Lecture 6 Spectral sequences (general & Hodge 2 de Rham)

What we have so far: for M cplx mfd. M, sheaves Ω^p of holomorphic p-forms

cohomology $H^q(M, \Omega^p_M)$ can be understood via Hodge theory & harmonic forms
can be computed via Dolbeault cplx / resol.
$\Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \to \cdots$

cohomology $H^p(M, \mathbb{C}) \simeq H^p(M, \Omega^*_M)$ ← sheaf hyper cohomology via K-injective resol. double complex ...

singular cohom.

from analytic Poincaré lemma $\mathbb{C} \simeq \Omega^*$ quasi-iso.

goal for today:
Related via Frölicher / Hodge-2-deRham spectral seq. $\boxed{H^q(M, \Omega^p) \Rightarrow H^{p+q}(M, \Omega^*)}$

Hodge filtration — how to build Ω^*_M from Ω^p_M's. $n = \dim M$.

Ω^*_M has filtration by subcomplexes $\Omega^{\geq q}_M$: $\cdots \to 0 \to \Omega^q \xrightarrow{d} \Omega^{q+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$

descending filtration $\Omega_M = \Omega^{\geq 0}_M \supseteq \Omega^{\geq 1}_M \supseteq \cdots \Omega^{\geq n}_M = \Omega^n_M \supseteq \Omega^{\geq n+1}_M = 0$

induces filtration on cohomology

$$\boxed{F^q H^p(M, \Omega^*_M) = \mathrm{Im}\left(H^p(M, \Omega^{\geq q}_M) \to H^p(M, \Omega^*)\right)}$$

How does this relate $H^p(M, \Omega^*)$ & $H^p(M, \Omega^q)$?

for fixed q have exact seq. of cplxs $0 \to \Omega^{\geq q+1}_M \to \Omega^{\geq q}_M \to \Omega^q_M \to 0$

⇝ long exact coh. seq

$$\cdots \to H^p(M, \Omega^{\geq q+1}) \to H^p(M, \Omega^{\geq q}) \to H^p(M, \Omega^q) \xrightarrow{\partial} H^{p+1}(M, \Omega^{\geq q+1}) \to \cdots$$

long exact seq encodes how coh. of $\Omega^{\geq q+1}$ & $\Omega^{\geq q}$ differ in terms of coh. of Ω^q.

Expl: C cpt. cplx curve Ω^*: $0 \to \mathcal{O}_C \xrightarrow{d} \Omega^1_C \to 0$

$\Omega^1[-1]$: Ω^1 in coh deg 1.

$$\cdots \to H^p(C, \Omega^1[-1]) \to H^p(C, \Omega^*) \to H^p(C, \mathcal{O}) \to H^{p+1}(C, \Omega^1) \to \cdots$$

$$\rightsquigarrow \left| \begin{array}{ccc} H^1(C, \Omega^1) & \simeq & H^2(C, \mathbb{C}) \\ H^1(C, \mathcal{O}) \oplus H^0(C, \Omega^1) & \simeq & H^1(C, \mathbb{C}) \\ H^0(C, \mathcal{O}) & \simeq & H^0(C, \mathbb{C}) \end{array} \right|$$

$H^0(C, \mathcal{O}) = \mathbb{C}$, $H^1(C, \mathcal{O})^\vee \simeq H^0(C, \Omega^1)$

Now would like to iterate this procedure, understand inductively how $\Omega^\bullet$ is built from Ω^q.

spectral seq. for filtered complex is a way organize the many long exact sequences.

(see Eric Peterson's chromotopy.org blog on spectral seq 8)

set up: cplx C^* w/ descending filtration by subcomplexes F^qC^*

$$\begin{tikzcd}
\cdots \arrow[r, hook] & F^3C^* \arrow[r, hook] & F^2C^* \arrow[r, hook] \arrow[d] & F^1C^* \arrow[r, hook] \arrow[d] & F^0C^* = C^* \arrow[d] \\
--- & & F^2/F^3 & F^1/F^2 & F^0/F^1
\end{tikzcd}$$

← for us $\Omega^{\geq q}$

← quotient maps

← for us Ω^q

apply cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(F^2C^*) \arrow[r] \arrow[d, "\pi_2"] & H^*(F^1C^*) \arrow[r] \arrow[d, "\pi_1"] & H^*(F^0C^*) = H^*(C^*) \arrow[d, "\pi_0"] \\
 & H^*(F^2/F^3) \arrow[ul] & H^*(F^1/F^2) \arrow[ul, "\partial_1"'] & H^*(F^0/F^1) \arrow[ul, "\partial_0"']
\end{tikzcd}$$

boundary maps from long exact seq have degree 1: $H^p(F^q/F^{q+1}) \to H^{p+1}(F^{q+1})$

$$\boxed{E_1\text{-page}: \quad E_1^{p,q} = H^p(F^q/F^{q+1})}$$

First approximation to $H^*(C^*)$, but:

- could see artificial cycle in F^q/F^{q+1} because actually nonzero boundary in F^{q+1}
- could fail to see boundaries from other filtration levels

keep track of these problems by differentials

$$\boxed{d_1: E_1^{p,q} \longrightarrow E_1^{p+1,q+1} \qquad d_1 = \pi_{q+1} \circ \partial_q}$$

$$\begin{tikzcd}
\cdots & H^*(F^2/F^3) \arrow[l] & H^*(F^1/F^2) \arrow[l, "d_1^{*,1}"'] & H^*(F^0/F^1) \arrow[l, "d_1^{*,0}"']
\end{tikzcd}$$

$$d_1 \circ d_1 = \pi_{*+2} \circ \underbrace{\partial_* \circ \pi_{*+1}}_{=0} \circ \partial_*$$

⇝ can take cohomology

$$\boxed{E_2\text{-page} \qquad E_2^{p,q} = \frac{\ker\, d_1^{p,q}: E_1^{p,q} \to E_1^{p+1,q+1}}{\operatorname{Im}\, d_1^{p-1,q-1}: E_1^{p-1,q-1} \to E_1^{p,q}}}$$

differentials

$$d_2^{p,q}: E_2^{p,q} \longrightarrow E_2^{p+1,q+2}$$

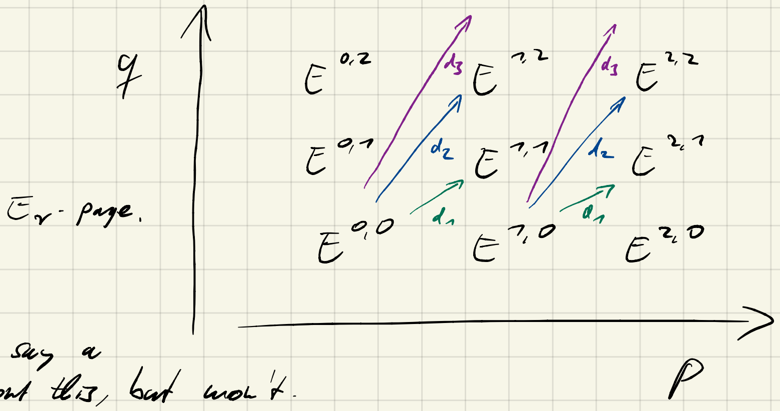
class from $E_2^{p,q}$ represented as class $x \in H^p(F^q/F^{q+1})$

$\underbrace{\partial_q(x) \in \ker \pi_{q+1}}_{d_1(x)=0}$ ⇝ $\partial_q(x) \in \operatorname{Im}\left(H^{p+1}(F^{q+2}) \to H^{p+1}(F^{q+1})\right)$, take π_{q+2} of preimage

Now iterate: E_{r+1}-page = cohomology of E_r-page

d_r-differentials $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+1, q+r}$.

Graphical representation



so it keeps going, then what?

Convergence issues could say a lot about this, but won't.

In simple cases – e.g. if only finitely many $H^p(F^q/F^{q+1})$ nonzero – there is an r s.th. all differentials on $E_{r'}$ for $r' > r$ are 0.

$\rightarrow E_{r'}^{p,q} = E_\infty^{p,q} = gr_F^q H^p := \frac{F^q H^p}{F^{q+1} H^p}$

$F^q H^p = \mathrm{Im}(H^p(F^q C^*) \to H^p(C^*))$

Then still have extension problems recovering $H^p(C^*)$ from graded pieces of $F^* H^p(C^*)$

Hodge-to-deRham / Frölicher spectral seq. apply above to Hodge filtration $F^* \Omega^*$

$\Rightarrow$ spectral seq $\boxed{E_1^{p,q} = H^p(M, \Omega^q) \Rightarrow H^{p+q}(M, \Omega^*)}$

w/ differentials $d_r^{p,q} : E_r^{p,q} \longrightarrow E^{p+1-r, q+r}$

About indexing: previous indices for $E_1^{p,q} = H^p(\Omega^{\geq q}/\Omega^{\geq q+1}) = H^p(\Omega^q[-q])$

but typical indexing for $E_1^{p,q} = H^p(\Omega^q) = H^{p+q}(\Omega^q[q])$

$\rightsquigarrow$ shift in coh. degree p.

in E_r page picture shift qth row left q steps.

degeneration

For M cpt. Kähler, harmonic forms α satisfy $d\alpha = 0$

all $d_r^{p,q} = 0$. spectral sequence degenerates at E_1-page

$$H^n(M, \Omega^*) \cong \bigoplus_{p+q=n} E_1^{p,q} = \bigoplus_{p+q=n} H^p(M, \Omega^q)$$

no extension problems vector spaces!

Rem: not true generally for cpt cplx mfd.

$\forall n \geq 2$ $\exists$ nilmanifolds w/ nontriv. d_n-differential (Rollenske)