Lecture 8 : Algebraic de Rham cohomology

de Rham cx on schemes

k = field

X sep finite type k-scheme , $\mathcal{O}_X$ = str. sheaf.

$$\Rightarrow \quad X = \bigcup_{i=1}^{r} \mathcal{U}_i \quad , \quad \mathcal{U}_i = \operatorname{Spec} A_i$$

$$A_i = \frac{k[t_1, \dots, t_n]}{I_i} = \mathcal{O}_X(\mathcal{U}_i)$$

Kähler differentials

$A = k$-alg , $M = A$-mod.

$\mathrm{Der}_k(A, M)$ = <u>derivations</u> $\partial : A \longrightarrow M$

i.e. ∂ is k-liner

- $\partial(ab) = a\,\partial(b) + b\,\partial(a)$ (Leibniz rule)
- $\partial(\lambda) = 0 \quad \forall \lambda \in k$

<u>Prop.</u> $\exists$ A-module $\Omega^1_{A/k}$ s.t.

$$\mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/k}, M) = \mathrm{Der}_k(A, M)$$

concretely:

$$\Omega^1_{A/R} = I/I^2 \qquad I = \ker(A \otimes_R A \to A)$$
$$a \otimes b \mapsto ab$$

A-module via $\underset{\in I}{a \cdot \beta} = (1\otimes a) \cdot \beta \equiv (a\otimes 1)\cdot \beta \mod I^2$

universal derivation: $d : A \to \Omega^1_{A/R}$
$$a \mapsto 1\otimes a - a \otimes 1 \mod I^2$$

[note $d(ab) = 1\otimes ab - ab\otimes 1 = 1\otimes ab - a\otimes b + a\otimes b - ab\otimes 1$
$= b\, d(a) + a\, d(b)$]

⇝ any Elt in $\Omega^1_{A/R}$ is a finite sum

$$\sum a_i \, db_i \quad , \; a_i, b_i \in A$$

and
$$\mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/R}, M) \longrightarrow \mathrm{Der}(A, M) \quad \text{bij.}$$
$$\varphi \longmapsto \varphi \circ d$$

For X a sep fin-type R-scheme

Def: $$\Omega^1_{X/R} := \Delta^{-1}\left(J/J^2\right)$$

where $\Delta : X \hookrightarrow X \times X$ and $J = \ker(\mathcal{O}_{X\times_R X} \to \mathcal{O}_X)$

$\rightsquigarrow$

$$\Gamma(U, \Omega^1_{X/k}) = \Omega^1_{A/k} \quad , \quad \begin{matrix} U = \operatorname{Spec} A \subset X \\ \text{open.} \end{matrix}$$

Note

$$\mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X) = \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X) =: \mathcal{T}_X$$

tangent sheaf.

Ex

$$\Omega^1_{k[x_1 \ldots x_n]/k} = \bigoplus_{i=1}^{n} k[x_1 \ldots x_n]\, dx_i$$

$$\Omega^1_{\mathbb{A}^n_k/k} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{A}^n_k}\, dx_i$$

Prop: X sep fin type $/k$

(1) $\Omega^1_{X/k}$ coh $\mathcal{O}_X$-mod.

(2) $f: X \to Y\ /k \Rightarrow f^*: \Omega^1_{Y/k} \to f_* \Omega^1_{X/k}$

$$a\, db \mapsto f^*(a)\, d f^*(b)$$

(3) $U \subset X$ open with $U \xrightarrow{u} \mathbb{A}^n_k$ étale

(chart)

$$\Rightarrow \Omega^1_{X/k}|U = \Omega^1_{U/k} = u^* \Omega^1_{\mathbb{A}^n_k/k} = \bigoplus_{i=1}^{n} \mathcal{O}_U\, dx_i$$

$$\Rightarrow \Omega^1_{X/k}|U \text{ free of rk } n = \dim U$$

(4) X smooth $/k$ ($\Leftrightarrow$ nonsingular if $k=\bar{k}$)
conn.

$\Rightarrow$ $X = \bigcup_i U_i$ open cov with $U_i \to \mathbb{A}^n_k$ ét.

$\Rightarrow$ $\Omega^1_{X/k}$ locally free $\mathcal{O}_X$-mod of rk n $= \dim X$

($\Leftrightarrow$ v.b)

and $\mathcal{T}_X = \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)$ sheaf of sections of the tangent bdl T_X

[Have $T_X = \underline{\mathrm{Spec}}(\mathrm{Sym}^\bullet \Omega^1_{X/k})$]

(5) $Y \overset{i}{\hookrightarrow} X$ closed im.

$\mathcal{J} = \mathrm{Ker}(\mathcal{O}_X \to i_* \mathcal{O}_Y)$

$\Rightarrow$ ex sequence

$$\mathcal{J}/\mathcal{J}^2 \to i^* \Omega^1_{X/k} \to \Omega^1_{Y/k} \to 0$$
$$\bar{a} \mapsto d\bar{a}$$

in part.
$$A = \frac{\mathbb{C}[x_1 .. x_n]}{(f_1, \dots, f_r)}$$

$$\Rightarrow \quad \frac{(f_1 \dots f_r)}{(f_1 \dots f_r)^2} \to A \otimes_{\mathbb{C}[x_1..x_n]} \Omega^1_{\mathbb{C}[x_1..x_n]/\mathbb{C}} \to \Omega^1_{A/\mathbb{C}} \to 0$$

$$A \otimes_{\mathbb{C}[x_1..x_n]} \Omega^1_{\mathbb{C}[x_1..x_n]/\mathbb{C}} = \bigoplus_{i=1}^n A\, dx_i$$

$$\bar{f}_j \mapsto \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i = df_j$$

$$\Rightarrow \quad \Omega^1_{A/\mathbb{C}} = \frac{\bigoplus_{i=1}^{n} A\,dx_i}{\left\{ (a_1, \dots, a_r)\, J \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} \;\middle|\; a_i \in A \right\}}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

$\rightsquigarrow$ if A is smooth J has in every pt max'l rk

$\Rightarrow$ $\Omega^1_{A/\mathbb{C}} \otimes_A A/m$ has the same v.sp dim $\forall$ max'l ideal m.

$\Rightarrow$ $\Omega^1_{A/\mathbb{C}}$ is locally free (i.e. fin. proj)

Ex: $\quad A = \dfrac{k[X, Y, Z]}{(X^p + Y^p + Z^p)}$

$k = \mathbb{C}$: $\quad \Omega^1_{A/\mathbb{C}} = \dfrac{A\,dx \oplus A\,dy \oplus A\,dz}{\langle p X^{p-1} dx + p Y^{p-1} dy + p Z^{p-1} dz \rangle}$ $\to$ free of rk 2 on $A[1/X]$, $A[1/Y]$, $A[1/Z]$

but not free at 0

and $\dim_{\mathbb{C}} \Omega^1_{A/\mathbb{C}} \otimes_A \dfrac{\mathbb{C}[X,Y,Z]}{(X,Y,Z)} = 3$

$k = \mathbb{F}_p$: $\quad \Omega^1_{A/\mathbb{F}_p} = A\,dx \oplus A\,dy \oplus A\,dz$: free of rk $3 > \dim A$

$\to$ not smooth.

Def: A <u>differential graded R-alg</u> (R-dga)

is a graded assoc. R-alg.

$B = \bigoplus_{n \geq 0} B^n$ together with a R-linear map.

$d : B \longrightarrow B$

s.t. (1) $d(B^n) \subset B^{n+1}$

(2) $d(ab) = d(a)\, b + (-1)^n a\, d(b)$, $a \in B^n$, $b \in B^r$

(3) $d \circ d = 0$

(4) $ab = (-1)^{nr} ba$ $\quad a \in B^n, b \in B^r$

and $a^2 = 0$ if $a \in B^n$ with n odd.

if B is a dga $\rightsquigarrow$ get cx

$B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2 \longrightarrow \cdots$

Constr.: $A = R\text{-alg.}$

$$\Omega^0_{A/R} := A \quad , \quad \Omega^n_{A/R} = \Lambda^n \Omega^1_{A/R} \quad , \quad n \geq 1$$

Then $\exists!$ map

$$d^n : \Omega^n_{A/R} \longrightarrow \Omega^{n+1}_{A/R}$$

s.t. $d^n(a_0\, da_1 \wedge \dots \wedge da_n) = (da_0) \wedge (da_1) \wedge \dots \wedge (da_n)$

to check welldefinedness is some work

but if A étale $/ R[t_1, \dots t_e]$

$$\Rightarrow \quad \Omega^n_{A/R} = \bigoplus_{1 \leq i_1 < \dots < i_n \leq e} A\, dt_{i_1} \wedge \dots \wedge dt_{i_n}$$

define $d^n(a\, dt_{i_1} \wedge \dots \wedge dt_{i_n}) := (da) \wedge dt_{i_1} \wedge \dots \wedge dt_{i_n}$

$\rightarrow$ OK.

$\rightsquigarrow$

$$\Omega^*_{A/R} := \Big(\bigoplus_{n \geq 0} \Omega^n_{A/R} \, , d \Big) \quad \text{is a } R\text{-dga}$$

Prop. let $(B = \bigoplus_{n \geq 0} B^n, \partial)$ be a k-dga

and assume $\exists\ \lambda : A \to B^0$ ring map

$\Rightarrow \exists!\ \lambda^* : \Omega^*_{A/k} \longrightarrow B$ map of dga's

s.t. $\lambda^0 = \lambda : A \to B^0$

Pf: $A \longrightarrow B^0 \xrightarrow{\partial} B^1$ is a k-derivation.

$$\rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "d"'] & B^0 \arrow[d, "\partial"] \\ \Omega^1_{A/k} \arrow[r, dashed, "\lambda^1"] & B^1 \end{tikzcd}$$

$$\rightsquigarrow \quad \underbrace{\Omega^1_{A/k} \otimes \cdots \otimes \Omega^1_{A/k}}_{n\text{-linear}} \longrightarrow B^n$$

$$\alpha_1 \otimes \cdots \otimes \alpha_n \longmapsto \lambda(\alpha_1) \cdots \lambda(\alpha_n) \qquad = 0 \text{ if } \alpha_i = \alpha_j \text{ for } i \neq j$$

$\to$ factors via $\Omega^n_{A/k} \longrightarrow B^n$

$\rightsquigarrow$ get map $\Omega^*_{A/k} \longrightarrow B$ compatible with d and ∂

$\square$

Note if X sm$/k$ of dim n

$\Rightarrow \Omega^1_{X/k}$ loc free of rk n

$\Rightarrow \Omega^N_{X/k} = 0 \quad \forall \quad N > n$

$\rightarrow$ get alg de Rham cx

$$\Omega^\bullet_{X/k} : 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{X/k} \to 0$$

$\rightsquigarrow$

Def: alg de Rham coh:

$$H^i_{dR}(X/k) = H^i(X, \Omega^\bullet_{X/k}) = H^i(\Gamma(X, I^\bullet))$$

$$[\Omega^\bullet_{X/k} \Rightarrow I^\bullet \text{ k.inj}]$$

Prop: X affine $/k$

$$\Rightarrow \quad H^i_{dR}(X/k) = H^i(\Gamma(X, \Omega^\bullet_{X/k}))$$

Pf: $E_1^{p,q} = H^p(X, \Omega^q_{X/k}) \Rightarrow H^i_{dR}(X/k)$

(Matthias)

$\Rightarrow E_1^{p,q} = 0, \quad p \neq 0$

and cx $\underbrace{E_1^{0,0} \xrightarrow{d} E_1^{0,1} \xrightarrow{d} E_1^{0,2} \to \dots}_{= \Gamma(X, \Omega^\bullet_{X/k})}$ $\quad \Rightarrow E_\infty^{p,q} = \begin{cases} E_2^{0,q}, & p = 0 \\ 0, & \text{else} \end{cases}$

$\square$

Ex:

$$H^i_{dR}(\mathbb{A}^1_R/R) = \begin{cases} \operatorname{Ker}(d : R[t] \to R[t]dt), & i = 0 \\ \dfrac{R[t]dt}{d(R[t])}, & i = 1 \\ 0, & \text{else} \end{cases}$$

$$\underset{\text{if } \operatorname{char}(R) = 0}{=} \begin{cases} R & i = 0 \\ 0 & \text{else} \end{cases}$$

$$\underset{\text{if } \operatorname{char}(R) = p > 0}{=} \begin{cases} R[t^p], & i = 0 \\ \bigoplus_{n \geq 1} R\, t^{pn-1} dt, & i = 1 \\ 0, & \text{else} \end{cases}$$

Assume $R = \mathbb{C}$ $\quad X$ sm$/\mathbb{C}$ $\quad \rightsquigarrow X(\mathbb{C})^{an}$ cx mfd

$\rightsquigarrow$ What is the relation

between $H^i_{dR}(X/\mathbb{C})$ and $H^i(X(\mathbb{C})^{an}, \mathbb{C})$?

$\rightarrow$ next time