Preliminaries

Let $\mathscr{C}$ be a category. We write $\mathrm{Hom}_{\mathscr{C}}$ or $\mathrm{Mor}(\mathscr{C})$ for the category of *morphisms*. Explicitly, its objects are morphisms $X \to Y$ in $\mathscr{C}$ and its morphisms are commutative squares

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'. \end{array}$$

Part I: Model Categories

The basics

Definition 1. A ***model category*** is a category $\mathscr{C}$, together with three distinguished classes of morphisms (more specifically, subcategories of $\mathrm{Hom}_{\mathscr{C}}$), namely

- weak equivalences ($\xrightarrow{\sim}$);
- fibrations ($\twoheadrightarrow$);
- cofibrations ($\hookrightarrow$),

which satisfy the following axioms:

MC1 The category $\mathscr{C}$ admits all small limits and colimits.

MC2 Weak equivalences satisfy the two-out-of-three property.

MC3 Weak equivalences, fibrations and cofibrations are closed under taking retracts.

MC4 Suppose we have a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y, \end{array}$$

then a lift $B \to X$ exists in case either [f is a cofibration and g is both a fibration and a weak equivalence] or [f is both a cofibration and a weak equivalence and g is a fibration].

MC5 Every morphism $X \to Y$ has a factorisation $X \xrightarrow{f} Z \xrightarrow{g} Y$ where [f is a cofibration and g both a weak equivalence and a fibration] as well as one where [f is both a weak equivalence and a cofibration and g a fibration].

Remark.
- Definitions vary. Some authors require $\mathscr{C}$ to only admit finite limits and colimits. Some authors also require the factorisation in MC5 to be functorial.
- Note that (by MC1) $\mathscr{C}$ admits an initial object $\varnothing$ and a terminal object $*$.
- We will call morphisms which are both cofibrations and weak equivalences *trivial cofibrations* (and similarly we speak of *trivial fibrations*). We will also often omit the morphisms from notation.

Before we deal with some examples, we introduce some more terms.

Definition 2. Let $\mathscr{C}$ be a model category. An object X is ***cofibrant*** if the morphism $\varnothing \to X$ is a cofibration. We call X ***fibrant*** if the morphism $X \to *$ is fibrant.

If $X \in \mathscr{C}$ is any object, we may factor the unique map $\varnothing \to X$ through some object QX as

$$\begin{array}{ccc} \varnothing & \hookrightarrow & QX \\ & \searrow & \downarrow \wr \\ & & X, \end{array}$$

so QX is cofibrant and weakly equivalent to X.

Definition 3. The datum consisting of QX and the map $QX \to X$ is a ***cofibrant replacement*** of X. Dually, one defines a ***fibrant replacement*** $X \to RX$.

Remark. Applying the lifting axiom to the diagram

$$\begin{array}{ccccc} \varnothing & \longrightarrow & & \longrightarrow & QY \\ \downarrow & & & & \downarrow \\ QX & \longrightarrow & X & \longrightarrow & Y \end{array}$$

yields a morphism $QX \to QY$ 'replacing' $X \to Y$. We cannot say anything about functoriality in $\mathscr{C}$, but we will be able to in its homotopy category.

Example 4. (i) We equip the category **Top** with the following model structure: weak equivalences are weak homotopy equivalences. Fibrations are Serre fibrations (i.e. maps with the RLP with respect to the inclusions $D^n \hookrightarrow D^n \times I$) and cofibrations are the maps which have the LLP with respect to all trivial fibrations.

(ii) We equip the category $\mathbf{Ch}_\bullet(\mathbf{Mod}_R)$ with the following model structure: weak equivalences are the quasi-isomorphisms. Fibrations are levelwise surjections. Cofibrations are levelwise injections with projective cokernel.

(iii) We equip the category **sSet** with the following model structure: weak equivalences are the maps whose geometric realisation is a weak equivalence in **Top**. Cofibrations are injections and fibrations are maps that have the RLP with respect to horn inclusions. We equip the category of simplicial R-modules $\mathbf{sMod_R}$ with the 'same' model structure; more specifically, we let weak equivalences and fibrations be the morphisms which are weak equivalences and fibrations respectively on simplicial sets. Cofibrations are maps which satisfy the LLP with respect to trivial fibrations.

Next to these meaningful examples there are the usual examples (e.g. products, opposite categories, trivial model structures).

Proposition 5. *Let $\mathscr{C}$ be a model category.*

(i) The trivial cofibrations are precisely the morphism that have the left lifting property with respect to the fibrations;

(ii) *The cofibrations are precisely the morphism that have the left lifting property with respect to the trivial fibrations;*

(iii) *The trivial fibrations are precisely the morphism that have the right lifting property with respect to the cofibrations;*

(iv) *The fibrations are precisely the morphism that have the right lifting property with respect to the trivial fibrations.*

Proof. We prove the first statement. The rest follows from analogous arguments. One direction is given from the definitions. Let $f\colon X \to Y$ be a morphism satisfying the left lifting property with respect to *all* fibrations. We factor f as $X \to Z \to Y$, where $i\colon X \to Z$ is a trivial cofibration and $p\colon Z \to Y$ is a fibration. Hence, we find a lift in the diagram

$$\begin{array}{ccc} X & \overset{\sim}{\underset{i}{\hookrightarrow}} & Z \\ f\big\downarrow & \overset{h}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow{\mathrm{id}_Y} & Y. \end{array}$$

With a bit of creativity we rewrite this diagram to

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown} & & \\ X & \xrightarrow[\mathrm{id}_X]{} & X & \xrightarrow[\mathrm{id}_X]{} & X \\ \big\downarrow f & & i\big\downarrow\wr & & \big\downarrow f \\ Y & \xrightarrow{h} & Z & \xrightarrow{p} & Y. \\ & & \underset{\mathrm{id}_Y}{\smile} & & \end{array}$$

Hence, we realise f as a retract of a trivial cofibration which makes it a trivial cofibration itself. ■

The homotopy category of a model category

Let $\mathscr{C}$ be a category. Let $S \subseteq \mathrm{Mor}(\mathscr{C})$ be a class of morphisms.

Definition 6. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is a ***localisation*** of $\mathscr{C}$ with respect to S if $F(f)$ is an isomorphism for all $f \in S$ and if for any other such functor $F'\colon \mathscr{C} \to \mathscr{D}'$ there exists a *unique* functor $G\colon \mathscr{D} \to \mathscr{D}'$ such that $G \circ F = F'$.

The category $\mathscr{D}$ is often written as $\mathscr{C}[S^{-1}]$. One observes that if a localisation exists, it is unique up to (unique) isomorphism. Now let $\mathscr{C}$ be a model category, its class of weak equivalences being denoted W.

Definition 7. The ***homotopy category*** of $\mathscr{C}$, denoted $\mathbf{Ho}(\mathscr{C})$, is the localisation (or more precisely, the codomain of the localisation) of $\mathscr{C}$ with respect to W.

It is nontrivial to show that such a localisation always exists. It however does for any model category $\mathscr{C}$; we will briefly describe its construction. One step in doing this requires us to put an equivalence relation (i.e. homotopy) on the classes of morphisms $\mathrm{Hom}_{\mathscr{C}}(A, X)$.

Definition 8. (i) A ***cylinder object*** for $A \in \mathscr{C}$ is an object $A \wedge I$, together with a factorisation $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$ of the map $\mathrm{id}_A + \mathrm{id}_A$.

(ii) Two maps $f\colon A \to X$ are ***left homotopic*** if there *exists* a cylinder object $A \wedge I$ for A and $f + g\colon A \sqcup A \to X$ factors through $A \wedge I$ via i.

Dually one defines ***path objects*** for X and ***right homotopies***. While one can obtain many detailed results regarding these objects and homotopies, we give a brief summary of the relevant results. The most important thing to keep in mind is: "Homotopies behave the nicest when A is cofibrant and X is fibrant."

- If A is cofibrant (resp. X fibrant), then left (resp. right) homotopy is an equivalence relation on $\mathrm{Hom}_{\mathscr{C}}(A, X)$;
- If A is cofibrant *and* X is fibrant, then left homotopy and right homotopy agree. We write $\pi(A, X)$ for the homotopy equivalence classes of maps $A \to X$. The composition in $\mathscr{C}$ induces well-defined compositions $\pi(A, X) \times \pi(X, Y) \to \pi(A, Y)$ and $\pi(B, A) \times \pi(A, X) \to \pi(B, X)$;
- If A and X are both cofibrant and fibrant, the weak equivalences $A \to X$ are *precisely* the morphisms which admit a homotopy inverse.

We are now ready to briefly summarise the construction of the homotopy category of $\mathscr{C}$. To do so, we define the following categories associated to $\mathscr{C}$: its full subcategories $\mathscr{C}_c, \mathscr{C}_f, \mathscr{C}_{cf}$ consisting of all cofibrant/fibrant/cofibrant-fibrant objects and their counterparts $\pi\mathscr{C}_c, \pi\mathscr{C}_f, \pi\mathscr{C}_{cf}$ having (left/right) homotopy classes as morphisms. The assignment of cofibrant replacements $X \to QX$ turns out to be a *functor* $Q\colon \mathscr{C} \to \pi\mathscr{C}_c$. A similar result holds for the fibrant replacements; the corresponding functor $R\colon \mathscr{C} \to \pi\mathscr{C}_f$. By restricting R to $\mathscr{C}_c$ it induces a functor $R'\colon \pi\mathscr{C}_c \to \pi\mathscr{C}_{cf}$. Now let $\mathscr{C}'$ be the category with $\mathrm{Ob}(\mathscr{C}') = \mathrm{Ob}(\mathscr{C})$ and $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \mathrm{Hom}_{\pi\mathscr{C}_{cf}}(R'QX, R'QY) = \pi(RQX, RQY)$. We obtain a canonical functor $\gamma\colon \mathscr{C} \to \mathscr{C}'$ given on the objects by the identity and on morphisms by $\gamma(f) = R'Q(f)$.

Proposition 9. *The category $\mathscr{C}'$ is the homotopy category of $\mathscr{C}$.*

Quillen functors

Let $\mathscr{C}$ and $\mathscr{D}$ be model categories. Let (F, U) be an adjoint pair of functors between $\mathscr{C}$ and $\mathscr{D}$.

Definition 10. The functor F (resp. U) is called a left (resp. right) ***Quillen functor*** if it preserves cofibrations and trivial cofibrations (resp. fibrations and trivial fibrations).

One can show that U is a right Quillen functor if and only if F is a left Quillen functor; in this case the adjunction (F, U) is called a *Quillen adjunction*.

Definition 11. A Quillen adjunction (F, U) is a ***Quillen equivalence*** if the natural isomophism $\mathrm{Hom}_{\mathscr{D}}(FX, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, UY)$ sends weak equivalences to weak equivalences for all cofibrant $X \in \mathscr{C}$ and fibrant $Y \in \mathscr{D}$.

Example 12. We give two examples of Quillen adjunctions. These adjunctions in fact turn out to be Quillen equivalences.

(i) Between topological spaces and simplicial sets, one has the adjunction

$$|\cdot| : \mathbf{Top} \rightleftarrows \mathbf{sSet} : \mathrm{Sing};$$

(ii) Between nonnegatively graded chain complexes of R-modules and simplicial R-modules, one has the Dold-Kan equivalence

$$K : \mathbf{Ch}_{\bullet}^{+}(\mathbf{Mod}_R) \rightleftarrows \mathbf{sMod_R} : N$$

(iii) The forgetful functor $U\colon \mathbf{sMod_R} \to \mathbf{sSet}$ is right Quillen.

Let $\mathscr{E}$ be another model category.

Definition 13. A ***left Quillen bifunctor*** is a functor $\Phi\colon \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ with the following properties:

(i) Φ preserves colimits separately in each variable;

(ii) if $i\colon C \to C'$ is a cofibration in $\mathscr{C}$ and $j\colon D \to D'$ is a cofibration in $\mathscr{D}$, then the induced map $\Phi(C, D') \sqcup_{\Phi(C,D)} \Phi(C', D) \to \Phi(C', D')$ is a cofibration in $\mathscr{E}$ which is trivial if either i or j is.

Part II: Homotopy on simplicial categories

Recall: A simplicial category is an **sSet**-enriched category (i.e. roughly spoken a category in which the sets of homomorphisms have the (extra) structure of a simplicial set). A great example is the enrichment of **Top**, given by $\mathrm{Hom}(X,Y)_n = \mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y)$.

Definition 14. Let $\mathscr{C}_\bullet$ be a simplicial category. Let $X, Y \in \mathscr{C}$. A ***homotopy*** from f to g in $\mathrm{Hom}_{\mathscr{C}}(X,Y)_0$ is a simplex $\sigma \in \mathrm{Hom}_{\mathscr{C}}(X,Y)_1$ with $d_1(\sigma) = f$ and $d_0(\sigma) = g$.

We will define an analogous notion of the nerve of a category which additionally 'encodes' the simplicial structure. For comparison, we include the definition of the nerve.

Definition 15. Let $[n]$ be the ordered set $\{0 < 1 < \cdots < n\}$. Then the elements of $[n]$ become the objects of a category when

(i) there is exactly one morphism between i and j precisely when $i \leq j$ (this is the order category);

(ii) the morphisms between i and j are the subsets $\{i < a_0 < \cdots < a_r < j\}$, composed via union (this is the path category).

Remark. We write $\mathrm{Path}[n]$ for the path category of $[n]$; in this case, each set of homomorphisms $\mathrm{Hom}(i,j)$ can be ordered by reverse inclusion, hence we can equip them with the structure of a category. The category $\mathrm{Path}[n]$ now becomes a simplicial category by setting $\mathrm{Hom}_\bullet(i,j) = N_\bullet(\mathrm{Hom}(i,j))$.

Definition 16. Let $\mathscr{C}$ ($\mathscr{C}_\bullet$) be a (simplicial) category.

(i) The ***nerve*** of $\mathscr{C}$, denoted $N_\bullet(\mathscr{C})$ is the simplicial set given by $N_n(\mathscr{C}) = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathscr{C})$;

(ii) The ***homotopy coherent nerve*** of $\mathscr{C}_\bullet$, denoted $N_\bullet^{\mathrm{hc}}(\mathscr{C})$, is the simplicial set given by $N_n^{\mathrm{hc}}(\mathscr{C}) = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathrm{Path}[n]_\bullet, \mathscr{C}_\bullet)$.

Remark. The nerve and homotopy coherent nerve agree on vertices and edges. However, 2-simplices of the nerve consist solely of commutative triangles, where 2-simplices of the homotopy coherent nerve consist of weakly commutative triangles (i.e. up to homotopy).

Proposition 17. *The homotopy coherent nerve of a simplicial category $\mathscr{C}_\bullet$ is an ∞-category if and only if $\mathscr{C}_\bullet$ is locally Kan i.e. if every simplicial set $\mathrm{Hom}_{\mathscr{C}}(X, Y)_\bullet$ is a Kan complex.*

We now 'combine' the theory of model categories and simplicial categories.

Definition 18. Let $\mathscr{C}_\bullet$ be a simplicial category whose underlying category is a model category. Then $\mathscr{C}_\bullet$ is a ***simplicial model category*** when

(i) The category $\mathscr{C}_\bullet$ is tensored and cotensored over **sSet** (e.g. we obtain 'tensor objects' $S_\bullet \otimes X$ such that $\mathrm{Hom}_{\mathbf{sSet}}(S_\bullet, \mathrm{Hom}_{\mathscr{C}}(X, Y)_\bullet)$ is canonically isomorphic to $\mathrm{Hom}_{\mathscr{C}}(S_\bullet \otimes X, Y)$ for all $Y \in \mathscr{C}_\bullet$);

(ii) The functor $- \otimes -\colon \mathbf{sSet} \times \mathscr{C} \to \mathscr{C}$ is a Quillen bifunctor.

Remark. For simplicial model categories, the homotopy relations associated to the model structure agree with the homotopy relations associated to the simplicial structure. Additionally, it is worthwhile to mention that $\mathrm{Hom}_{\mathscr{C}}(X, Y)_\bullet$ is a Kan complex in a simplicial category whenever X is cofibrant and Y is fibrant.

Part III: Relation to ∞-categories

The theories of model categories and infinity categories are strongly related. We will see in a moment that we may assign to any model category an infinity category which behaves like the localisation of the nerve with respect to the weak equivalences. The converse is almost true, as every *presentable* infinity category arises in such a way from a model category. Here a presentable infinity category is one that is equivalent to a full subcategory of the category of presheaves on a small category.

Definition 19. Let $\mathscr{C}$ be an infinity category and let W be a set of morphisms. A morphism $f\colon \mathscr{C} \to \mathscr{D}$ exhibits $\mathscr{D}$ as *the ∞-category obtained from $\mathscr{C}$ by inverting the set of morphisms W* if, for every ∞-category $\mathscr{E}$, composition with f induces a fully faithful embedding $\mathrm{Fun}(\mathscr{D}, \mathscr{E}) \hookrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{E})$ which essential image consists of the functors that map elements of W to isomorphisms.

We also denote the ∞-category $\mathscr{D}$ in the previous definition by $\mathscr{C}[W^{-1}]$.

Definition 20. Let $\mathbf{C}$ be a model category and let $\mathscr{D}$ be an ∞-category. A functor $f\colon N_\bullet(\mathbf{C}_c) \to \mathscr{D}$ exhibits $\mathscr{D}$ as the ***underlying ∞-category*** of $\mathbf{C}$ whenever it induces an equivalence

$$N_\bullet(\mathbf{C}_c)[W^{-1}] \cong \mathscr{D}.$$

Here $\mathbf{C}_c$ is the full subcategory of cofibrant objects and W is the collection of weak equivalences between cofibrant objects.

Remark. One obtains equivalent ∞-categories by replacing $\mathbf{C}_c$ by either $\mathbf{C}, \mathbf{C}_f$ $\mathbf{C}_{cf}$ and replacing W by the weak equivalences between the coresponding objects. In particular, one observes that the underlying ∞-category of a model category is only dependent on its weak equivalences.

Theorem 21. *Let $\mathbf{C}_\bullet$ be a simplicial model category. Then the underlying ∞-category of $\mathbf{C}$ is the homotopy coherent nerve $N^{\mathrm{hc}}_\bullet(\mathbf{C})$.*