Aim (next 2 talks) Construct Lusztig's parametrisation of $\mathrm{Irr}(G^F)$ via <u>Lusztig series</u> & <u>dual gps</u>

- a F^*-stable conj. class of semi-simple elts (s) in the dual gp G^*
- unip. char of $C_{G^*}(s)^{F^*}$

Today focus on the first part.

Setup: G conn red gp over $k=\overline{\mathbb{F}_p}$, $F: G \to G$ Steinberg endo

T a F-stable max^l torus, $\theta \in \mathrm{Irr}(T^F)$ we obtain a virtual character $R_T^G(\theta)$ "D-L character".

$R_T^G(\theta)(1) = \varepsilon_G \varepsilon_T |G^F : T^F|_{p'}$

Prop $\rho \in \mathrm{Irr}(G^F)$ & $s \in G^F$ s.s. Then

(D-L 76 7.6)

(G-M 2.2.18)

$$\rho(s) = \frac{1}{|H|_p} \sum_{(T,\theta)} \varepsilon_H \varepsilon_T \langle R_T^G(\theta), \rho \rangle \, \theta(s)$$

where $\varepsilon_H, \varepsilon_T \in \{\pm 1\}$, $H = C_G^\circ(s)$ & sum is over $\{T$ F-stable max^l torus, $s \in T$, & $\theta \in \mathrm{Irr}(T^F)$.

There is a character formular for a general elt but more involved (G-M 2.2.16)

If g is unip then $R_T^G(\theta)(g)$ does not depend on θ.

$\rightsquigarrow$ For $\rho \in \mathrm{Irr}(G^F)$ $\exists$ pair (T,θ) s.t. $\langle R_T^G(\theta), \rho \rangle \neq 0$ ($s=1$ so $\rho(s) \neq 0$)

Prop (Scalar Prod)

(D-L 76 §6)

G-M 2.2.8)

$$\langle R_T^G(\theta), R_{T'}^G(\theta') \rangle = \frac{1}{|T^F|} \# \{ g \in G^F \mid ({}^gT, {}^g\theta) = (T', \theta') \}$$

$\rightsquigarrow$ - The $R_T^G(\theta)$ indexed by G^F-conj classes in

$$\mathcal{X}(G,F) := \{ (T,\theta) \mid T \text{ F-stable max}^l, \theta \in \mathrm{Irr}(T^F) \}$$

form an set of orthogonal class functions.

- If no elt of $N_G(T)^F$ fixes θ, then $R_T^G(\theta)$ or $-R_T^G(\theta)$ is in $\mathrm{Irr}(G^F)$

In general $\langle\ ,\ \rangle = 0$ but still share an irred constituent as virtual characters.

Def^n $(T,\theta), (T',\theta') \in \mathcal{X}(G,F)$ are called geometrically conj (by $g \in G$) if $\exists\, n \in \mathbb{N}_{>0}$ s.t. & $g \in G^{F^n}$ s.t. ${}^gT = T'$ & g conj $\theta \circ N_{F^n/F}$ to $\theta' \circ N_{F^n/F}$

(G-M 2.3.1)

$$N_{F^d/F}: T \to T, \quad t \mapsto t \cdot F(t) \cdots F^{d-1}(t), \qquad N_{F^d/F}(T^{F^d}) = T^F \Rightarrow \mathrm{Irr}(T^F) \hookrightarrow \mathrm{Irr}(T^{F^d}), \quad \theta \mapsto \theta \circ N_{F^d/F}$$

Thm (Exclusion thm) If $R_{T_1}^G(\theta_1)$ & $R_{T_2}^G(\theta_2)$ have an irred. const. in common then (T_1,θ_1) & (T_2,θ_2) are geom. conj.

(D-L 76 § 5.4 + 6.3)

(G-M 2.3.2)

We want to interpret "geom conj" via bijections of $\mathcal{X}(G,F)/\sim_{G^F}$ to other sets.

Weyl gps

Recall: Let T F-stable max^l torus, W weyl gp defined w.r.t. T_0 & $g \in G$ s.t. ${}^gT_0 = T$.

Also denote by σ the action of F on W.

Then $g^{-1}F(g) \in N_G(T_0)$ $\quad (F(g)F(T_0)F(g)^{-1} = F(T) = T = gT_0g^{-1})$ $\quad$ (G-M 1.6.4)

If ${}^{g_1}T_0 = {}^{g_2}T_0$ is F-stable, with $\dot{w}_i = g_i^{-1}F(g_i)$ for $w_i \in W$, then $\exists x \in W$ s.t. $w_2 = x w_1 \sigma(x)^{-1}$

$\rightsquigarrow$ $\{G^F\text{-conj cl of } F\text{-stable max}^l \text{ tori}\} \xleftrightarrow{1-1} \{\sigma\text{-conj cl of } W\}$

$$x \cdot w \cdot \sigma(x)^{-1}$$

$${}^gT_0 = T_w \longleftarrow\!\shortmid \; w$$

for $g^{-1}F(g) = \dot{w}$

Push this a little further to consider $\mathcal{X}(G,F)$

$({}^gT_0)^F = g\,(T_0[w])\,g^{-1}$ for $T_0[w] := T_0^{wF} = \{ t \in T_0 \mid F(t) = \dot{w}^{-1} t \dot{w} \}$

$\hookrightarrow (wF)(g) := \dot{w}F(g)\dot{w}^{-1}$

$\mathcal{X}(W,\sigma) := \{ (w,\theta) \mid w \in W,\ \theta \in \text{Irr}(T_0[w]) \} \curvearrowleft W$ $\quad$ (G-M 2.3.20)

$x \cdot (w,\theta) := (x w \sigma(x)^{-1}, {}^{\dot{x}}\theta)$ $\quad$ $[\, w' = x w \sigma(x)^{-1}$ then $T_0[w] \xrightarrow{\sim} T_0[w']$, $t \longmapsto {}^{\dot{x}}t = \dot{x} t \dot{x}^{-1}\,]$

refine the previous ~~the~~ bijection

$$\mathcal{X}(G,F)/\!\sim_{G^F} \xleftrightarrow{1-1} \mathcal{X}(w,\sigma)/\!\sim_W$$

$$(T,\theta') \longleftarrow\!\shortmid (w,\theta)$$

$T = {}^gT_0$, $g^{-1}F(g) = \dot{w}$

${}^g\theta'(t) = \theta(t) \quad \forall t \in T_0[w]$

Rem
- $\langle R^G_{T_1}(\theta_1'), R^G_{T_2}(\theta_2') \rangle = \# \{ x \in W \mid x\cdot(w_1,\theta_1) = (w_2,\theta_2) \}$ $\quad$ $(T_i,\theta_i') \longleftrightarrow (w_i,\theta_i)$

(G-M 2.3.22)

- $\rho(1) = \frac{1}{|W|} \sum_{(w,\theta) \in \mathcal{X}(W,\sigma)} \langle R^G_T(\theta), \rho \rangle R^G_T(\theta)(1)$ $\quad$ $(w,\theta) \sim (T,\theta)$ $\quad$ (G-M 2.3.23)

Dual gps

Def^{n} $(G,F), (G^*,F^*)$ G, G^* conn red gp F, F^* steinberg endo. Then (G,F) & (G^*, F^*) are said to be in duality if $\exists$ max^l tori $T_0 \subset G$ & $T_0^* \subset G^*$ s.t. for the corres. root datum $\mathcal{R} = (X, R, Y, R^\vee)$ & $\mathcal{R}^* := (X^*, R^*, Y^*, R^{*\vee})$:

- $\exists$ isom $\delta : X \longrightarrow Y^*$ s.t. $\delta(R) = R^{*\vee}$ & $\langle \lambda, \alpha^\vee \rangle = \langle \alpha^*, \delta(\lambda) \rangle$ $\forall \lambda \in X, \alpha \in R$
  $\alpha^* \rightsquigarrow \delta(\alpha) = (\alpha^*)^\vee$
- $\delta(\lambda \circ F) = F^* \circ \delta(\lambda)$ $\forall \lambda \in X$
  i.e. if $\lambda \in X$ & $\nu \in Y^*$ corr. under δ, then $\lambda \circ F$ corr to $F^* \circ \nu$.

Rem δ defines an isom of root data between $\mathcal{R}$ & the dual of $\mathcal{R}^*$. (1.5.17 G-M)
Recall conn red gps are determined up to isom by their root datum
$\rightsquigarrow$ $(X, R, Y, R^\vee) \overset{\text{duality}}{\rightsquigarrow} (Y, R^\vee, X, R)$

Ex $GL_2^* \cong GL_2$ $\qquad SL_2^* \cong PGL_2$

Rem Let (G,F) & (G^*,F^*) be in duality $\rightsquigarrow$ isom

$$\begin{array}{ccc} N_G(T_0)/T_0 & & N_{G^*}(T_0^*)/T_0^* \\ \downarrow & & \swarrow \\ W & \longrightarrow & W^* \\ w & \longmapsto & w^* \end{array}$$

s.t. $\sigma(w)^* = (\sigma^*)^{-1}(w^*)$

$$\begin{array}{ccc} \{\sigma\text{-conj classes in } W\} & \overset{1-1}{\longleftrightarrow} & \{\sigma^*\text{-conj classes in } W^*\} \\ \updownarrow & & \updownarrow \\ \{G^F \text{ c.c. } F\text{-st max}^l \text{ tori of } G\} & & \{G^{*F^*} \text{ c.c. of } F^*\text{-st. max}^l\text{.tori in } G^*\} \\ T & \longmapsto & T^* \end{array}$$

$T = T_w$ then $T^* = T_{(w^*)^{-1}}$ (Moreover $|T_0[w]| = |T_0^*[(w^*)^{-1}]|$ & so $|T^F| = |T^{*F^*}|$)

Prop $\{\text{geom-conj cl on } \mathcal{X}(G,F)\} \overset{1-1}{\longleftrightarrow} \{F^*\text{-stable conj cl of s.s. elts in } G^*\}$
(D-L 76)
(G-M 2.5.4 + 2.6.29)

Idea: Construct a common labelling system.
Fix $\psi : k^\times \hookrightarrow \mathbb{C}^\times$ & isom $i : k^\times \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$ $\qquad$ ($\psi = \exp \circ\, i$ $\quad \exp(x + \mathbb{Z}) := e^{2\pi i x}$)

- (λ, n) s.t. $\lambda \in X(T_0)$, $n \in \mathbb{N}_{>0}$ coprime to $p = \mathrm{char}(k)$
  Set $Z_{\lambda,n} = \{ w \in W \mid \exists\, \lambda_w \in X(T_0) \text{ with } \lambda_w(t)^n = \lambda(F(t)\, \underbrace{\dot{w}^{-1} t^{-1} \dot{w}}_{T_0[w]}) \ \forall t \in T_0 \}$
  & $\Delta(\lambda, n) = \{ (\lambda, n) \mid Z_{\lambda,n} \neq \emptyset \}$
  There is an equiv relation on $\Delta(\lambda,n)$ $\qquad (\lambda_1, n_1) \sim (\lambda_2, n_2)$ if $n_1 = n_2$ & $\exists x \in W$ $\lambda_2 = x \cdot \lambda_1 + n_1 \nu$
  $\nu \in X(T_0)$

- $\Lambda(\lambda,n)/\sim \xleftrightarrow{1-1} \mathcal{X}(W,\sigma)/\sim$ las geom conj

$(\lambda,n) \longmapsto (w,\theta)$

where $w \in Z_{\lambda,n}$ & $\theta = \psi \circ \lambda_w|_{T_0[w]} \in \operatorname{Irr}(T_0[w])$

(note $\lambda_w|_{T_0[w]}(t)^n = 1$ by definition)

- $\Lambda(\lambda,n)/\sim \xleftrightarrow{1-1} \{F^*\text{-stable conj cl of s.s. elts in } G^*\}$

$(\lambda,n) \longmapsto t_{\lambda,n} := \delta(\lambda)(\tau^{-1}(\frac{1}{n}+\mathbb{Z}))$

one has to show $(\lambda_1,n_1)\sim(\lambda_2,n_2) \iff t_{\lambda_1,n_1}$ & t_{λ_2,n_2} are G^*-conjugate. ☑

Lm (D-L 76) 1) For $w\in W$ $\exists!$ isom $\operatorname{Irr}(T_0[w]) \longrightarrow T_0^*[(w^*)^{-1}]$

$\theta \longmapsto s_\theta$

s.t. if $(\lambda,n)\mapsto(w,\theta)$ then $s_\theta = t_{\lambda,n}$.

Shown G^* class of $t_{\lambda n}$ is F^*-stable $\Rightarrow \exists\, x\in W^*$ s.t. $\dot{x}\, t_{\lambda n}\, \dot{x}^{-1} = F(t_{\lambda n})$

$\rightsquigarrow$ ~~$\exists$~~ $w\in Z_{\lambda,n}$ then $F(t_{\lambda,n}) = \dot{w}^* t_{\lambda n} (\dot{w}^*)^{-1}$

$\Rightarrow t_{\lambda,n} \in T_0^*[(w^*)^{-1}]$

2) Moreover $(w_i,\theta_i)\in\mathcal{X}(W,\sigma)$ are geom conj $\iff s_{\theta_2} = \dot{y}\, s_{\theta_1}\, \dot{y}^{-1}$ for some $y\in W^*$

Pf 1)

2) $(\lambda_i,n_i)\mapsto(w_i,\theta_i)$. Then $(\lambda_1,n_1)\sim(\lambda_2,n_2) \iff t_{\lambda_1,n_1}$ & t_{λ_2,n_2} are G^*-conj

$\iff$ $N_{G^*}(T_0^*)$-conj

e.g. $-\ GL_2$ (1.3.7 G-M)

$T_0 = \{(\lambda_1\ \lambda_2)\}$ $\quad X(T_0) = \mathbb{Z}^2$, $x_i(\lambda_1\lambda_2) = \lambda_i$ $\quad Y(T_0) = \mathbb{Z}^2$, $v_1(\lambda) = (\lambda\ 1)$, $v_2(\lambda) = (1\ \lambda)$

$N_G(T) = \langle T, n\rangle \quad n = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\overset{\sim}{\rightarrow}$ Fei $\quad R = \{\pm\alpha\} \quad \alpha(\lambda_1\lambda_2) = \lambda_1\lambda_2^{-1} \quad -\alpha(\lambda_1\lambda_2) = \lambda_1^{-1}\lambda_2$

$R^\vee = \{\pm\alpha^\vee\} \quad \alpha^\vee(\lambda) = (\lambda\ \lambda^{-1}), \quad -\alpha^\vee(\lambda) = (\lambda^{-1}\ \lambda)$

$\mathbb{Z}R = \langle(1,-1)\rangle \subseteq X, \quad \mathbb{Z}R^\vee = \langle(1,-1)\rangle \subseteq Y$

$-\ SL_2 \quad T_0' = \{(\lambda\ \lambda^{-1})\}$

$R' = \{\pm\alpha'\} \quad \alpha'$ restrict of α to T_0' $\quad \alpha'(\lambda\ \lambda^{-1}) = \lambda^2$

$R'^\vee = \{\pm\alpha'^\vee\} \quad \alpha'^\vee(\lambda) = (\lambda\ \lambda^{-1})$

$X(T_0) = \mathbb{Z} \quad x(\lambda\ \lambda^{-1}) = \lambda \quad \mathbb{Z}R' = 2\mathbb{Z}$

$Y(T_0) = \mathbb{Z} \quad v(\lambda) = (\lambda\ \lambda^{-1}) \quad \mathbb{Z}R'^\vee = \mathbb{Z}$

$-\ PGL_2 \cong GL_2(k)/Z, \quad \bar{T}_0 = T_0/Z = \overline{(\lambda\ 1)} \quad X(\bar{T}_0) = \mathbb{Z}, \ x(\overline{\lambda\ 1}) = \lambda \quad Y = \mathbb{Z}, \ v(\lambda) = \overline{(\lambda\ 1)}$

$Z \subset \ker(\alpha) \Rightarrow \bar\alpha \in X(\bar{T}_0) \quad \bar\alpha(\bar t) := \alpha(t) \quad \bar\alpha(\overline{\lambda\ 1}) = \lambda$

$\bar\alpha^\vee(\lambda) := \overline{\alpha^\vee(\lambda)}$

$\bar\alpha^\vee(\lambda) = \cancel{\alpha^\vee}\,\overline{(\lambda\ \lambda^{-1})} = \overline{(\lambda^2\ 1)}$

$\bar R = \{\pm\bar\alpha\} \quad \mathbb{Z}\bar R = \mathbb{Z}$

$\bar R^\vee = \{\pm\bar\alpha^\vee\} \quad \mathbb{Z}\bar R^\vee = 2\mathbb{Z}$

$\exists\ \alpha^\vee \in Y$ s.t. $\quad w_\alpha \cdot x = x - (x, \alpha^\vee)\alpha \quad \forall\ x \in X$ $\quad$ 1.3.1 pg 23

$w_\alpha \cdot x(t) = x(\dot w^{-1} t \dot w) = x(\lambda_2\ \lambda_1) = \lambda_2^{n_1}\lambda_1^{n_1}$ $\quad w_\alpha \cdot x = n_2 x_1 + n_1 x_1$

$\alpha^\vee = v_1 - v_2 \quad \cancel{\alpha = x_1 + x_2}$ $\quad x = n_1 x_1 + n_2 x_2$ $\quad x_i(v_j(z)) = z^{\langle x_i, v_j\rangle}$

$x - \langle x, \alpha^\vee\rangle\alpha = n_1 x_1 + n_2 x_2$ $\quad \langle \lambda_i, v_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j\end{cases}$

$\quad - (n_1 - n_2)(x_1$

$\langle x, \alpha^\vee\rangle = n_1 - n_2$

$= (n_1 x_1 + n_2 x_2) - (n_1 - n_2)(x_1 - x_2) = n_2 x_1 + n_1 x_2$

$t\, u_\alpha(z)\, t^{-1}$
$\parallel$
$u_\alpha(\alpha(t)z)$

$u_\alpha(z) = \begin{pmatrix}1 & z\\ & 1\end{pmatrix}$

$\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix} u_\alpha(z) \begin{pmatrix}\lambda_1^{-1} & \\ & \lambda_2^{-1}\end{pmatrix}$

$= \begin{pmatrix}1 & \lambda_1\lambda_2^{-1}z\\ & 1\end{pmatrix}$

$= u_\alpha(\lambda_1\lambda_2^{-1}z)$

A refinement of the exclusion thm

Notation $\mathcal{Y}(G,F) = \{(T,s) \mid T \text{ F-stable max}^l \text{ torus}, s \in T^F\} \circlearrowleft G^F \qquad g\cdot(T,s) = ({}^gT, {}^gs)$

$\mathcal{Y}(W,\sigma) = \{(w,t) \mid w \in W, t \in T_0[w]\} \circlearrowleft W \qquad x\cdot(w,t) = (xw\sigma(x)^{-1}, \dot{x}t)$

$$\rightsquigarrow \quad \mathcal{Y}(G,F)/\sim_{G^F} \xleftrightarrow{1-1} \mathcal{Y}(w,\sigma)/\sim_W$$

$$(T, s) \longleftarrow (w,t)$$

$T = {}^gT_0 \quad g^{-1}F(g) = \dot{w}$

$\&\ s = {}^gt$

Cor (2.5.12) The isom $\theta \mapsto s_\theta$ from before induces a bijection

$$\mathcal{X}(W,\sigma)/\sim_W \xleftrightarrow{1-1} \mathcal{Y}(W^*,\sigma^*)/\sim_{W^*}$$

$$(w,\theta) \longmapsto ((w^*)^{-1}, s_\theta)$$

Defn For $(T^*,s) \in \mathcal{Y}(G^*,F^*)$ with $(T^*,s) \sim (T,\theta) \in \mathcal{X}(G,F)$, then set $R^G_{T^*}(s) = R^G_T(\theta)$

Prop (2.5.15 G-M) $\langle R^G_{T_1^*}(s_1), R^G_{T_2^*}(s_2)\rangle = \frac{1}{|T_1^{*F^*}|}\#\{x \in G^{*F^*} \mid {}^x(T_1^*,s) = (T_2^*,s)\}$

Pf $(T_i^*,s_i) \sim (T_i,\theta_i)$

$\quad\ \wr \qquad\qquad\quad \wr$

$((w_i^*)^{-1}, s_i') \quad (w_i,\theta_i')$

LHS $= \#\{x \in W \mid x\cdot(w_1,\theta_1') = (w_2,\theta_2')\}$

RHS $= \#\{y \in W^* \mid y\cdot((w_1^*)^{-1}, s_1') = ((w_2^*)^{-1}, s_2')\}$

As $s_i' = s_{\theta_i'}$, & action of W on LHS under isom is compatible w/ W^* on RHS. ☐

Cor (2.5.10 (b)) $R^G_{T_i^*}(s_i)$ have an irred constit. in common $\Rightarrow$ s_1 & s_2 are G^*-conj.

Thm (Lusztig 77) (2.5.21 G-M) If $R^G_{T_i^*}(s_i)$ $i=1,2$ have an irred const. in common then s_1 & s_2 are G^{*F^*}-conj.

Pf uses regular embedding $\tilde{G}$, $Z(\tilde{G}) = Z^\circ(\tilde{G}) \rightarrow$ centralisers are connected.

Lusztig Series

Defn $s \in G^{*F^*}$ s.s. $\mathcal{E}(G^F,s) := \{\rho \in \mathrm{Irr}(G^F) \mid \langle R^G_{T^*}(s), \rho\rangle \neq 0 \text{ for some } (T^*,s) \in \mathcal{Y}(G^*,F^*)\}$

Thm (L. 77) $\mathrm{Irr}(G^F) = \bigsqcup_s \mathcal{E}(G^F,s)$ conj. classes of s.s. elts in G^{*F^*}

Pf $\mathrm{Irr}(G^F) = \bigcup_s \mathcal{E}(G^F,s)$

Let $s_2 = xs_1x^{-1}$ & T_1^* max torus containing s_1. Then $T_2 := {}^xT_1 \ni s_2$ so (T_2^*,s_2) & (T_1^*,s_1) are G^{*F^*}-conj

$\Rightarrow R^G_{T_1^*}(s_1) = R^G_{T_2^*}(s_2)$ & so $\mathcal{E}(G^F,s_1) \subseteq \mathcal{E}(G^F,s_2)$. The other inclusion is analogous.

$\mathcal{E}(G^F,s_1) \cap \mathcal{E}(G^F,s_2) \neq \emptyset \Rightarrow s_1$ & s_2 are G^{*F^*}-conj $\Rightarrow$ equality. ☐

Example $Gl_2(q)$ $\underline{G} \cong \underline{G}^*$ $W = C_2$, $F^*(a_{ij}) = (a_{ij}^q)$

F^* acts trivially on W^* $\Rightarrow$ $\exists$ 2 G^{*F^*}-conj classes of F^*-stable tori.

I_0, I_w

$\hookrightarrow$ diag matrices

$N_{\underline{G}}(I_0) = \langle I_0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ $\quad I_0^{F^*} = (\mathbb{F}_q^\times)^2$, $\quad I_w^{F^*} \cong I_0[w] \cong \mathbb{F}_{q^2}^\times$

Know ord of Σ is $q(q-1) = q^l |Z^\circ(G)^F|$

s.s. conj classes: $\begin{pmatrix} a & \\ & a \end{pmatrix}$ $\cancel{a \neq 0}$ $a \in \mathbb{F}_q^\times$ $\quad \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b \in \mathbb{F}_q^\times$ $\quad \begin{pmatrix} a & \\ & a^q \end{pmatrix}$ $a \in \mathbb{F}_{q^2}^\times$, $a \neq a^q$

$C_{\underline{G}^*}(s)$ $\quad q-1$ $\quad \frac{(q-1)(q-2)}{2}$ $\quad \frac{q(q-1)}{2}$

$\quad G^*$ $\quad I_0$ $\quad I_w$

$\underline{s = \begin{pmatrix} a & \\ & b \end{pmatrix} \; a \neq b}$,

$s \in I_0$

$\langle R_{I_0}^G(s), R_{I_0}^G(s) \rangle = 1$ $\Rightarrow$ irred char.

$\mathcal{E}(G^F, s) = \{\rho_s\}$ & $\rho_s(1) = |G^F|_{p'} / |I_0^{F^*}|$ $\varepsilon_G \varepsilon_{I_0} = \pm(q+1)$

Deg formula

$R_T^G(\theta)(1)$

$= \varepsilon_G \varepsilon_T \, |G^F : T^F|_{p'}$

2.2.12

$\underline{s \sim \begin{pmatrix} a & \\ & a^q \end{pmatrix} \; a^q \neq a}$,

$s \in I_w$, $\langle \quad \rangle = 1$ $\Rightarrow$ irred

$\mathcal{E}(G^F, s) = \{\rho_s\}$ & $\rho_s(1) = |G^F|_{p'}/|T| \quad \pm(q-1)$

$\underline{s = \begin{pmatrix} a & \\ & a \end{pmatrix} \; a \in \mathbb{F}_q^\times}$,

$\langle \quad \rangle = N_{\underline{G}}(I_0)^F / I_0^F = 2$ $\Rightarrow$ $R_{I_0}^G(s)$ sum of 2 irred chars.

$R_{I_0}^G(s)(1) = \pm(q+1)$

(Check: Gl_2 has $q-1$ characters of degree 1 & all lie in one such $R_{I_0}^G(s)$)

$\Rightarrow$ sum of a deg 1 & deg q char.

$R_T^G(\theta)$:

$T \subseteq B$ borel subgp, $U = R_u(B)$.

$\mathcal{L}_F^{-1}(U) = \{x \in G \mid x^{-1}F(x) \in U\}$

$G^F \times T^F$ acts on $\mathcal{L}_F^{-1}(U)$ $\quad (g,h).x = gxh^{-1}$

$\rightsquigarrow$ $H_c^i(\mathcal{L}_F^{-1}(U), \bar{\mathbb{Q}}_\ell)$ is a $\bar{\mathbb{Q}}_\ell(G^F \times T^F)$-module $\quad$ "ℓ-adic cohom. wit compact support"

Then for $\theta \in \mathrm{Irr}(T^F)$ obtain

$$R_T^G(\theta): G^F \longrightarrow \mathbb{C}$$

$$g \longmapsto \frac{1}{|T^F|} \sum_{t \in T^F} \Big(\underbrace{\sum_i (-1)^i \,\mathrm{Tr}\left((g,h), H_c^i(\mathcal{L}_F^{-1}(U), \bar{\mathbb{Q}}_\ell)\right)}_{\text{Lefschetz numbers in } \mathbb{Z}.} \Big) \cdot \theta(t)$$

is a virtual character of G^F