Lecture 4: Analytic de Rham and Dolbeault complex on complex mfd's

Def: M top space, connected, Hausdorff

is a <u>complex mfd of dim n</u> $:\Longleftrightarrow$

$\exists$ complex atlas $(\mathcal{U}_i, \varphi_i : \mathcal{U}_i \to \mathbb{C}^n)$

i.e. • $M = \bigcup_i \mathcal{U}_i$ open cov

• $\varphi_i : \mathcal{U}_i \xrightarrow[\text{homeom.}]{\sim} \varphi_i(\mathcal{U}_i) \underset{\text{open}}{\subset} \mathbb{C}^n$

• $\varphi_i \circ \varphi_j^{-1} : \underset{\mathbb{C}^n}{\varphi_j(\mathcal{U}_j \cap \mathcal{U}_i)} \to \mathcal{U}_j \cap \mathcal{U}_i \to \underset{\mathbb{C}^n}{\varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)}$

is holomorphic

Recall if we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ ($\mathbb{R}$-lin)

$$(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$$

$$x = \frac{1}{2}(z + \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z})$$

then a C^∞ function $f : \underset{\mathbb{C}^n}{\mathcal{U}} \to \mathbb{C}$ is holomorphic

$\Longleftrightarrow \frac{\partial f}{\partial \bar{z}_j} = 0 \quad \forall j = 1 \dots n \Leftrightarrow f = \sum a_\alpha z^\alpha$ (power series) locally around any pt

Ex

1) $\mathbb{C}^n$ = cx mfd.

2) $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^*$ is a cx mfd.

$$= \bigcup_j \underbrace{\{(z_0:\dots:z_n) \mid z_j \neq 0\}}_{=\mathcal{U}_j}$$

$$\mathcal{U}_j \longrightarrow \mathbb{C}^n$$
$$(z_0:\dots:z_n) \longmapsto \left(\frac{z_0}{z_j}, \dots, \frac{z_n}{z_j}\right)$$

$$\rightsquigarrow \quad \mathbb{C}^n\setminus\{z_1=0\} \xrightarrow{\varphi_0^{-1}} \mathcal{U}_0 \cap \mathcal{U}_1 \xrightarrow{\varphi_1} \mathbb{C}^n\setminus\{z_0=0\}$$
$$(z_1,\dots,z_n) \longrightarrow (1:z_1:\dots:z_n) \longrightarrow \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1}\right)$$

hol

3) $\Lambda \subset \mathbb{C}^n$ lattice, i.e. a free $\mathbb{Z}$-mod s.t. $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^n$

$\Rightarrow$ $\mathbb{C}^n/\Lambda$ cx mfd. (cx torus)

4) let X smooth finite-type $\mathbb{C}$-scheme

$\Rightarrow$ $X(\mathbb{C})$ is a mfd (inverse function Thm)

Def: M cx mfd with atlas $(\varphi_i : U_i \to \mathbb{C}^n)_i$

1) $\mathcal{O}_M$ = sheaf of holomorphic fctions on M

i.e. $\mathcal{O}_M(U) = \{ f : U \to \mathbb{C} \text{ cts} \mid \underset{\subset \mathbb{C}^n}{\varphi_i(U_i \cap U)} \xrightarrow{\varphi_i^{-1}} U_i \cap U \xrightarrow{f} \mathbb{C} \text{ hol } \forall i \}$

$U \subset M$ open

$\rightsquigarrow \mathcal{O}_{M,p} \cong \mathbb{C}\{z_1 \ldots z_n\}$ cvgt power series.

2) Ω^1_M sheaf of hol 1-forms:

$\varphi : U \to \mathbb{C}^n$ chart.

$$\Omega^1_U := \bigoplus_{j=1}^n \mathcal{O}_U \, dz_j$$

If $\psi : V \to \mathbb{C}^n$ is another chart with coordinates $(w_1 \ldots w_n)$

$$g_{U,V} = (g^1_{U,V}, \ldots, g^n_{U,V}) : \psi(V \cap U) \xrightarrow{\psi^{-1}} V \cap U \xrightarrow{\varphi} \varphi(V \cap U) \subset \mathbb{C}^n$$

$$\to g^*_{UV} : \Omega^1_{U|U \cap V} \longrightarrow \Omega^1_{V|U \cap V}$$

$$dz_j \longmapsto d(z_j \circ g_{UV}) = \sum_{k=1}^n \frac{\partial g^j_{UV}}{\partial w_k} dw_k$$

$\rightsquigarrow$ glues to give Ω^1_M

3) $\Omega^q_M = \Lambda^q \Omega^1_M$

4) $\Omega^\bullet_M = \mathcal{O}_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \to \dots \to \Omega^n_M$

$$d\,(f\, dz_I) = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}\, dz_k \wedge dz_I$$

$(dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_q} \qquad I = (i_1, \dots, i_q))$

<u>Analytic Poincaré lemma</u>

$\underset{\text{const sheaf}}{\mathbb{C}_M} \xrightarrow{\simeq} \Omega^\bullet_M$ qis

<u>Ex</u>: <u>$n=1$</u> $\rightsquigarrow$ local question $\rightsquigarrow$ stalks

statement equivalent to

$0 \to \mathbb{C} \to \mathbb{C}\{z\} \to \mathbb{C}\{z\}\,dz \to 0$ exact. $\rightsquigarrow$ OK

$f \mapsto f'\,dz$

<u>Cor</u>: $H^j(M, \mathbb{C}) = H^j(M, \Omega^\bullet_M)$

sheaf cohomology defined using K-inj

We get additional str on $H^i(X, \mathbb{C})$:

1)
$$\begin{tikzcd}
\Omega_M^{\geq q} : \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \Omega_M^q \arrow[r] \arrow[d, equal] & \dots \arrow[r] & \Omega_M^n \arrow[d, equal] \\
\Omega_M^{\geq 0} = \Omega_M^\bullet : & \mathcal{O}_M \arrow[r] & \dots \arrow[r] & \Omega_M^{q-1} \arrow[r] & \Omega_M^q \arrow[r] & \dots \arrow[r] & \Omega_M^n
\end{tikzcd}$$

set $\quad F^q H^i(M, \Omega_M^\bullet) = \mathrm{Im}\left(H^i(M, \Omega_M^{\geq q}) \to H^i(M, \Omega_M^\bullet)\right)$

$\rightsquigarrow$ Hodge filtration

$$H^i(M, \mathbb{C}) = F^0 \supset F^1 \supset \dots \supset F^n$$

2) $\quad d\log : \quad \mathcal{O}_M^\times[-1] \longrightarrow \Omega_M^{\geq 1} \longrightarrow \Omega_M^\bullet$

$$\rightsquigarrow \quad \begin{aligned} \mathrm{Pic}(M) = H^1(M, \mathcal{O}_M^\times) &\longrightarrow F^1 \subset H^2(M, \mathbb{C}) \\ L &\longmapsto c_1(L) \end{aligned}$$

<u>Ex</u>: $\quad H^{2j}(\mathbb{P}^n(\mathbb{C}), \mathbb{C}) = \mathbb{C} \cdot (c_1(L))^j$

But $H^i(M, \Omega_M^\bullet)$ and $H^p(M, \Omega_M^q)$ is sheaf coh

and we don't have partition of unity in the holomorphic world $\to \Omega_M^q$ is not a fine sheaf $\rightsquigarrow$ not Γ-acyclic

→ solution go back to C^∞ - world.

<u>Def:</u> view M as C^∞ - mfd.

$$(\varphi: U_j \to \mathbb{C}^n \cong \mathbb{R}^{2n})$$

1) $\mathcal{A}^0_M$ = sheaf of cx-valued C^∞ - fct

$$\mathcal{A}^0_M(U) = \{ f : \underset{\text{cls}}{U} \to \mathbb{C} \mid \underset{\cap\, \mathbb{R}^{2n}}{\varphi(U_j \cap U)} \xrightarrow{\varphi_j^{-1}} U_j \cap U \to \mathbb{C} \text{ is } C^\infty \}$$

2) $\mathcal{A}^1_M$ = cx-valued C^∞ 1-forms on M

on chart $U \to \mathbb{C}^n \cong \mathbb{R}^{2n}$
$(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$

$$\mathcal{A}^1_U = \bigoplus_{j=1}^n \mathcal{A}^0_U \, dx_j \oplus \bigoplus_{j=1}^n \mathcal{A}^0_U \, dy_j \qquad \left[\begin{array}{l} x = \frac{1}{2}(z + \bar z) \\ y = \frac{1}{2i}(z - \bar z) \end{array} \right.$$

$$= \underbrace{\bigoplus_{j=1}^n \mathcal{A}^0_U \, dz_j}_{\mathcal{A}^{1,0}_U} \oplus \underbrace{\bigoplus_{j=1}^n \mathcal{A}_U \, d\bar z_j}_{\mathcal{A}^{0,1}_U}$$

g = change of coordinates $z \to w$

$$\Rightarrow d(z_j \circ g) = \sum \frac{\partial g^j}{\partial w_k} dw_k + \underbrace{\sum \frac{\partial g^j}{\partial \bar w_k} d\bar w_k}_{=0 \text{ since } g \text{ is hol.}}$$

$$d(\bar z_j \circ g) = \underbrace{\sum \frac{\partial \bar g^j}{\partial w_k} dw_k}_{=0} + \sum \frac{\partial \bar g^j}{\partial \bar w_k} d\bar w_k$$

⇒ get global decomposition $\mathcal{A}^1_M \cong \mathcal{A}^{1,0}_M \oplus \mathcal{A}^{0,1}_M$

3) $\mathcal{A}^j_M = \Lambda^j \mathcal{A}^1_M \cong \bigoplus_{p+q=j} \mathcal{A}^{p,q}_M$

$\mathcal{A}^{p,q}_M \ni \omega$ local section

$\Rightarrow \omega \underset{\text{locally}}{=} \sum f_{I,J}\, dz_I \wedge d\bar{z}_J \quad (f_{I,J} \in \mathcal{A}^0)$

with $I = (i_1 < \dots < i_p)$

$J = (j_1 < \dots < j_q)$

$d : \mathcal{A}^j_M \longrightarrow \mathcal{A}^{j+1}_M$

decomposes into sums of

$\partial : \mathcal{A}^{p,q}_M \longrightarrow \mathcal{A}^{p+1,q}_M$, $f\, dz_I \wedge d\bar{z}_J \mapsto \sum_k \frac{\partial f}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J$

$\bar{\partial} : \mathcal{A}^{p,q}_M \longrightarrow \mathcal{A}^{p,q+1}_M$, — " — $\mapsto \sum_k \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$

4) For p fixed we get the <u>Dolbeault cx</u>

$\mathcal{A}^{p,\bullet}_M : \quad \mathcal{A}^{p,0}_M \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_M \longrightarrow \mathcal{A}^{p,2}_M \to \dots \to \mathcal{A}^{p,n}_M$

note each $\mathcal{A}^{p,q}_M$ is a $\mathcal{A}^0 =$ cx-valued C^∞-module

$\hookrightarrow$ has part. of unity

$\to$ fine sheaf.

Def.:

$$H^{p,q}_{\bar\partial}(M) = \text{Dolbeault cohomology}$$

$$= \frac{\operatorname{Ker}\left(\bar\partial : A^{p,q}(M) \longrightarrow A^{p,q+1}(M)\right)}{\bar\partial\left(A^{p,q-1}(M)\right)}$$

Dolbeault lemma ($\bar\partial$ = Poincaré-lemma)

$$H^{p,q}_{\bar\partial}(B_N(0)) = \begin{cases} \Omega^p(B_N(0)) & q = 0 \\ 0 & , q \neq 0 \end{cases}$$

$0 < N \leq \infty$

Ideas in pf.:

- suff to treat $p = 0$
- $f \in A^0$ is hol $\Leftrightarrow \bar\partial f = 0$ ($\rightsquigarrow$ "$q=0$" part)

Take $q \geq 1$

- show for any $\alpha \in A^{0,q}(B_N(0))$ with $\bar\partial(\alpha) = 0$ and any $r < N$ $\exists$ $\beta_r \in A^{0,q-1}(B_r(0))$ with $\bar\partial(\beta_r) = \alpha$

(uses integration)

- construct $\beta \in A^{0,q}(B_N(0))$ from the β_r

Cor:

$$\Omega^p_M \xrightarrow{\sim} A^{p,\bullet}_M \quad \text{is a qis}$$

↑ fine sheaves

$\rightsquigarrow$
last time

Cor (Dolbeault Thm)

$$\Rightarrow \quad H^q(M, \Omega^p_M) = H^{p,q}_{\bar\partial}(M)$$

Final comment

if $C^\cdot$ $C^0 \to \dots \to C^n$ is a cx

and

$$\begin{matrix} C^j & \to & K^{j,\cdot} \\ \downarrow & & \downarrow \\ C^{j+1} & \to & K^{j+1,\cdot} \end{matrix}$$

is a res $\forall j$.

$$\Rightarrow \quad C^\cdot \xrightarrow[q\,is]{\sim} \mathrm{Tot}(K^{\cdot,\cdot}) = \text{total cx},$$

where $\mathrm{Tot}(K^{\cdot,\cdot})^j = \bigoplus_{a+b=j} K^{a,b} \longrightarrow \bigoplus_{a+b=j+1} K^{a,b}$

$$f_{a,b} \mapsto d_1(f_{a,b}) + (-1)^a d_2(f_{a,b})$$

Cor: $H^j(M, \Omega^\cdot_M) \cong H^j(\mathrm{Tot}(A^{\cdot,\cdot}(M)))$

//