Lecture 5 — harmonic forms & Hodge decomposition

Recollection from Kay's lecture

geometric objects where complex analysis works! → cplx mfd: locally like open $U \subseteq \mathbb{C}^n$ + holomorphic transition maps

e.g. $\mathbb{C}P^n$, Riemann surface Σ, ...

sheaves Ω^p of holomorphic p-forms

sheaves $\mathcal{A}^n = \bigoplus_{p+q=n} \mathcal{A}^{p,q}$ of cplx-valued smooth forms

Locally $\omega = \sum a_{I,J}\, dz^I \wedge d\bar{z}^J$

$d : \mathcal{A}^p \to \mathcal{A}^{p+1}$ decomposes into $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$
& $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$

$\rightsquigarrow$ Dolbeault cplx as resolution (from Dolbeault Lemma)

$$\Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \to \cdots$$

Dolbeault cohomology
$$H^q(M,\Omega^p) = H^{p,q}_{\bar\partial}(M) = H^q(\mathcal{A}^{p,*})$$

[goal for next 3 lectures: Hodge theory — representing cohomology by harmonic forms
Hodge decomposition $H^n(X,\mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$ for cpt Kähler

more general: Hdg2dR spectral seq. $H^p(X,\Omega^q) \Rightarrow H^{p+q}(X,\Omega^\bullet)$
& mixed Hodge structures

lit: Griffiths & Harris, principles of alg. geom.]

Hermitian metrics, Laplace operator & harmonic forms

cplx mfd M $\rightsquigarrow$ tangent bdl. $TM \otimes \mathbb{C} \cong TM^{1,0} \oplus T^{0,1}M$

from cplx structure $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$; holom. tangent bdl.

hermitian metric h: reduction of structure gp of $T^{1,0}M$ to $U(n) \subseteq GL_n(\mathbb{C})$.

$V \otimes \bar{V} \to \mathbb{C}$, $h(z,w) = \overline{h(w,z)}$

($\hat{=}$ holomorphic tangent spaces equipped w/ pos. def. Hermitian inner product)

real part $g = \frac{1}{2}(h+\bar{h})$ — Riemannian metric, — imag. part $\omega = \frac{i}{2}(h-\bar{h})$ (1,1)-form.

Kähler manifold $d\omega = 0$
($\rightsquigarrow \omega$ symplectic form)

Riem. geom. $O(n)$ — cplx geom. $U(n)$ — sympl. geom. $Sp(2n)$

expl. Fubini-Study-metric on $\mathbb{C}P^n$

$$\mathbb{C}P^n \simeq (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^\times = S^{2n+1}/S^1$$

hermitian metric on $\mathbb{C}^{n+1}$ $ds^2 = \sum dz_i \otimes d\bar{z}_i$, standard euclidean metric on $\mathbb{R}^{2n}$
restriction to S^{2n+1} invariant under S^1-action
$\rightsquigarrow$ induces hermitian metric on $\mathbb{C}P^n$

$\rightsquigarrow$ closed submfd $M \subseteq \mathbb{C}P^n$ $\hat{=}$ smooth proj. cplx alg. var. } these are all Kähler.

M cpt ! hermitian metric $\rightsquigarrow$ hermitian form on all $T^\vee_{(p,q)} = \Lambda^p T^\vee_{(1,0)} \otimes \Lambda^q T^\vee_{(0,1)}$

$\rightsquigarrow$ inner product on forms $(\varphi, \psi) = \int_M h(\varphi(z), \psi(z)) \frac{\omega^n}{n!}(z)$ — volume form from h. — norms for forms

so $\mathcal{A}^{p,q}$ is normed vector space

Q: To understand cohomology $H^{p,q}_{\bar\partial}(M)$, how can we get "nice" representatives of cohomology classes?

$\bar\partial$-closed form $\varphi \in Z^{p,q}_{\bar\partial}$ minimal norm $\iff \bar\partial^* \varphi = 0$ — $\bar\partial^*$ adjoint of $\bar\partial$ $(\bar\partial^*\varphi, \psi) = (\varphi, \bar\partial\psi)$

$\rightsquigarrow$ combine $\bar\partial\varphi = 0$, $\bar\partial^*\varphi = 0$

this is assuming $\mathcal{A}^{p,q}$ is Hilbert space & $\bar\partial$ bounded

$\|\varphi + \bar\partial\eta\|^2 = \|\varphi\|^2 + \|\bar\partial\eta\|^2 + 2\,\mathrm{Re}(\varphi, \bar\partial\eta) > \|\varphi\|^2$.

into Laplace equation $\boxed{\Delta_{\bar\partial}\varphi = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\varphi = 0}$

solutions are called harmonic forms $\sim$ unique representatives of cohomology classes.

Why does this make sense? actual definition of $\bar\partial^*$ via

Hodge star operator $* : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{n-p,n-q}(M)$ — $n = \dim M$

def^d by $(\varphi(z), \eta(z))\frac{\omega^n}{n!} = \varphi(z) \wedge *\eta(z)$

then def $\bar\partial^* = -*\bar\partial*$, adjunction follows from Stokes.

Rmk: on $\mathbb{C}^n$ w/ standard metric & vol. form $dz = dz_1 \wedge \dots \wedge dz_n$, $p = q = 0$ i.e. $f \in C^\infty_c(\mathbb{C}^n)$

$$\Delta(f) = \bar\partial^*\bar\partial f = \bar\partial^*\left(\sum \frac{\partial f}{\partial \bar z_i} d\bar z_i\right) = \dots = -2\sum \frac{\partial^2 f}{\partial z_i \partial \bar z_i} = -\frac{1}{2}\sum\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right) f$$

standard Laplace operator

The Hodge theorem — M cpt !

1. $H^{p,q}_{\bar\partial}(M)$ is finite-dimensional
2. have Green operator $G : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}$ s.th. get Hodge decomposition of forms

$$\varphi = \mathcal{H}^{p,q}(\varphi) + \bar\partial(\bar\partial^* G(\varphi)) + \bar\partial^*(\bar\partial G(\varphi))$$

$\mathcal{H}^{p,q} : \mathcal{A}^{p,q} \to H^{p,q}_{\bar\partial}$ orthogonal proj.

harmonic repr. of φ $H^{p,q}_{\bar\partial}(M)$; $\cap\ \bar\partial\mathcal{A}^{p,q-1}(M)$; $\cap\ \bar\partial^*\mathcal{A}^{p,q+1}(M)$

pf idea: functional analysis — on manifolds, i.e. global analysis

need to solve Laplace eq $\Delta\varphi = \eta$ for η w/ $\mathcal{H}^{p,q}(\eta) = 0$ — solution $\varphi = G(\eta)$

I. first get formal solution ψ in Hilbert space completion of $\mathcal{A}^{p,q}$ — as distribution

w/ $(\psi, \Delta\varphi) = (\eta, \varphi)$ for all $\varphi \in \mathcal{A}^{p,q}$

II. regularity: formal solution is actually a C^∞-function! i.e. actual solution

III. spectral theory: finite-dim eigenspaces

Applications & examples

M cpt cplx mfd for I, II,
M cpt Kähler for rest

(I) finite-dimensionality

$$\mathcal{H}^{p,q}(M) \xrightarrow{\simeq} H^q(M,\Omega^p)$$ is a finite-dim vector space

(applies e.g. to global holomorphic differentials $\Omega^p(M)$)

(II) Kodaira–Serre duality $\quad H^n(M,\Omega^n) \simeq \mathbb{C}$

& Hodge star induces nondeg. pairing $H^q(M,\Omega^p) \otimes H^{n-q}(M,\Omega^{n-p}) \to H^n(M,\Omega^n)$

(III) Hodge decomposition

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M) \cong \bigoplus_{p+q=r} H^q(M,\Omega^p)$$

relative version: decomposition thm BBD

$$H^{p,q}(M) = \overline{H^{q,p}(M)}$$

Consequences:
- holomorphic forms are harmonic
- odd Betti numbers $b_{2q+1}(M)$ are even
- even Betti numbers $b_{2q}(M) > 0$ (ω^q closed non-exact (q,q) form)

Hodge-diamond $\quad$ diagram for Hodge numbers

$$\begin{array}{ccccc}
 & & (n,n) & & \\
 & & (n-p,n-p) & & \\
(n,0) & & & & (0,n) \\
 & (1,0) & & (0,1) & \\
 & & (0,0) & &
\end{array}$$

Hodge star: symmetry around center
cplx conj.: refl. on vertical line

expl: $\mathbb{P}^n \quad h^{p,q} = \begin{cases} 1 & p=q \\ 0 & \text{otherwise} \end{cases}$

expl. K3 surface

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

no global holomorphic forms

The Hodge conjecture

X smooth projective complex variety

$\leadsto$ cycle class map $\quad CH^p(X) \longrightarrow H^{2p}(X,\mathbb{Z})$

(closed subvariety $Z \subseteq X \longmapsto$ Poincaré dual of fundamental class $[Z]$)

image contained in $H^{p,p}(X)$

Hodge conjecture: rationally, classes in $H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$ are algebraic

connection between alg. geom. & topology via period integrals.