Lecture 13: Proof of Deligne-Illusie

We work over $\mathbb{F}_p$ (makes notation easier)

X sm $/\mathbb{F}_p$, $\Omega^j_X := \Omega^j_{X/\mathbb{F}_p}$

in this case $X = X^{(p)}$ (as id = Frob : $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{F}_p$)

and $F : X \to X^{(p)} = X$ is the absolute Frob.

(i.e. = id on top space and $F^*: \mathcal{O}_X \to \mathcal{O}_X$, $a \mapsto a^p$)

<u>Recall</u> We have an $\mathcal{O}_X$-linear isomorphism

$$C^{-1} : \Omega^j_X \xrightarrow{\sim} \mathcal{H}^j(F_* \Omega^\bullet_X)$$

satisfying
- $C^{-1}(\alpha \wedge \beta) = C^{-1}(\alpha) \wedge C^{-1}(\beta)$
- $C^{-1}(a\,db) = a^p b^{p-1} db$ $\quad (a, b \in \mathcal{O}_X)$

[$\Leftrightarrow$ $C^{-1}(a\, d\log u) = a^p\, d\log u$ $\quad (a \in \mathcal{O}_X, u \in \mathcal{O}_X^*)$]

- $q = 0 \rightsquigarrow C^{-1} = F^*$

We want to prove

<u>Thm</u> (D-I) Assume X has a sm lift $\tilde{X}/\mathbb{Z}/p^2\mathbb{Z}$

$\Rightarrow$ $\exists$ can isom in $D(\mathcal{O}_X)$

$$\varphi_{\tilde{X}} : \bigoplus_{j<p} \Omega^j_X[-j] \xrightarrow{\sim} \tau_{\leq p-1} F_* \Omega^\bullet_X$$

s.t. $\mathcal{H}^j(\varphi_{\tilde{X}}) = C^{-1}$, $j \leq p-1$

Rks:

0) Thm $\Rightarrow$ H-to-dR sp seq deg (in case X lifts$/W_2$, $\dim X < p$)

1) $\varphi_{\tilde{X}}$ will not be a map of cx'es

in fact we will construct

$$\bigoplus_{j<p} \Omega^j_X[-j] \longrightarrow \tau_{\leq p-1} \mathcal{C}^\bullet \xleftarrow[\text{qis}]{\sim} \tau_{\leq p-1} F_* \Omega^\bullet_X$$

2) it suffices to construct $\varphi_{\tilde{X}}$ with

$\mathcal{H}^j(\varphi_{\tilde{X}}) = C^{-1}$, then it is a qis

by the properties of C^{-1}

3) Recall

$$\tau_{\leq p-1} \Omega^\bullet_X = \mathcal{O}_X \xrightarrow{d} \Omega^1_X \longrightarrow \dots \longrightarrow \Omega^{p-2}_X \longrightarrow \underset{\substack{\parallel \\ \mathrm{Ker}\, d}}{Z\Omega^{p-1}_X} \to 0$$

define $\varphi^0_{\tilde X} := F^* : \mathcal{O}_X \longrightarrow Z\Omega^0_X = \ker(d : F_*\mathcal{O}_X \to F_*\Omega^1_X)$

$\alpha \longmapsto \alpha^p$

(indep of $\tilde X$)

Assume we defined

$$\begin{tikzcd} \varphi^1_{\tilde X} : & \Omega^1_X \arrow[r] & Z\Omega^1_X \arrow[r] & 0 \\ & 0 \arrow[u] \arrow[r] & F_*\mathcal{O}_X \arrow[u, "d"] & \end{tikzcd} \qquad (*)$$

with $\mathcal{H}^1(\varphi^1_{\tilde X}) = C^{-1}$

then define for $1 \le j < p$ by

$$\varphi^j_{\tilde X} : \Omega^j_X[-j] \longrightarrow (\Omega^1_X)^{\otimes j}[-j] \xrightarrow[\text{[in } D(\mathcal{O}_X)]]{(\varphi^1_{\tilde X})^{\otimes j}} (\tau_{\le 1} F_*\Omega^\cdot)^{\otimes j} \xrightarrow{\wedge} \tau_{\le j}(F_*\Omega^\cdot) \to \tau_{\le p-1}(F_*\Omega^\cdot)$$

$$\alpha_1 \wedge \dots \wedge \alpha_j \longmapsto \frac{1}{j} \sum_{\sigma \in S_j} \mathrm{sgn}(\sigma)(\alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(j)})$$

$\alpha_i \in \Omega^1_X$

multiplicativity of $C^{-1} \Rightarrow \mathcal{H}^j(\varphi^j_{\tilde X}) = C^{-1}$

$\to$ suff to construct $\varphi^1_{\tilde X}$ as in $(*)$

Excursion Frobenius lifts

Let $\tilde{A}$ be a flat $\mathbb{Z}/p^2\mathbb{Z}$-alg

and $A = \tilde{A}/p\tilde{A}$

$\Rightarrow$ on A have Frob

$$F: A \to A \quad , \quad a \mapsto a^p$$

Frobenius lift to $\tilde{A}$ is a ring hom

$$\tilde{F}: \tilde{A} \to \tilde{A} \quad \text{s.t.} \quad \tilde{F}(a) = \tilde{a}^p \mod pA$$

1) assume $\tilde{A} = \mathbb{Z}/p^2[X_1 \dots X_n]$

$\Rightarrow \tilde{F}: \tilde{A} \to \tilde{A}$, $x_i \mapsto x_i^p$ is a Frob lift
[also $x_i \mapsto x_i^p + p g_i$ any g_i]

2) Assume $\tilde{B}$ is <u>standard étale</u> over $\tilde{A}$, i.e.

$$\tilde{B} = \frac{\tilde{A}[T]_g}{(f)} \quad \text{with} \quad f' = \frac{\partial f}{\partial T} \in \tilde{B}^\times$$

assume we have $\tilde{F}_A$ a Frob lift on $\tilde{A}$
then to give $\tilde{F}_B : \tilde{B} \to \tilde{B}$ Frob lift over $\tilde{F}_A$

is eq to find $u(T) \in \tilde{A}[T]_g$ with

$$f_1(T^p + p u(T)) \in (f) \quad \text{where } f_1 := \tilde{F}_A(f)$$

Then define $\tilde{F}_B(h(T)) = h_1(T^p + p u(T))$

Have $f_1(T^p + p\,u(T)) \underset{p}{=} f_1(T^p) + p\,u(T)\,f_1'(T^p)$, $f_1'(T^p) \in \tilde{B}^\times$

$p^2 = 0$ in $\tilde{B}$ $\qquad\qquad\qquad\qquad$ $\overset{\shortparallel}{(f'(T))^p + p\,\ell}$

and $f_1(T^p) = f(T)^p + p\,h$, some $h \in A[T]$

$\rightarrow$ set $u = -\dfrac{h}{f_1'(T^p)} \in \tilde{B}$ $\rightarrow$ OK

we find.

<u>Lem</u>. X sm $/\mathbb{F}_p$, $\tilde{X}/_{\mathbb{Z}/p^2}$ sm lift

$\Rightarrow$ $\exists$ open affine covering $\tilde{X} = \bigcup_{i=1}^{n} \tilde{U}_i$

s.t. $\exists$ $\tilde{F}_i : \tilde{U}_i \rightarrow \tilde{U}_i$ lift of

$F : U_i \rightarrow U_i = \tilde{U}_i \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p$

a) <u>Construction of $\varphi^1_{\tilde{X}}$ if</u>

<u>$\exists \quad \tilde{F} : \tilde{X} \longrightarrow \tilde{X}$ Frob lift. :</u>

Have $\quad \underline{p} : \mathcal{O}_X \longrightarrow \mathcal{O}_{\tilde{X}} \quad , \quad a \longmapsto p\tilde{a}$

is inj (as $\tilde{X}$ flat / $\mathbb{Z}/p\mathbb{Z}$)

and $\forall \quad a \in \mathcal{O}_{\tilde{X}} \quad \exists \quad u(a) \in \mathcal{O}_X$ s.t.

$$\tilde{F}^*(a) = a^p + \underline{p}\, u(a)$$

$$\Rightarrow \quad \tilde{F}^* : \Omega^1_{\tilde{X}} \longrightarrow \underline{p}\left(\Omega^1_{\tilde{X}}\right)$$
$$db \longmapsto d\left(b^p + \underline{p}\, u(a)\right) = \underline{p}\left(\bar{b}^{p-1} d\bar{b} + du(a)\right)$$

$$\begin{tikzcd}
\Omega^1_{\tilde{X}} \arrow[r, "\tilde{F}^*"] \arrow[d] & \Omega^1_{\tilde{X}} \\
\Omega^1_X \arrow[r, dashed, "\exists!\, f"] & \Omega^1_X \arrow[u, hook, "\underline{p}"']
\end{tikzcd}$$

with $f(a\,db) = a^p\left(b^{p-1}db + d\,u(a)\right)$

$\Rightarrow \quad \bullet \quad f(\alpha) \equiv C^{-1}(\alpha) \mod d(\mathcal{O}_X)$

$\bullet \quad d(f) = 0$

$\Rightarrow$ get morph.

$$\begin{tikzcd}
& 0 \\
f : \Omega^1_X \arrow[r] & Z\Omega^1_X \arrow[u] \\
0 \arrow[u] \arrow[r] & \mathcal{O}_X \arrow[u]
\end{tikzcd}$$

b) Assume $\tilde{F}_1, \tilde{F}_2 : \tilde{X} \to \tilde{X}$ are two F-lifts

Claim: $\exists \; \theta : \mathcal{O}_X \to F_* \mathcal{O}_X$ derivation

$$\text{s.t. } (\tilde{F}_2^* - \tilde{F}_1^*)(a) = \underline{p}\, \theta(\bar{a}) \quad , a \in \mathcal{O}_{\tilde{X}}$$

indeed:
$$\tilde{F}_i^*(a) = a^p + \underline{p}\, u_i(a)$$

with $u_i : \mathcal{O}_{\tilde{X}} \to \mathcal{O}_X$ map

$$\Rightarrow \quad \tilde{F}_2^*(a) - \tilde{F}_1^*(a) = \underline{p}\,(u_2(a) - u_1(a))$$

and $\tilde{F}_2^*(pb) - \tilde{F}_1(pb) = 0$

$\Rightarrow \exists \; \theta : \mathcal{O}_X \to \mathcal{O}_X$ s.t. $\theta(a) = u_2(\tilde{a}) - u_1(\tilde{a})$

have $\theta(a+b) = \theta(a) + \theta(b)$

and $\underline{p}\,\theta(ab) = \tilde{F}_2^*(\tilde{a})\,(\tilde{F}_2^*(\tilde{b}) - \tilde{F}_1^*(\tilde{b})) + \tilde{F}_1^*(\tilde{b})\,(\tilde{F}_2^*(\tilde{a}) - \tilde{F}_1^*(\tilde{a}))$

$= \underline{p}\,(F^*(a)\,\theta(b) + F^*(b)\,\theta(a))$

$\Rightarrow \theta(ab) = F^*(a)\,\theta(b) + F^*(b)\,\theta(a)$ Leibniz rule

$\Rightarrow$ get $\mathcal{O}_X$-linear map

$$h : \Omega^1_X \to F_* \mathcal{O}_X \quad , \; a\,db \mapsto a^p\,\theta(b)$$

let $f_i = \underline{p}^{-1}\tilde{F}_i^* : \Omega^1_X \longrightarrow Z\Omega^1_X$ as in a)

$$a\,db \longmapsto a^p\left(b^{p-1}db + du_i(\tilde{b})\right)$$

then

$$f_2 - f_1 = d \circ h_{12}$$

$(h = h_{12})$

if $\tilde{F}_i$ $i = 1, 2, 3$ are lifts

$$\Rightarrow h_{13} = h_{12} + h_{23}$$

(c) The map $\varphi^1_{\tilde{X}}$ in general:

$\tilde{X} = \bigcup_i \tilde{U}_i$ with $\tilde{F}_i : \tilde{U}_i \to \tilde{U}_i$ F-lift.

$$U_i := \tilde{U}_i \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p$$

consider (on X)

$$\begin{aligned}
\mathcal{C}^0 &= \bigoplus_i \mathcal{O}_{U_i} \\
\delta^0 \downarrow & \\
\mathcal{C}^1 &= \bigoplus_i \Omega^1_{U_i} \oplus \bigoplus_{i,j} \mathcal{O}_{U_{ij}} \qquad U_{ij} = U_i \cap U_j \\
\delta^1 \downarrow & \\
\mathcal{C}^2 &= \bigoplus_i \Omega^2_{U_i} \oplus \bigoplus_{i,j} \Omega^1_{U_{ij}} \oplus \bigoplus_{i,j,k} \mathcal{O}_{U_{ijk}}
\end{aligned}$$

(Čech cx of $\Omega^{\cdot}_X$)

with

$$\delta^0((a_i)_i) = \left((d a_i)_i \, , \, (a_{j|U_{ij}} - a_{i|U_{ij}})_{i,j} \right)$$

$$d^1((\alpha_i),(b_{ij})) = \left((d\alpha_i)_i \, , \, (\alpha_{j|U_{ij}} - \alpha_{i|U_{ij}} + d b_{ij})_{i,j} \, , \, (b_{jk} - b_{ij} + b_{ik}) \right)$$

Have

$$\begin{tikzcd} Z\Omega^1 \arrow[r] & Z\mathcal{C}^1 & da \arrow[r, mapsto] & ((da)_{|U_i}, 0) \\ \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{C}^0 \arrow[u] , & a \arrow[r, mapsto] \arrow[u, mapsto] & (a_{|U_i})_i \arrow[u, mapsto] \end{tikzcd}$$

induces qis $\tau_{\leq} \Omega^\cdot \xrightarrow{\sim} \tau_{\leq} \mathcal{C}^1$

$\widetilde{F}_i : \widehat{U}_i \to \widehat{U}_i$ yield $[L_Y \quad a) + b) \quad]$

$f_i : \Omega^1_{U_i} \to Z\Omega^1_{U_i}$

$h_{ij} : \Omega^1_{U_{ij}} \to F_* \mathcal{O}_{U_{ij}}$

s.t. $f_j - f_i = d h_{ij}$ and $h_{ij} + h_{jk} = h_{ij}$ on U_{ijk}

$\Rightarrow$

$$\begin{tikzcd}
\Omega^1_X \arrow[r] & Z\,\mathcal{C}^1 & \tau_{\leq} F_*(\Omega^\bullet) \arrow[l, "\sim"'] \\
0 \arrow[u] \arrow[r] & \mathcal{C}^0 \arrow[u] &
\end{tikzcd}$$

$$\alpha \longmapsto \left(f_i(\alpha u_i),\ f_{ij}(\alpha u_{ij}) \right)$$

define

$$\varphi^1_X : \quad \Omega^1_X[-1] \longrightarrow \tau_{\leq}(F_* \Omega^\bullet)$$

in $D(\mathcal{O}_X)$

with $\mathcal{H}^1(\varphi^1_X) = C^{-1}$ $\square$