Lecture 7 mixed Hodge structures

goal today: (mixed) Hodge structures are additional structures on cohomology of algebraic varieties (axiomatizes & generalizes Hodge decomposition)

discuss definitions, examples & links/applications to geometry

Def pure Hodge structure of weight n

abelian grp $H_{\mathbb{Z}}$ & decomposition $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{p+q=n} H^{p,q}$

s.th. $H^{p,q} = \overline{H^{q,p}}$.

Rem: alternative def: finite decreasing filtration F^pH on $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$

s.th. $\forall\ p+q = n+1$ $\quad F^p \cap \overline{F^q H} = 0$ & $F^pH \oplus \overline{F^qH} = H$.

translation: $H^{p,q} = F^pH \cap \overline{F^qH}$ / $F^pH = \bigoplus_{i \geq p} H^{i,n-i}$

Rem: $H^{p,q}$ are cplx subspaces, easiest example w/ decomp. $H_{\mathbb{Z}} = \mathbb{Z}^2$

$\mathbb{Z}^2 \otimes \mathbb{C} = \mathbb{C}^2 \simeq H^{1,0} \oplus H^{0,1}$

$H^{1,0} = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle \quad H^{0,1} = \left\langle \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\rangle$

in particular, decomposition is not defined over $\mathbb{Z}$ or even $\mathbb{R}$.

examples / geometric origin: M cpt. Kähler

$\rightsquigarrow$ pure Hodge structure of wt n $\quad H_{\mathbb{Z}} = H^n(M, \mathbb{Z})$, wt $H^{p,q} = H^q(M, \Omega^p)$ from Hodge decomposition

Hodge filtration from Frölicher spectral seq

Tate Hodge structure

$\mathbb{Z}(1)$: $\quad H_{\mathbb{Z}} = 2\pi i \mathbb{Z} \leq H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C} = H^{-1,-1}$ $\qquad$ pure of wt -2

$\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ $\quad$ wt $-2n$. $\qquad H^*(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Z}(-n)$.

$2\pi i$ factors make $\frac{1}{(2\pi i)^d}\int_M$ independent of choice of orientation

Def: polarization / polarized Hodge structure (of wt n)

pure Hodge structure $H_{\mathbb{Z}}$, $H^{p,q}$ of wt n

+ non-deg. bilinear form $Q: H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$

s.th.
1. $Q(\varphi, \psi) = (-1)^n Q(\psi, \varphi)$
2. $Q(\varphi, \psi) = 0$ for $\varphi \in H^{p,q}$, $\psi \in H^{p',q'}$, $p \neq q'$
3. $i^{p-q} Q(\varphi, \overline{\varphi}) > 0$ for $0 \neq \varphi \in H^{p,q}$

Hodge-Riemann bilinear relations

examples / geometric origin: M cpt Kähler w/ Kähler form ω

e.g. $X(\mathbb{C})$ for $X \subseteq \mathbb{P}^n$ smooth proj. var. / $\mathbb{C}$
ω dual to class of hyperplane section.

$$\boxed{Q(\varphi, \psi) = \int_M \varphi \wedge \psi \wedge \omega^k \quad \text{on } H^{n-k}}$$

written for forms, but works for integer cohom.: $\omega \in H^2(M, \mathbb{Z})$!

idea: polarization $\hat{=}$ projective embedding of cpt mfd.

Periods & moduli spaces (side note)

period domains = parameter spaces for polarized Hodge structures

e.g.: $\mathbb{H} = SL_2\mathbb{R} / SO(2)$ as parameter space for polarized Hodge structure on H^1 of elliptic curve.

in this case, get analytic model of moduli space as quotient by autom. gp $SL_2\mathbb{Z}$ changing polarization

[Similar picture for K3 surfaces]

(related stuff on (intermediate) jacobians in a later lecture)

Def mixed Hodge structures

triple $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$

- $H_{\mathbb{Z}}$ f.g. ab. gp
- $W_\bullet$ increasing $\mathbb{Z}$-filtration on $H_{\mathbb{Z}} \otimes \mathbb{Q}$ (weight filtration)
- $F^\bullet$ decreasing $\mathbb{N}$-filtration on $H_{\mathbb{Z}} \otimes \mathbb{C}$ (Hodge filtration)

s.th.
$$gr_k^W H_{\mathbb{Q}} = \frac{W_k H_{\mathbb{Q}}}{W_{k-1} H_{\mathbb{Q}}} \quad \text{w/ induced } F^\bullet \text{ filtration}$$
is pure Hodge structure of wt k.

Idea: mix together pure HS of different weights, e.g. arising from extensions in long exact sequences.

Rem: morphisms of MHS preserve filtrations $f(W_k) \subseteq W_k'$ & $f(F^p) \subseteq F^{p'}$

$\rightsquigarrow$ abelian category of homological dim 1.

Example:
how to get mixed Hodge structures (following Deligne, Hodge II & III)

M cpt mfd, $D \subseteq M$ normal crossing div.

Logarithmic p-form ω on $X \setminus D$:
ω & $d\omega$ have pole order ≤ 1 along D

Expl: $d(\log z) = \frac{dz}{z}$ on $\mathbb{C} \setminus \{0\}$.

$\rightsquigarrow$ sheaf $\Omega^p_M(\log D)$ of p-forms w/ log poles along D.

$\rightsquigarrow$ logarithmic de Rham cplx $\Omega^\bullet_M(\log D)$

Concretely: locally around $x \in M$, (M, D) looks like $(\mathbb{C}^n, V(z_1 \cdots z_k))$

$$\Omega^1_M(\log D) = \mathcal{O}_M \cdot \frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_M \cdot \frac{dz_k}{z_k} \oplus \mathcal{O}_M \cdot dz_{k+1} \oplus \cdots \oplus \mathcal{O}_M\, dz_n$$

$$\Omega^p_M(\log D) = \Lambda^p \Omega^1_M(\log D)$$

Logarithmic de Rham thm (Deligne) M cpt mfd, $D \subseteq M$ normal crossing div.

$$\Omega^\bullet_M(\log D) \xrightarrow{\simeq} j_* \Omega^\bullet_{M \setminus D} \quad \text{quasi-iso.}$$

see Kay's upcoming lectures $\rightsquigarrow$ $H^\bullet(M, \Omega^\bullet_M(\log D)) \cong H^\bullet_{sing}(M \setminus D, \mathbb{C})$

mixed Hodge structure on coh. $H^n(U, \mathbb{C})$:

choose compactification $U \subseteq M$ w/ D normal crossing s.th. $U = M \setminus D$.

induced by filtrations on $\Omega^\bullet_M(\log D)$:

Hodge filtration: $F^q \Omega^p_M(\log D) = \begin{cases} \Omega^p_M(\log D) & q \leq p \\ 0 & \text{otherwise} \end{cases}$

Weight filtration: $W_n \Omega^p_M(\log D) = \begin{cases} \Omega^p_M(\log D) & p \leq n \\ \Omega^{p-n}_M \wedge \Omega^n_M(\log D) & 0 \leq n \leq p \\ 0 & \text{otherwise} \end{cases}$

Rem: works more generally for arbitrary cplx varieties (Deligne Hodge III)

for singular: use resol. of sing. to replace proj. sing. var. X by simplicial variety $X_\bullet$ w/ X_n smooth, proj.

get MHS from filtrations of $X_\bullet$ ...