Lecture 3: de Rham Thm via Hypercohomology

Recall last time:

de Rham Thm X - real C^∞ mfld

$$\Rightarrow \quad H^j_{dR}(X) \cong H^j_{sing}(X, \mathbb{R}) \qquad \forall j$$

more precisely

$$\Omega^j(X) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(C_{j,sm}(X,\mathbb{R}), \mathbb{R}) =: C^j_{sm}(X,\mathbb{R})$$

$$\omega \in \Omega^j(X), \qquad \gamma \in C_{j,sm}(X,\mathbb{R}), \quad \gamma = \sum a_i \gamma_i, \quad \gamma_i : \Delta^j \to X \ \ C^\infty \text{ in a nbhd of } \Delta^j, \quad \Delta^j \subset \mathbb{R}^{j+1}$$

[locally $X \supset U \cong \mathbb{R}^n \to (x_1, \dots, x_n)$ coord.

$$\omega|_U = \sum_{I = (i_1 < \dots < i_j)} b_I \, dx_I, \text{ where } dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_j} \Big]$$

$$\omega \longmapsto \Big(\gamma \mapsto \int_\gamma \omega = \sum_i a_i \int_{\Delta^j} \gamma_i^* \omega \in \mathbb{R} \Big)$$

induces a quasi-isom of complexes

$$\Omega^\bullet(X) \xrightarrow{\simeq} C^\bullet_{sm}(X, \mathbb{R})$$

$\rightsquigarrow$
- + nice!
- very specific to C^∞-mfd's
- not clear whether there is an analogue for complex mfd's / schemes ...

§ (Hyper) cohomology

(X, A) ringed space (e.g. $(X, \mathcal{O}_X)$ scheme / variety, (X, C^∞), $(X, \mathbb{Z})$
C^∞ mfld, top sp, const sheaf)

$F : Sh(A) =$ (sheaves of A-modules on X) $\longrightarrow$ (ab gps)
left exact functor.

e.g. $F = \Gamma(X, -)$ = global section functor.

$\rightsquigarrow$ extends to

$$F : \underset{\text{complexes in } Sh(A)}{\underset{\shortparallel}{C(Sh(A))}} \longrightarrow \underset{\text{complexes in gps.}}{\underset{\shortparallel}{C(\text{ab gps})}}$$

$$K^\bullet = \begin{pmatrix} \vdots \\ \downarrow \\ K^i \\ \downarrow \\ K^{i+1} \\ \downarrow \\ K^{i+2} \\ \downarrow \\ \vdots \end{pmatrix} \longmapsto F(K^\bullet) = \begin{pmatrix} \vdots \\ \downarrow \\ F(K^i) \\ \downarrow \\ F(K^{i+1}) \\ \downarrow \\ F(K^{i+2}) \\ \downarrow \\ \vdots \end{pmatrix}$$

- a morph of cx'es $K^\bullet \xrightarrow{\varphi} L^\bullet$ $\left(\leftrightarrow \begin{array}{ccc} K^i & \xrightarrow{\varphi^i} & L^i \\ \downarrow & & \downarrow \\ K^{i+1} & \xrightarrow[\varphi^{i+1}]{} & L^{i+1} \end{array} \right)$

  is a <u>quasi-isom (qis)</u>

  $:\Longleftrightarrow$ φ induces isom $\varphi : \underset{\frac{\mathrm{Ker}(d : K^i \to K^{i+1})}{\mathrm{Im}(d : K^{i-1} \to K^i)}}{\underset{\shortparallel}{H^i(K^\bullet)}} \longrightarrow H^i(L^\bullet)$ $\forall i$

$\rightsquigarrow$ make life easier : consider complex up to qis

$\rightsquigarrow$ Problem if the functor F is not exact it does not map quasi-isomorphic complexes to quasi-isomorphic cx'es

$\rightsquigarrow$ <u>solution : right derived functors</u>

- $M^\cdot \in C(Sh(A))$ <u>acyclic</u> $\Leftrightarrow$ $H^i(M^\cdot) = 0 \quad \forall i$

- $I^\cdot \in C(Sh(A))$ <u>K-injective</u> $\Leftrightarrow$

  $\forall M^\cdot$ acyclic $\quad \mathrm{Hom}(M^\cdot, I^\cdot)/_{\text{homotopy}} = 0$

  $[f \sim g \Leftrightarrow f - g = d_I \circ s - s \circ d_M]$

<u>Have</u>

(1) $\forall \quad L^\cdot \in C(Sh(A)) \quad \exists \quad L^\cdot \xrightarrow{\sim} I^\cdot$ qis (unique up to homotopy), $I^\cdot$ K-inj

(2) $\forall$

$$\begin{tikzcd} K^\cdot \arrow[r, "\sim"] \arrow[d] & M^\cdot \arrow[d, dotted, "\exists"] \\ L^\cdot \arrow[r, "\sim"] & I^\cdot \end{tikzcd}$$

qis, $\exists$ (unique up to homotopy)

<u>Def</u>: $RF(L^\cdot) := F(I^\cdot)$ well def up to homotopy

and $R^j F(L^\cdot) := H^j(F(I^\cdot))$ independent of choice of $I^\cdot$

<u>specifically</u> let $L^\bullet \in C(Sh(A))$

$$H^j(X, L^\bullet) := R^j\Gamma(X,-)(L^\bullet) = R^j\Gamma(X, L^\bullet)$$

<u>hyper/cohomology of $L^\bullet$ w.r.t $\Gamma(X,-)$</u>

<u>Properties:</u>

(a) if $L^\bullet = G$ = sheaf in degree 0

$0 \to G \to I^0 \to I^1 \to I^2 \to \dots$ inj res.

(i.e. seq is exact and each I^j is inj, i.e. $\mathrm{Hom}(-, I^j)$ exact functor)

$$\Rightarrow \quad G \xrightarrow{\sim} \underset{\text{K-inj}}{I^\bullet} \Rightarrow H^j(X,G) = \frac{\mathrm{Ker}(\Gamma(X,I^j) \to \Gamma(X,I^{j+1}))}{\mathrm{Im}(\Gamma(X,I^{j-1}) \to \Gamma(X,I^j))}$$

$(\Rightarrow H^0(X,G) = \Gamma(X,G))$

(b) $L^\bullet \to K^\bullet$ morphism of functors

$$\Rightarrow \quad R^jF(L^\bullet) \longrightarrow R^jF(K^\bullet)$$

canonical morphism (ex by (2), indep of choices)

(c) $0 \to L^\bullet \to K^\bullet \to M^\bullet \to 0$ ex seq of cx'es (or dist-Δ)

$\rightsquigarrow$ long exact seq

$$\dots \to R^jF(L^\bullet) \to R^jF(K^\bullet) \to R^jF(M^\bullet) \xrightarrow[\partial]{} R^{j+1}F(L^\bullet) \to \dots$$

boundary map (functorial)

(d) $L^\bullet \xrightarrow{\sim} K^\bullet$ qis

$$\Rightarrow \quad R^jF(L^\bullet) \xrightarrow{\sim} R^jF(K^\bullet)$$

Important: Forget about K-inj's

→ only used to define $R^j F$

→ never try to use them to compute $R^j F(L^\cdot)$

instead use:

Leray's acyclicity lemma

$K^\cdot$ bounded below (i.e. $K^j = 0 \quad \forall j \ll 0$)

$K^\cdot \xrightarrow{\sim} A^\cdot$ with $A^\cdot$ bounded below
and A^j is F-acyclic $\forall j$
↳ i.e. $R^n F(A^j) \quad \forall n > 0$

$$\Rightarrow \quad R^n F(K^\cdot) = H^n(F(A^\cdot))$$

Examples: on $(X, \mathcal{A})$ ringed space

- $\mathcal{G}$ = flasque sheaf, i.e. $\forall\, V \subset \underset{\text{open}}{\mathcal{U}} \subset X$: $\mathcal{G}(\mathcal{U}) \to \mathcal{G}(V)$ surj.

  $\Rightarrow$ $\mathcal{G}$ is $\Gamma(X, -)$ acyclic

- Assume $\mathcal{A}$ has the property:

partition of 1: $X = \bigcup_{i \in I} \mathcal{U}_i$ open cov $\Rightarrow \exists\ \rho_i \in \mathcal{A}(X)$, $i \in I$,
s.t. $\operatorname{supp}(\rho_i) \subset \mathcal{U}_i$ and $\sum_i \rho_i = 1$ (locally fin sum)

Then any $\mathcal{G} \in Sh(\mathcal{A})$ is $\Gamma(X, -)$ acyclic
(⇝ fine sheaf)

§ Singular cohomology as sheaf cohomology

<u>Thm</u> Let A be a ring

every pt has a basis of contr open nbhds

$X =$ loc. contractible top space
(e.g. $X = C^0$ mfd)

get $(X, A) =$ ringed space
A ↑ constant sheaf

Then $H^j(X, A) = H^j_{sing}(X, A) = H^j(C^\bullet_{sing}(X, A))$

<u>Pf</u>: Set $\mathcal{C}^j_{sing}(A) =$ the sheaf of A-modules

$$X \underset{open}{\supset} U \longmapsto C^j_{sing}(U, A)$$

$\rightsquigarrow$ • $\mathcal{C}^\bullet_{sing}(A)$: $\mathcal{C}^0_{sing}(A) \longrightarrow \mathcal{C}^1_{sing}(A) \longrightarrow \mathcal{C}^2_{sing}(A) \to \dots$
cx in $Sh(A)$

• $\mathcal{C}^j_{sing}(A)$ is flasque: $V \subset U$

$\Rightarrow \Gamma(U, \mathcal{C}^j_{sing}(A)) = \prod_{\gamma : \Delta^j \to U} \mathrm{Hom}(\mathbb{Z}\gamma, A) \xrightarrow{proj} \prod_{\gamma : \Delta^j \to V} \mathrm{Hom}(\mathbb{Z}\gamma, A) = \Gamma(V, \mathcal{C})$

$+$ U contractible $\Rightarrow$ $H^j_{sing}(U, A) = \begin{cases} A & j = 0 \\ 0, & j \geq 1 \end{cases}$

$\Rightarrow$ $\underset{\substack{\uparrow \\ \text{sheaf in deg } 0}}{A} \xrightarrow{\sim} \mathcal{C}^\bullet_{sing}(A)$ qis $\rightsquigarrow$ acyclic resol

$\Rightarrow$ $H^j(X, A) \underset{Leray}{=} H^j(\Gamma(X, \mathcal{C}^\bullet_{sing}(A))) \underset{defn}{=} H^j_{sing}(X, A)$ □

§ dR cohomology as hypercoh

<u>Thm.</u> X real C^∞-mfd.

$\rightsquigarrow$ $(X, \mathcal{E}^\infty)$ ringed space

and $\Omega_X^\bullet = (\underset{\mathcal{E}^\infty}{\overset{\|}{\Omega^0}} \to \Omega^1 \to \Omega^2 \to \dots) \in Sh(\mathcal{E}^\infty)$

<u>Then</u> $$H^j(X, \underset{\substack{\uparrow \\ \text{const sheaf}}}{\mathbb{R}}) = H^j(X, \Omega_X^\bullet) = H^j_{dR}(X)$$

<u>Pf:</u>

- <u>Recall Poincaré-lemma:</u> $U \subset \mathbb{R}^n$ star shaped

$$\Rightarrow \quad H^i_{dR}(U) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i \geq 1 \end{cases}$$

$$\rightsquigarrow \quad 0 \to \mathbb{R} \to \underset{\mathcal{E}^\infty}{\overset{\|}{\Omega^0}} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \dots \quad \text{exact}$$

i.e. $\mathbb{R} \xrightarrow{\sim} \Omega^\bullet$ qis in

- $\mathcal{E}^\infty$ has partition of 1 $\Rightarrow$ $\Omega^\bullet$ complex of $\Gamma(X,-)$-acyclic sheaves

$\square$

<u>Cor</u> (de Rham Thm)

$$H^j_{dR}(X) \cong H^j(X, \mathbb{R}) \cong H^j_{sing}(X, \mathbb{R}) = H^j_{sing}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$$