

Lecture 5

harmonic forms & Hodge decomposition

Recollection from Kay's lecture

geometric objects where complex analysis works! \rightarrow cplx mfd: locally like open $U \subseteq \mathbb{C}^n$ + holomorphic transition maps
 e.g. $\mathbb{C}P^n$, Riemann surface Σ , ...

sheaves \mathcal{O}^p of holomorphic p -forms

sheaves $\mathcal{A}^n = \bigoplus_{p+q=n} \mathcal{O}^{p,q}$ of cplx-valued smooth forms

$d: \mathcal{A}^p \rightarrow \mathcal{A}^{p+1}$ decomposes into $\partial: \mathcal{O}^{p,q} \rightarrow \mathcal{O}^{p+1,q}$ & $\bar{\partial}: \mathcal{O}^{p,q} \rightarrow \mathcal{O}^{p,q+1}$
 locally $\omega = \sum a_{i_1, \dots, i_p} dz^{i_1} \wedge \dots \wedge dz^{i_p}$

\rightarrow Dolbeault cplx as resolution

$$\mathcal{O}^p \rightarrow \mathcal{O}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{O}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{O}^{p,2} \rightarrow \dots$$

Dolbeault cohomology

$$H^q(M, \mathcal{O}^p) = H^q_{\bar{\partial}}(M) = H^q(\mathcal{O}^{p,*})$$

from Dolbeault Lemma.

goal for next 3 lectures: Hodge theory \rightarrow representing cohomology by harmonic forms
 Hodge decomposition $H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$ for cplx Kähler

more general: Hodge 2-dim spectral seq. $H^p(X, \mathcal{O}^q) \Rightarrow H^{p+q}(X, \mathbb{R})$
 & mixed Hodge structures

Griffiths & Harris principles of alg geom.

Hermitian metrics, Laplace operator & harmonic forms

cplx mfd $M \rightarrow$ tangent bundle $TM \otimes \mathbb{C} \cong T\mathbb{R}^{2n} \oplus T\mathbb{R}^{2n}$
 for cplx structure \downarrow $\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \otimes \mathbb{C}$

Hermitian metric h : reduction of structure gp of $T\mathbb{R}^{2n}$ to $U(n) \subseteq GL_n(\mathbb{C})$.

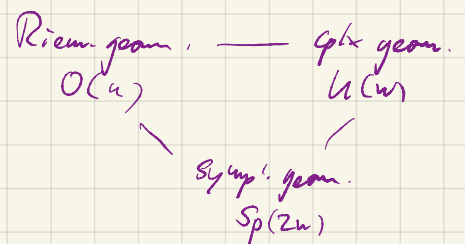
holom. tangent bundle \rightarrow $v \otimes \bar{u} \rightarrow \mathbb{C}$
 $h(z, u) = \overline{h(u, z)}$

($\hat{=}$ holomorphic tangent spaces equipped w/ pos. def. Hermitian inner product)

real part $g = \frac{1}{2}(h + \bar{h})$ - Riemannian metric, -imag. part $\omega = \frac{i}{2}(h - \bar{h})$ (1,1)-form.

Kähler manifold $d\omega = 0$

($\rightarrow \omega$ symplectic form)



expl. Fubini-Study-metric on $\mathbb{C}P^n$

$$\mathbb{C}P^n \cong (\mathbb{C}^{4n+1} \setminus \{0\}) / \mathbb{C}^* = S^{4n+1} / S^1$$

Hermitian metric on \mathbb{C}^{4n+1} $ds^2 = \sum dz_i \otimes d\bar{z}_i$, standard euclidean metric on \mathbb{R}^{4n+2}
 restriction to S^{4n+1} invariant under S^1 -action

\rightarrow induces Hermitian metric on $\mathbb{C}P^n$

\uparrow closed submfd $M \subseteq \mathbb{C}P^n$
 $\hat{=}$ smooth proj. cplx alg. var. } these are all Kähler.

M opt 5 Hermitian metric \leadsto Hermitian form on all $T_{(p,q)}^v = \wedge^p T_{(2,0)}^v \otimes \wedge^q T_{(0,1)}^v$
 \leadsto inner product on forms $(\varphi, \psi) = \int_M h(\varphi(z), \psi(z)) \frac{\omega^n}{n!}(z)$ volume form for forms
 so $\mathcal{O}^{p,q}$ is normed vector space volume form from h .

Q: To understand cohomology $H_{\bar{\partial}}^{p,q}(M)$, how can we get "nice" representatives of cohomology classes?

$\bar{\partial}$ -closed form $\varphi \in \mathcal{O}^{p,q}_{\bar{\partial}}$ minimal norm $\iff \bar{\partial}^* \varphi = 0$ $\bar{\partial}^*$ adjoint of $\bar{\partial}$
 $(\bar{\partial}^* \varphi, \psi) = (\varphi, \bar{\partial} \psi)$

\leadsto combine $\bar{\partial} \varphi = 0$
 $\bar{\partial}^* \varphi = 0$ this is assuming $\mathcal{O}^{p,q}$ is Hilbert space & $\bar{\partial}$ bounded
 $\|\varphi + \bar{\partial} \psi\|^2 = \|\varphi\|^2 + \|\bar{\partial} \psi\|^2 + 2 \operatorname{Re}(\varphi, \bar{\partial} \psi) > \|\varphi\|^2$

into Laplace equation $\Delta_{\bar{\partial}} \varphi = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) \varphi = 0$

solutions are called **harmonic forms** \sim unique representatives of cohomology classes.

Why does this make sense? actual definition of $\bar{\partial}^*$ via

Hodge star operator $*$: $\mathcal{O}^{p,q}(M) \rightarrow \mathcal{O}^{n-p, n-q}(M)$ $n = \dim M$
 def'd by $(\varphi(z), \psi(z)) \frac{\omega^n}{n!} = \varphi(z) \wedge * \psi(z)$

then def $\bar{\partial}^* = - * \bar{\partial} *$, adjunction follows from Stokes.

Remark: on \mathbb{C}^n w/ standard metric, $p=q=0$ i.e. $f \in C_c^\infty(\mathbb{C}^n)$
 & vol. form $dz = dz_1 \wedge \dots \wedge dz_n$

$\Delta(f) = \bar{\partial}^* \bar{\partial} f = \bar{\partial}^* \left(\sum \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \right) = \dots = -2 \sum \frac{\partial^2 f}{\partial \bar{z}_i \partial z_i} = -\frac{1}{2} \sum \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) f$ Standard Laplace operator

The Hodge theorem (1) $H_{\bar{\partial}}^{p,q}(M)$ is finite-dimensional M opt 5

(2) have **Green operator** $G: \mathcal{O}^{p,q} \rightarrow \mathcal{O}^{p,q}$ s.t. get Hodge decomposition of forms

$\varphi = \mathcal{H}^{p,q}(\varphi) + \bar{\partial}(\bar{\partial}^* G(\varphi)) + \bar{\partial}^*(\bar{\partial} G(\varphi))$ $\mathcal{H}^{p,q}: \mathcal{O}^{p,q} \rightarrow H_{\bar{\partial}}^{p,q}$
orthogonal proj.
harmonic repr. of φ $H_{\bar{\partial}}^{p,q}(M)$ $\bar{\partial} \mathcal{O}^{p,q-1}(M)$ $\bar{\partial}^* \mathcal{O}^{p,q+1}(M)$

pf idea: functional analysis used to solve Laplace eq $\Delta \varphi = \eta$
 on manifolds, i.e. global analysis for η w/ $\mathcal{H}^{p,q}(\eta) = 0$ solution $\varphi = G(\eta)$

- (I) first get formal solution φ in Hilbert space completion of $\mathcal{O}^{p,q}$
 w/ $(\varphi, \Delta \varphi) = (\eta, \varphi)$ for all $\varphi \in \mathcal{O}^{p,q}$ as distribution
- (II) regularity: formal solution is actually a C^∞ -function!
 i.e. actual solution
- (III) spectral theory: finite-dim eigenspaces

Applications & examples

M opt cplx vfd for I, II ,

M opt Kähler for rest

I finite-dimensionality

$$\mathcal{H}^{p,q}(M) \xrightarrow{\cong} H^q(M, \Omega^p) \text{ is a finite-dim vector space}$$

(applies e.g. to global holomorphic differentials $\Omega^p(M)$)

II Kodaira-Serre duality

$$H^n(M, \Omega^n) \cong \mathbb{C}$$

& Hodge star induces nondeg. pairing $H^q(M, \Omega^p) \otimes H^{n-q}(M, \Omega^{n-p}) \rightarrow H^n(M, \Omega^n)$

III Hodge decomposition

$$H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(M) \cong \bigoplus_{p+q=r} H^q(M, \Omega^p)$$

$$H^{p,q}(M) = \overline{H^{q,p}(M)}$$

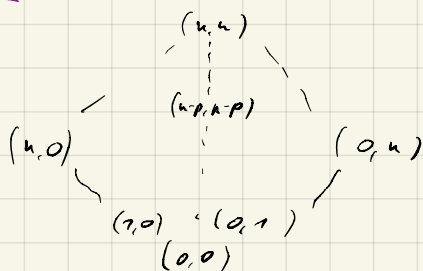
relative version: decomposition thru Hodge Riemann bilinear decomposition (HBD)

Consequences:

- holomorphic forms are harmonic
- odd Betti numbers $b_{2q+1}(M)$ are even
- even Betti numbers $b_{2q}(M) \geq 0$ (ω^q closed non-exact (q,q) form)

Hodge-diamond

Diagram for Hodge numbers



Hodge star: symmetry around center
 cplx conj.: refl. on vertical line

expl: P^n $h^{p,q} = \begin{cases} 1 & p=q \\ 0 & \text{otherwise} \end{cases}$

expl: K3 surface

		1		
	0		0	
1		20		1
	0		0	
		1		

no global holomorphic forms

The Hodge conjecture

X smooth projective complex variety

$$\text{Cycle class map } CH^p(X) \longrightarrow H^{2p}(X, \mathbb{Z})$$

(closed subvariety $Z \subseteq X$) \longmapsto Poincaré dual of fundamental class $[Z]$

image contained in $H^{p,p}(X)$

Hodge conjecture: rationally, classes in $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ are algebraic

connection between alg. geom. & topology via period integrals.