# Lecture 2 — de Rham thm on mflds

## Cohomology: de Rham & (smooth) singular

smooth mflds w/ boundary

$\mathbb{N}$-graded $\mathbb{R}$-vector spaces or other coefficients

What is cohomology? idea: contravariant functor $H^\bullet : \mathrm{Mfd}^{op} \longrightarrow \mathbb{R}\text{-vect}^{\mathbb{N}}$

satisfying

(1) homotopy invariance $H^\bullet(M\times[0,1]) \simeq H^\bullet(M)$

makes stuff invariant under homotopy equivalence

(2) Mayer-Vietoris seq for covering $M = U\cup V$ + conditions ...

$$\to H^i(M) \to H^i(U)\oplus H^i(V) \to H^i(U\cap V) \xrightarrow{\partial} H^{i+1}(M) \to \cdots$$

makes stuff computable via decomposing spaces

(3) additivity $H^\bullet(\coprod M_i) \cong \bigoplus H^\bullet(M_i)$

helpful to deal w/ infinity

## Example 1: deRham cohomology

$M \longmapsto H^\bullet_{dR}(M)$, functoriality from pullback of forms.

- homotopy invariance ~ version of Poincaré lemma.
- Mayer-Vietoris: for covering $M = U\cup V$ get exact seq of deRham cplxs.

$$\boxed{0 \to \Omega^\bullet(M) \to \Omega^\bullet(U)\oplus\Omega^\bullet(V) \to \Omega^\bullet(U\cap V) \to 0}$$

⇝ induces long exact seq of cohomology

homological algebra

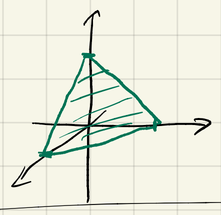
## Example 2: (smooth) singular cohomology

$X \longmapsto H^\bullet_{sing}(X,R)$ (generally for top. space $X$ & coeff. ring $R$)

top. standard simplex $\Delta^n = \{(t_0,\dots,t_n)\in\mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\}$

(convex hull of standard basis vectors $e_1,\dots,e_{n+1}$)

2-simplex $\Delta^2 \subseteq \mathbb{R}^3$



singular "n-simplex": cts. map $\Delta^n \longrightarrow X$

singular n-chains: $C_n(X,R) = R[\sigma \mid \sigma:\Delta^n\to X]$ R-linear combinations of singular simplices

boundary map $C_n(X,R) \to C_{n-1}(X,R) : \sigma \longmapsto \sum_{i=0}^{n} (-1)^i d_i(\sigma)$ restriction of $\sigma:\Delta^n\to X$ to i-th boundary face.

⇝ singular (co)chain cplxs $C^{sing}_\bullet(X,R)$ & dual $C^\bullet_{sing}(X,R)$

singular cohomology $H^i_{sing}(X,R) = H^i(C^\bullet_{sing}(X,R))$

[measures "higher-dim holes" in space built out of simplices]

**Properties:** homotopy invariance
(as before, chain homotopy built from combinatorial homotopy operator)

Mayer-Vietoris (as before, from exact seq of cochain complexes

$$0 \to C^\bullet(M) \to C^\bullet(U)\oplus C^\bullet(V) \to C^\bullet(U\cap V) \to 0$$

## Variation for deRham thm: for smooth mfld $M$

use smooth simplices $\sigma : \Delta^n \longrightarrow M$ (def'd by smooth map on open nbhd of $\Delta^n \subseteq \mathbb{R}^{n+1}$)

Whitney: continuous maps homotopic to smooth maps

$\rightsquigarrow \quad H^*_{sing}(M,\mathbb{R}) \simeq H^*_{sm}(M,\mathbb{R})$

can integrate $k$-forms over smooth $k$-chains, b/c
(s.th. version of Stokes thm holds)

# Theorem of de Rham

## The comparison homomorphism

$$\Psi : \Omega^k(M) \longrightarrow C^k_{sm}(M,\mathbb{R})$$
$$\omega \longmapsto \left( \gamma \longmapsto \int_\gamma \omega \right)$$

Properties: – induces chain maps $\Omega^*(M) \to C^*_{sm}(M,\mathbb{R})$
(linearity + Stokes thm)

– induced maps $H^*_{dR}(M) \longrightarrow H^*_{sm}(M,\mathbb{R})$

compatible w/ pullbacks, boundary maps in long exact sequences & products
($\wedge$ vs $\cup$)

**Theorem (de Rham):** for every smooth mfd $M$,

$H^*_{dR}(M) \longrightarrow H^*_{sm}(M,\mathbb{R})$ is an isom. of graded $\mathbb{R}$-alg.

pf sketch: $\Psi : H^*_{dR}(-) \longrightarrow H^*_{sm}(-,\mathbb{R})$ is a nat. transf of cohomology theories on sm mfds.
which is iso for $M = pt$.

1. de Rham iso holds for convex subsets in $\mathbb{R}^n$
(Poincaré Lemma)

2. if $M = U \cup V$ & de Rham iso holds for $U, V, U \cap V$ then iso holds for $M$ (Mayer-Vietoris sequence & 5-Lemma)

3. for family $U_\alpha$, deRham iso for $U_\alpha$ implies de Rham iso for $\coprod_\alpha U_\alpha$

$\Rightarrow$ every $M$ which has finite cover by open sets diffeo to convex subset of $\mathbb{R}^n$ satisfies de Rham iso.

exhaustion function trick (cf. Lee or Tu books on smooth mfds)

$\Rightarrow$ extend to all open $\subseteq \mathbb{R}^n$
& all smooth mfds

[ sheaf-theoretic pf, see Kay's lecture ]

# Periods

cplx numbers z s.th. Re z & Im z can be presented

def: numbers that can be expressed as integrals of rat'l diff. forms over rational (semi) algebraic varieties

(lit of Kontsevich–Zagier paper)

$$\int_{P(x_1,\dots,x_n)\geq 0} Q(x_1,\dots,x_n)\,dx_1\dots dx_n$$

$P$ poly, $Q$ rat'l w/ $\mathbb{Q}$-coeff.

Expl: – $\ln 2 = \int_1^2 \frac{dx}{x}$

– $\pi = \iint_{x^2+y^2\leq 1} dx\,dy = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

can allow algebraic functions here!

– elliptic integrals (hence the name ....)

– $\zeta(3) = \iiint_{0<x<y<z<1} \frac{dx\,dy\,dz}{(1-x)yz}$ & all other (multiple) zeta values

– conjectural periods (Deligne–Beilinson–Scholl): special $L$-values

conjectural non-period: $e$

motivic picture (Kontsevich–Zagier, Nori, Huber-Klawitter–Müller-Stach)

conjectural presentation (of period algebra $\hat{P}$) w/ generators $(X, D, \omega, \gamma)$

$X$ smooth var $/\mathbb{Q}$, $D$ snc divisor, $d = \dim X$

$\omega \in \Omega^d(X)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$

representing $\int_\gamma \omega$

modulo
1. linearity
2. change of var.
3. Stokes formula.

period algebra $\hat{P}$ as algebra of functions on pro-alg. torsor of isos between

$$H^*_{\text{Betti}}: X \longmapsto H^*_{\text{sing}}(X(\mathbb{C}); \mathbb{Q}) \quad \& \quad H^*_{dR}: X \longmapsto \mathbb{H}^*(X, \Omega^\bullet_X)$$

concretely: periods are entries of the base-change matrices relating the two $\mathbb{Q}$-structures under de Rham iso

$$\boxed{H^*_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq H^*_{dR}(X(\mathbb{C}))}$$

in the background, there is the motivic Galois gp

→ motives as representations of motivic Galois.