Lecture 9: Grothendieck's Thm

"algebraic dR Coh = analytic dR Coh"

X finite type $\mathbb{C}$-scheme

$\rightarrow \quad (X^{an}, \mathcal{O}_{X^{an}})$ analytic space

with $X^{an} = X(\mathbb{C})$ (as set)

<u>top</u> via $\mathcal{U} \subset X$ open with $\mathcal{U} \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ closed.

$$\rightsquigarrow \underset{\{f_1 = \ldots = f_r = 0\}}{\overset{}{\mathcal{U}(\mathbb{C})}} \hookrightarrow \underset{\uparrow \text{ analytic top}}{\mathbb{C}^n} \quad \text{induced top}$$

$\rightsquigarrow$ well defined top on $X(\mathbb{C})$

<u>sheaf $\mathcal{O}_{X^{an}}$</u> s.t. $\mathcal{O}_{X^{an}}|\mathcal{U}(\mathbb{C}) = \dfrac{\mathcal{O}^{an}_{\mathbb{C}^n}}{(f_1, \ldots, f_r)}$

$\rightarrow$ have $\varepsilon : X^{an} \longrightarrow X$ cts

and $\mathcal{O}_X \longrightarrow \varepsilon_* \mathcal{O}_{X^{an}}$

$\rightarrow \varepsilon : (X^{an}, \mathcal{O}_{X^{an}}) \rightarrow (X, \mathcal{O}_X)$

map of ringed spaces

<u>Universal Property</u>:

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, dotted, "\exists!"'] & X \\ X^{an} \arrow[ur] & \end{tikzcd}$$

(ana sp) (morph of loc ringed sp / $\mathbb{C}$)

<u>Rmk</u>: X sm/$\mathbb{C}$ $\iff$ X^{an} mfd

- $\mathcal{F}$ $\quad \mathcal{O}_X$-mod

define $\mathcal{F}^{an} = \varepsilon^* \mathcal{F} = \varepsilon^{-1}\mathcal{F} \otimes_{\varepsilon^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}}$

$\rightarrow$ get nat map $\mathcal{F} \rightarrow \varepsilon_* \mathcal{F}^{an}$

$\rightarrow \quad \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X^{an}, \mathcal{F}^{an}) \quad (*)$

$\rightarrow \quad H^j(X, \mathcal{F}) \longrightarrow H^j(X^{an}, \mathcal{F}^{an})$

(apply $(*)$ to an inj. resolution)

<u>Thm</u> (GAGA, Serre 1953)

X <u>proj</u> $\mathbb{C}$-scheme, $\mathcal{F}$ quasi-coh $\mathcal{O}_X$-mod.

$\Rightarrow \quad H^j(X, \mathcal{F}) \cong H^j(X^{an}, \mathcal{F}^{an}) \quad \forall j$

<u>Pf</u>: 1) <u>wlog $X = \mathbb{P}^r_{\mathbb{C}}$</u> : $\quad X \overset{i}{\hookrightarrow} \mathbb{P}^r_{\mathbb{C}} \quad H^j(X, \mathcal{F}) = H^j(\mathbb{P}^r_{\mathbb{C}}, i_*\mathcal{F})$
same with "an"

2) <u>wlg $\mathcal{F}$ coh</u> : $\quad \mathcal{F} = \underset{\nearrow}{\operatorname{colim}}\, \underset{\substack{\uparrow \\ \text{coh sub sheaves}}}{\mathcal{F}_\nearrow}$ and "an" and "H^j" commute with colim

3) <u>Thm OK for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}$</u> : Čech computation
(both sides vanish for $j \neq 0$)

4) <u>Thm OK for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}(n)$, all n:</u>

Let $H \subset \mathbb{P}^r_{\mathbb{C}}$ hyperplane

$\Rightarrow$ ex seq $0 \to \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}(n-1) \to \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}(n) \to \mathcal{O}_H(n) \to 0$

also for "an"

by l.e.s + Ind/r, n (OK for $n=0$) + 5-lemma $\Rightarrow$ OK

5) $\mathcal{F}$ coh on $\mathbb{P}^r_{\mathbb{C}}$

Have $\mathcal{F} \otimes \mathcal{O}(n)$ gen by global sections for $n \gg 0$

$\Rightarrow$ s.e.s

$$0 \to K \to \underbrace{\bigoplus_{i=1}^{m} \mathcal{O}(-n)}_{= E} \to \mathcal{F} \to 0$$

$\to$ l.e.s

$$\begin{tikzcd}
H^j(X,K) \arrow[r] \arrow[d, "\varepsilon_1"] & H^j(X,E) \arrow[r] \arrow[d, "\varepsilon_2"] & H^j(X,\mathcal{F}) \arrow[r] \arrow[d, "\varepsilon_3"] & H^{j+1}(X,K) \arrow[r] \arrow[d, "\varepsilon_4"] & H^{j+1}(X,\mathcal{F}) \arrow[d, "\varepsilon_5"] \\
H^j(X^{an},K^{an}) \arrow[r] & H^j(X^{an},E^{an}) \arrow[r] & H^j(X^{an},\mathcal{F}^{an}) \arrow[r] & H^{j+1}(X^{an},K^{an}) \arrow[r] & H^{j+1}(X^{an},\mathcal{F}^{an})
\end{tikzcd}$$

$\left.\begin{array}{l}\text{descending ind}/j \Rightarrow \varepsilon_4, \varepsilon_5 \text{ isom.} \\ 4) \Rightarrow \varepsilon_2 \text{ isom}\end{array}\right\} \Rightarrow \varepsilon_3$ surj $\forall \mathcal{F}$

$\Rightarrow \varepsilon_1$ is surj $\Rightarrow \varepsilon_3$ inj

$\square$

Cor: X sm proj $/\mathbb{C}$

$$\Rightarrow \quad H^j_{dR}(X/\mathbb{C}) = H^j_{dR}(X^{an}/\mathbb{C})$$

⌜ Recall: $H^j_{dR}(X/\mathbb{C}) = H^j(X, \Omega^\bullet_{X/\mathbb{C}}) = H^j(\Gamma(X, K^\bullet))$ ⌝

$$\Omega^\bullet_{X/\mathbb{C}} \xrightarrow{\simeq} K \quad K\text{-inj}$$

$$H^j_{dR}(X^{an}/\mathbb{C}) = H^j(X^{an}, \Omega^\bullet_{X^{an}/\mathbb{C}}) \underset{P.L}{=} H^j(X, \mathbb{C})$$ ⌟

⌞

Pf: Have nat map of cx's

$$\Omega^\bullet_{X/\mathbb{C}} \longrightarrow \varepsilon_* \Omega^\bullet_{X^{an}/\mathbb{C}} \quad \text{comp with stupid filtr.}$$

$\Rightarrow$ get map of s.s

$$\begin{array}{ccc} E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) & \Longrightarrow & H^{p+q}_{dR}(X/\mathbb{C}) \\ \downarrow \cong \text{GAGA} & \Longrightarrow & \cong \downarrow \\ E_1^{p,q} = H^q(X^{an}, \Omega^p_{X^{an}/\mathbb{C}}) & \Longrightarrow & H^{p+q}_{dR}(X^{an}/\mathbb{C}) \end{array}$$

(Note $\Omega^p_{X^{an}/\mathbb{C}} = (\Omega^p_{X/\mathbb{C}})^{an}$)

□

Cor: K field of char 0, X sm proj $/K$

Then

$$E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}(X, \Omega^\bullet_{X/K})$$

degenerates, i.e.,

for $F^p := \operatorname{Im}\left(H^n(X, \Omega^{\geq p}_{X/K}) \longrightarrow H^n(X, \Omega^\bullet_{X/K})\right)$

we have

$$F^p/F^{p+1} = H^{n-p}(X, \Omega^p_{X/K}) \qquad \forall\, p, n$$

equivalently

$$\dim_K H^n_{dR}(X/K) = \sum_{p+q=n} h^{p,q}(X)$$

where $h^{p,q}(X) = \dim_K H^q(X, \Omega^p_{X/K})$

Furthermore $h^{p,q}(X) = h^{q,p}(X)$

Pf: base change to $\mathbb{C}$ + GAGA + Hodge theory $\square$

Note the statements above are completely algebraic but the proof is analytic (Hodge Theory)

Moreover the statements do not hold in general in pos characteristic ($\to$ January)

Thm (Grothendieck 1963 / Deligne 1970 for reg sing case)

X sm / $\mathbb{C}$ (not nec proj or proper)

$$\Rightarrow \quad H^j_{dR}(X/\mathbb{C}) = H^j_{dR}(X^{an}/\mathbb{C}) = H^j(X^{an}, \mathbb{C})$$

(pf see below)

$\Rightarrow$ last time

Cor: X sm affine / $\mathbb{C}$

$$\Rightarrow \quad H^j(X^{an}, \mathbb{C}) = \frac{\ker(d : \Omega^j(X) \longrightarrow \Omega^{j+1}(X))}{d(\Omega^{j-1}(X))}$$

Ex: $f \in \mathbb{C}[x,y]$

$$\Rightarrow \quad H^1(\{f=0\}^{an}, \mathbb{C}) = \frac{\Omega^1_{\mathbb{C}[x,y]}}{f\,\Omega^1_{\mathbb{C}[x,y]} + \mathbb{C}[x,y]\,df + d(\mathbb{C}[x,y])}$$

For the pf of the Thm

- X sm $/ \mathbb{C}$ $\quad D = \sum_{i=1}^{r} D_i \subset X$ red eff div

D is SNCD $:\Leftrightarrow$ $\bigcap_{i \in I} D_i$ sm of codim $|I|$

(simple normal crossing divisor) $\quad \forall \; I \subset \{1 \ldots r\}$

it locally $\quad X = \operatorname{Spec} \mathbb{C}[t_1, \ldots, t_n]$

$$D = V(t_1 \cdots t_r)$$

Set $j : U = X \setminus D \hookrightarrow X$

define:

$\Omega^1_X(\log D) \subset j_* \Omega^1_U$ locally by

$$\Omega^1_X(\log D) = \bigoplus_{i=1}^{r} \mathcal{O}_X \, d\log t_i \oplus \bigoplus_{i=r+1}^{n} \mathcal{O}_X \, dt_i$$

$\Rightarrow$

$\Omega^q_X(\log D) = \Lambda^q \Omega^1_X(\log D)$ free $\mathcal{O}_X$-mod with basis $\quad d\log t_I \wedge dt_J$

$$I = (1 \le i_1 < \ldots < i_a \le r) \qquad a+b=q$$

$$J = (r+1 \le j_1 < \ldots < j_b \le n)$$

subcx $\quad \Omega^\bullet_X(\log D) \subset j_* \Omega^\bullet_U \quad ,$ same for "an"

Thm
$$H^n(X^{an}, \Omega^\bullet_{X^{an}}(\log D)) \xrightarrow{\sim} H^n(U^{an}, \Omega^\bullet_{U^{an}})$$

(used to put mixed HS on $H^n(U^{an}, \mathbb{C})$, see Matthias)

Also true for "alg" instead of "an"

pf: Have $j : X \setminus D \hookrightarrow X$ is affine $(\Rightarrow j_* = Rj_*)$

→ to show $\Omega^\bullet_{X^{an}}(\log D) \xhookrightarrow{i} j_* \Omega^\bullet_{U^{an}}$ qis

in dim = 1

$$\begin{tikzcd} \mathbb{C}\{z\}\, d\log z \arrow[r, shift left, "i"] & \mathbb{C}\{z, 1/z\}\, d\log z \arrow[l, shift left, "r"] \arrow[d, bend left, "H"] \\ \mathbb{C}\{z\} \arrow[u, "d"] \arrow[r, shift left, "i"] & \mathbb{C}\{z, 1/z\} \arrow[u, "d"] \arrow[l, shift left, "r"] \end{tikzcd}$$

cvgt power series in nbhd of 0

cvgt Laurentseries on punctured disc

$r =$ "cut poles" $\Rightarrow r \circ i = id$

and $id - i \circ r = d \circ H + H \circ d$

where

H in degree 1 $\quad H\left(\sum_{n \in \mathbb{Z}} a_n z^n \, d\log z\right) = \sum_{n<0} \frac{1}{n} a_n z^n$

else $H = 0$

$f = \sum_n a_n z^n \Rightarrow f - i \circ r(f) = \sum_{n<0} a_n z^n = Hd(f)$

$f \, d\log z \Rightarrow f \, d\log z - i \circ r(f \, d\log z) = \sum_{n<0} a_n z^n d\log z = dH(f)$ $\square$

Pf of Grothendieck's Thm

<u>To show:</u> the nat map

$$(*) \quad H^i_{dR}(X/\mathbb{C}) \longrightarrow H^i_{dR}(X^{an}/\mathbb{C}) \quad \text{is an isom.}$$

Have map between conjugate sp. seq^les

$$\begin{array}{ccc} E_2^{p,q} = H^p(X, \mathcal{H}^q(\Omega^\bullet_{X/\mathbb{C}})) & \Longrightarrow & H^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \\ \downarrow & & \downarrow \\ E_2^{p,q} = H^p(X^{an}, \mathcal{H}^q(\Omega^\bullet_{X^{an}/\mathbb{C}})) & \Longrightarrow & H^{p+q}(X^{an}, \Omega^\bullet_{X^{an}/\mathbb{C}}) \end{array}$$

$\quad$ sheaf assoc to $\underset{X^{an}}{U \subset} \mapsto H^q(U, \Omega^\bullet_{X^{an}/\mathbb{C}})$

$\leadsto$ suff to show $(*)$ is an isom for X affine

$\underset{\text{Hironaka}}{\Rightarrow}$ $\exists$ $j: X \hookrightarrow \overline{X}$ with $\overline{X}$ sm, proj

and $\overline{X} \setminus X = D$ SNCD

$\to$ Have

$$\begin{tikzcd}
H^n(\bar{X}, j_* \Omega^\bullet_{X/\mathbb{C}}) \arrow[rr, "j \text{ affine}", "\cong"'] & & H^n(X, \Omega^\bullet_{X/\mathbb{C}}) \arrow[ddd, "\Rightarrow\ \cong"'] \\
H^n(\bar{X}, \Omega^\bullet_{\bar{X}}(\log D)) \arrow[u, "\text{Term}", "\cong"'] \arrow[d, "\text{GAGA}"', "\cong"] & & \\
H^n(\bar{X}^{an}, \Omega^\bullet_{\bar{X}^{an}}(\log D)) \arrow[d, "\cong"', "\text{Term}"] & & \\
H^n(\bar{X}^{an}, j_* \Omega^\bullet_{X^{an}/\mathbb{C}}) \arrow[rr, "\cong", "j \text{ affine}"'] & & H^n(X^{an}, \Omega^\bullet_{X^{an}/\mathbb{C}})
\end{tikzcd}$$

$\square$