subset (0,\infty)$ and the bijective map of sets

$$\begin{aligned} I &\to qI \\ y &\mapsto q \cdot y \end{aligned}$$

and the commutative diagram

$$\begin{array}{ccc} B^b & \xrightarrow[\sim]{\varphi} & B^b \\ \downarrow & & \downarrow \\ B_I & \xrightarrow[\sim]{\tilde{\varphi}} & B_{qI} \end{array}$$

with the maps

$$\varphi : B^b \xrightarrow{\sim} B^b$$

$$\sum_{n>>-\infty}^{\infty} [x_n]\pi^n \mapsto \sum_{n>>-\infty}^{\infty} [x_n^q]\pi^n$$

$$B^b \to B_I$$

$$x \mapsto (x + \mathscr{U})_{\mathscr{U}\in\mathscr{F}_I}$$

$$\tilde{\varphi} : B_I \xrightarrow{\sim} B_{qI}$$

$$(\sum_{n>>-\infty}^{\infty} [x_{n,\mathscr{U}}]\pi^n + \mathscr{U})_{\mathscr{U}\in\mathscr{F}_I} \mapsto (\sum_{n>>-\infty}^{\infty} [x_{n,\mathscr{U}}^q]\pi^n + \mathscr{U})_{\mathscr{U}\in\mathscr{F}_{qI}}$$

and the obvious inverse map

$$\psi : B^b \to B^b$$
$$\sum_{n>>-\infty}^{\infty} [y_n]\pi^n \mapsto \sum_{n>>-\infty}^{\infty} [(y_n)^{\frac{1}{q}}]\pi^n$$
$$\tilde{\psi} : B_{qI} \to B_I$$
$$(\sum_{n>>-\infty}^{\infty} [x_{n,\mathscr{U}}]\pi^n + \mathscr{U})_{\mathscr{U}\in\mathscr{F}_{qI}} \mapsto (\sum_{n>>-\infty}^{\infty} [x_{n,\mathscr{U}}^{-q}]\pi^n + \mathscr{U})_{\mathscr{U}\in\mathscr{F}_I}$$

is the extension by continuity of B^b.

Question: Why is $\tilde{\varphi}$ well-defined?

Answer: Let

$$\forall \mathscr{U} = \bigcap_{i=0}^{k} v_i^{-1}([m,\infty)) \in \mathscr{F}_I : \sum_{n>>-\infty}^{\infty} [x_{n,\mathscr{U}}]\pi^n \in \mathscr{U}.$$

We have to show

$$\sum_{n>>-\infty}^{\infty} [(x_{n,\mathscr{U}})^q]\pi^n \in \mathscr{V} = \bigcap_{j=0}^{l} v_{r_j}^{-1}([q\cdot m,\infty)) \in \mathscr{F}_{qI}.$$

For all $j \in \{0,\dots,l\}$ it holds

$$v_{r_j}(\sum_{n>>-\infty}^{\infty} [(x_{n,\mathscr{U}})^q]\pi^n) = \inf_{n\in\mathbb{Z}}\{v(x_{n,\mathscr{U}}^q) + n\cdot r_j\} = \inf_{n\in\mathbb{Z}}\{q\cdot v(x_{n,\mathscr{U}}) + q\cdot n\cdot r_i\} = q\cdot \inf_{n\in\mathbb{Z}}\{v(x_{n,\mathscr{U}}) + n\cdot r_i\} \geq q\cdot m.$$

The reason or this inquality is that $r_j \in qI$ for all $j \in \{0,\dots,l\}$ and v is a valuation on F.

If $I = (0,+\infty)$ then we have the commutative diagram

$$\begin{array}{ccc} B^b & \xrightarrow{\varphi} & B^b \\ \downarrow & & \downarrow \\ B & \xrightarrow[\sim]{\tilde{\varphi}} & B \end{array}$$

An alternative definition of B is used in the following examples and is given by

Definition 4. *Let $I \subset (0,1)$ be an interval. The completion of B^b for the family of norms $(|.|_\rho)_{\rho\in I}$ is given by*

$$B_I = \varprojlim_{\mathscr{U}\in\mathscr{F}} B^b/\mathscr{U}$$

and the open neighbourhoods $\mathscr{U}$ of 0 are given by the topology of the complete norm

$$||.||_I = \sup_{\rho\in I} |.|_\rho : B_I \to [0,+\infty].$$

Here is

$$|.|_\rho : B^b \to \mathbb{R} \cup \{+\infty\}$$
$$\sum_{n>>-\infty}^{\infty} [x_n]\pi^n \mapsto \sup_{n\in\mathbb{Z}} |x_n|\rho^n$$

for every $\rho \in (0,1)$. *For every* $\rho \in (0,1)$ *there exists an unique element* $r \in (0,+\infty)$, *such that* $\rho = q^r$ *and* $|.|_\rho = \rho^{-\nu_r(.)}$, *where*

$$\nu_r : B^b \to \mathbb{R} \cup \{+\infty\}$$
$$\sum_{n>>-\infty}^{\infty} [x_n]\pi^n \mapsto \inf_{n\in\mathbb{Z}} \{\nu_r(x_n) + nr\}$$

are the previous valuations $\forall r > 0$. *The change from a norm* $|.|_\rho$ *and the corresponding valuation* ν_r *for a* $\rho \in (0,1)$ *and* $r \in (0,\infty)$ *such that* $\rho = q^r$ *changes the open intervals* $(0,1)$ *and* $(0,+\infty)$. *In this sense the definitions are compatible.*

Beispiel 5. *Let* $I = [\rho_1, \rho_2] \subset (0,1]$ *with* $a, b \in F$ *s.th.* $|a| = \rho_1$ *and* $|b| = \rho_2$. *Furthermore* $B := B_{(0,1)}$ *in this situation. Then it follows*

$$B(0,1) = \{x \in B^b | \, ||x||_I \leq 1\} = \mathbb{A}_{\inf}[\frac{[a]}{\pi}, \frac{\pi}{[b]}]$$

with $B^b = \mathbb{A}_{\inf}[\frac{1}{\pi}, \frac{1}{[\omega]}]$, *the norm* $||.||_I : \sup_{\rho\in I} |.|_\rho : B_I \to [0,+\infty]$ *and* $B_I = \widehat{\mathbb{A}_{\inf}[\frac{[a]}{\pi}, \frac{\pi}{[b]}]}[\frac{1}{\pi}]$ *in the* π*-adic topology.*
Here is

$$|.|_\rho : B^b \to [0,+\infty]$$
$$\sum_{n>>-\infty}^{\infty} [x_n]\pi^n \mapsto \inf_{n\in\mathbb{N}} \{\nu(x_n)\rho^n\}$$

$\forall \rho \in I$.

Definition 6. *The schematic Fargues-Fontaine curve is defined as*

$$X := X_{E,F} := \mathrm{Proj}(\bigoplus_{d\geq 0} B^{\varphi=\pi^d})$$

with $B^{\varphi=\pi^d} = \{x \in B | \varphi(x) = \pi^d \cdot x\}$ *for all* $d \geq 0$.

Beispiel 7. *We consider the simpler setting* $E = \mathbb{Q}_p \subset F = \mathrm{Quot}(\mathscr{O}_C^b)$ *with* $C = \hat{\overline{\mathbb{Q}}}_p$ *and* $B = B_{(0,1)}$ *to look at the graded ring structure of the ring* $\bigoplus_{n\in\mathbb{Z}} B^{\varphi=p^n}$. *Let* $f \in B^{\varphi=p^m}, g \in B^{\varphi=p^n}$ *and then the graded ring structure is given by*

$$\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g) = (p^m \cdot f) \cdot (p^n \cdot f) = p^{m+n} \cdot f \in B^{\varphi=p^{m+n}}.$$

For $f = \sum_{n\in\mathbb{Z}} [c_n] \cdot p^n \in \bigoplus_{n\in\mathbb{Z}} B^{\varphi=p^n}$ *converging, i.e.* $c_n \in C^b$ *for all* $n \in \mathbb{Z}$ *with* $\lim_{n>0}\sup |c_n|^{\frac{1}{n}} \leq 1$ *and* $\lim_{n\to\infty} |c_{-n}|^{\frac{1}{n}} = 0$, *it follows by Definition*

$$\sum_{n\in\mathbb{Z}} [c_n^p] \cdot p^n = \varphi(f) = p^k \cdot f = \sum_{n\in\mathbb{Z}} [c_n^p] \cdot p^{n+k} = \sum_{n\in\mathbb{Z}} [c_{n-k}] \cdot p^n.$$

Hence $c_{n-k} = c_n^p$ *for all* $n, k \in \mathbb{Z}$.

1.case: Suppose that $k < 0$. *For all* $n \in \mathbb{Z}$ *we consider the sequence* $c_{n+k} = c_n^{\frac{1}{p}}, c_{n+2k} = c_n^{\frac{1}{p^2}}, c_{n+3k} = c_n^{\frac{1}{p^3}}, \ldots$

2.case: Let $k = 0$. *Hence* $c_n = c_n^p$ *for all* $n \in \mathbb{Z}$, *i.e.* $c_n \in \mathbb{F}_p \subset C^b$. *This induce a map*

$$\begin{aligned} \mathbb{F}_p &\to W_{\mathcal{O}_E}(\mathbb{F}_p) \hookrightarrow W_{\mathcal{O}_E}(\mathbb{F}_p)[\frac{1}{p}] := W(\mathbb{F}_p)[\frac{1}{p}] = \mathbb{Q}_p \to B^{\varphi=\pi} \\ c_n &\mapsto [c_n] \mapsto [c_n] \mapsto [c_n] \cdot p^n \end{aligned}$$

3.case: Let $k > 0$. *Then from the condition* $c_{n-k} = c_n^p$ *is follows*

$$\begin{aligned} c_k^p = c_0 &\Rightarrow c_k = c_0^{\frac{1}{p}} \\ c_{k+1}^p = c_1 &\Rightarrow c_{k+1} = c_1^{\frac{1}{p}} \\ c_{k+2}^p = c_2 &\Rightarrow c_{k+2} = c_2^{\frac{1}{p}} \\ c_{k+3}^p = c_3 &\Rightarrow c_{k+3} = c_3^{\frac{1}{p}} \\ &. \\ &. \\ &. \\ c_{2k}^p = c_k &\Rightarrow c_{2k} = c_k^{\frac{1}{p}} = c_0^{\frac{1}{p^2}} \\ c_{2k+1}^p = c_{k+1} &\Rightarrow c_{2k+1} = c_{k+1}^{\frac{1}{p}} = c_1^{\frac{1}{p^2}} \\ c_{2k+2}^p = c_{k+2} &\Rightarrow c_{2k+1} = c_{k+1}^{\frac{1}{p}} = c_2^{\frac{1}{p^2}} \\ &. \\ c_{3k}^p = c_{2k} &\Rightarrow c_{3k} = c_k^{\frac{1}{p^2}} = c_0^{\frac{1}{p^3}} \\ c_{3k+1}^p = c_{2k+1} &\Rightarrow c_{3k+1} = c_{2k+1}^{\frac{1}{p^2}} = c_1^{\frac{1}{p^3}} \\ &. \end{aligned}$$

Therefore $c_0, \ldots, c_{k-1}$ *determines the sequence* $(c_n)_{n \in \mathbb{Z}}$.

claim:
$c_0, \ldots, c_{k-1} \in \mathfrak{m}_C^b \subset \mathcal{O}_C^b$.

proof:
This follows directly by the $1:1$*-correspondence*

$$\begin{aligned} \mathfrak{m}_C^b &\to B^{\varphi=p} \\ c &\mapsto \sum_{n \in \mathbb{Z}} [c^{q^{-n}}] p^n \end{aligned}$$

(cf. Annschütz [10] in Proposition 4.2.1. part (2))
where $f \in B^{\varphi=p}$.

Lemma 8. *It holds* $(B^b)^{\varphi=\pi^d} = \begin{cases} E, & d=0 \\ 0, & d \neq 0 \end{cases}$

Beweis. We are separating the proof in two cases:

1. $d = 0$

2. $d \neq 0$

<u>1.Case:</u> Let $f = \sum_{i>>-\infty}^{\infty} [x_i]\pi^i \in (B^b)^{\varphi=1}$. By Definition $\forall i \in \mathbb{Z}$ we get $\varphi(x_i) = x_i$, i.e. $x_i \in \mathbb{F}_q$.
This implies $f \in E$.

<u>2.Case:</u> Let $d \neq 0$ and $0 \neq f \in B^{\varphi=\pi^d}$. Then $\forall x \in \mathbb{R}$ we have

$$q\text{Newt}(f)(x) = \text{Newt}(\varphi(f))(x) = \text{Newt}(\pi^d f)(x) = \text{Newt}(f)(x-d).$$

Thus $\forall n \in \mathbb{N}$ we have

$$q^n\text{Newt}(f)(x) = \text{Newt}(\varphi^n(f))(x) = \text{Newt}(\pi^{dn} f)(x) = \text{Newt}(f)(x-dn).$$

Now we distinguish the two subcases
i.) $d > 0$
ii.) $d < 0$

Ad i.): Now it follows $q^n\text{Newt}(f)(x) = \text{Newt}(f)(x-nd) = \infty$, $\forall n >> 0$. But this implies the contradiction $f = 0$.

Ad ii.): Now there exists a $x_0 << 0$ such that $\text{Newt}(f)(x_0) \geq \text{Newt}(f)(x)$ $\forall x \in$ R. As $\text{Newt}(f)$ is non-decreasing, we get

$$\text{Newt}(f)(x) \geq \text{Newt}(f)(x_0 - nd) = q^n\text{Newt}(f)(x_0) \geq q^n\text{Newt}(f)(x)$$

$\forall n >> 0$, $\forall x \in \mathbb{R}$. This implies $\text{Newt}(f)(x) = \infty$, for all $x \in \mathbb{R}$ i.e. $f = 0$. □

Lemma 9. *Let* $(x_n)_{n\in\mathbb{Z}} \subset F^{\mathbb{Z}}$ *a sequence s.th.*

$$\lim_{|n|\to\infty} v(x_n) + nr = \infty$$

for all $r \in (0,\infty)$. *Then* $\sum_{n\in\mathbb{Z}} [x_n]\pi^n$ *converges in B.*

Beweis. Hence it suffices to show that

$$v_r([x_n]\pi^n) \to \infty$$

for $|n| \to \infty$ and for all $r \in (0,\infty)$ holds. Because this is equivalent to $|[x_n]\cdot\pi^n|_\rho = 0$ $\forall |n| \to \infty, \forall \rho \in (0,1)$. But this follows directly by the only assumption with

$$v_r([x_n]\pi^n) = v(x_n) + rn \to \infty$$

for all $|n| \to \infty$ and for all $r \in (0,\infty)$. □

Bemerkung 10. *We construct a bijektiv map* $\mathfrak{m}_F \to B^{\varphi=\pi}$ *in the following way:*
Let $a \in \mathfrak{m}_F$, *i.e.* $v(a) > 0$. *Then*

$$f_a := \sum_{i\in\mathbb{Z}} [a^{q^{-i}}]\pi^i$$

converges in B, *because of the previous Lemma with the satisfyed assumption:*

$$v(a^{q^{-i}}) + ir = q^{-i}v(a) + ir \to \infty$$

for all $|i| \to \infty$ *and* $v(a) > 0$. *Moreover*

$$\varphi(f_a) = \sum_{i\in\mathbb{Z}} [a^{-(i-1)}]\pi^i = \pi \cdot f_a$$

after an Index shift, induce the map

$$\mathfrak{m}_F \to B^{\varphi=\pi}$$
$$a \mapsto f_a$$

Definition 11. *For an open interval* $I \subset (0,\infty)$ *and* $f \in B_I$ *we define* $\mathrm{Newt}_I^0(f)$ *as the decreasing convex function whose Legendre transformation is*

$$\mathscr{L} : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$$
$$r \mapsto \begin{cases} v_r(f) & , r \in I \\ -\infty & , r \notin I \end{cases}$$

with

$$\mathrm{Newt}_I(f) := \{(x, \mathrm{Newt}_I^0(f)(x)) \in \Gamma(\mathrm{Newt}_I^0(f)) | \exists! i \in \mathbb{N}_0 : \ x \in [i, i+1] \ : \lambda_i \in -I\} \subset \mathbb{R}^2.$$

Here λ_i *is the slope of* $\mathrm{Newt}_I^0(f)$ *on the interval* $[i, i+1]$ *for all* $i \in \mathbb{N}_0$.

Bemerkung 1. *Let* $K \subset I$ *and* $(f_n)_{n\in\mathbb{Z}}$ *a sequence in* B^b *and* f *an element in* B^b *with* $f \neq 0$, *s.th.*

$$\lim_{n\to\infty} f_n = f.$$

I.e.

$$\forall r \in K : \forall \varepsilon \geq 0 : \exists N \in \mathbb{N} : \forall n \geq N : |f_n - f|_r \leq \varepsilon.$$

Daraus folgt

$$\exists N \in \mathbb{N} : \forall n \geq N : \forall r \in K : v_r(f_n) = v_r(f).$$

Daraus folgt $\mathscr{L}(\mathrm{Newt}_I^0(f))$ *is a concave function with integral slopes. Therefore* Newt_I^0 *is a decreasing convex polygon with integral break points. If* $f \in B$ *and* λ_i *the slope of* Newt *for all* $i \in \mathbb{N}$, *then*

1.) $\lambda_i < 0$

2.) $\lim_{i\to\infty} \lambda_i = 0$

3.) $\lim_{i\to-\infty} \lambda_i = \infty$

If the interval I is compact, then the definition of the Newton polygon must be modificated.

Definition 12. *Let $I = [a, b] \subset (0, \infty)$ a compact interval, $f \in B_I$ and $f \neq 0$. We define $\mathrm{Newt}_I^0(f)$ as the decreasing convex function whose Legendre transform is*

$$\mathscr{L} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$$
$$r \mapsto \begin{cases} v_r(f) & , r \in I \\ v_a(f) + (r-a)\partial_g v_a(f) & , r < a \\ v_b(f) + (r-b)\partial_d v_b(f) & , r > b \end{cases}$$

with

$$\mathrm{Newt}_I(f) := \{(x, \mathrm{Newt}_I^0(f)(x)) \in \Gamma(\mathrm{Newt}_I^0(f)) | \exists! i \in \mathbb{N}_0 : \ x \in [i, i+1] \ : \lambda_i \in -I\} \subset \mathbb{R}^2.$$

The λ_i is the slope of $\mathrm{Newt}_I^0(f)$ on the interval $[i, i+1]$ for all $i \in \mathbb{N}_0$ just like before.

Bemerkung 13. *For an element $f \in B^b$ we consider in the definition before the fonction*

$$\mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$$
$$r \mapsto v_r(f)$$

with their left and right derivatives

$$\partial_g v_a(f) = \lim_{r \to a^-} \frac{v_r(f) - v_a(f)}{r - a} \geq \frac{v_r(f) - v_a(f)}{r - a}$$

for all $r < a$ and

$$\partial_d v_b(f) = \lim_{r \to b^+} \frac{v_r(f) - v_b(f)}{r - b} \leq \frac{v_r(f) - v_b(f)}{r - b}$$

for all $b < r$. This is equivalent to $v_r(f) \leq v_a(f) + \partial_g v_a(f)(r-a)$ for all $r < a$ and

$$v_r(f) \geq v_b(f) + \partial_g v_b(f)(r-b)$$

for all $b < r$. Furthermore let λ be a slope of $\mathrm{Newt}(f)$, then $m_\lambda(\mathrm{Newt}(f)) = \partial_g v_{-\lambda}(f) - \partial_d v_{-\lambda}(f)$, where $m_\lambda(\mathrm{Newt}(f))$ is the multiplicity of the slope λ of $\mathrm{Newt}(f)$. The mutiplicity formula explains the difference between the open and the compact case and the right multipicities of the slopes $-a, -b$ are in the previous definition $m_{-a}(\mathrm{Newt}(f)) = \partial_g v_a(f) - \partial_d v_a(f)$ and $m_{-b}(\mathrm{Newt}(f)) = \partial_g v_b(f) - \partial_d v_b(f)$.