Lecture 12 — Deligne - Illusie

Lifting / W_2

k = perfect field, $\mathrm{char}(k) = p > 0$

$W_2(k)$ = unique ring s.t. $\exists$

Witt vectors of length 2

1) $\exists$ $W_2(k) \twoheadrightarrow k$ surj. ring hom.

with kernel $pW_2(k)$

2) and $k \longrightarrow pW_2(k)$ is bij.

$\lambda \longmapsto p\tilde{\lambda}$

Ex: 1) $k = \mathbb{F}_p \Rightarrow W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$

2) in general: $W_2(k) = \{(a_0, a_1) \mid a_i \in k\}$

with $(a_0, a_1) + (b_0, b_1) = \left(a_0 + b_0, \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p} + a_1 + b_1\right)$

$(a_0, a_1) \cdot (b_0, b_1) = \left(a_0 b_0, a_0^p b_1 + b_0^p a_1 + \cancel{p a_1 b_1}\right)$

3) We also have $W(k)$ ← Witt vectors, p-torsionfree, complete DVR with max'l ideal (p) and residue field $\frac{W(k)}{p} = k$

$W_n(k) := \frac{W(k)}{p^n}$

- let X be a smooth k scheme

We say X has a <u>smooth lift over W_2</u>

$\Longleftrightarrow \exists\ \tilde{X} \to \operatorname{Spec} W_2(k)$ smooth s.t.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \tilde{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} W_2(k) \end{tikzcd}$$

cl im induced by $W_2(k) \twoheadrightarrow k$

<u>Examples:</u>

- $\mathbb{A}^n_{W_2(k)}$, $\mathbb{P}^n_{W_2(k)}$ smooth lift $/W_2$ of $\mathbb{A}^n_k$, $\mathbb{P}^n_k$

- $F = \sum_{|I|=d} a_I x^I \in k[x_0, \dots, x_n]$ with $H = V_+(F) \subset \mathbb{P}^n_k$ smooth, $a_I \in k$

Set $\tilde{F} = \sum_{|I|=d} \tilde{a}_I x^I \Rightarrow \tilde{H} = V_+(\tilde{F}) \subset \mathbb{P}^n_k$ sm. lift

lift of a_I to $W_2(k)$

- let $X/\mathbb{C}$ smooth

$\Rightarrow \exists\ A \subset \mathbb{C}$, fin gen $/\mathbb{Z}$, $\exists\ \overline{X} \to \operatorname{Spec} A$ sm s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{X} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Shrinking A we can assume $\operatorname{Spec} A \to \mathbb{Z}$ is sm.

let $\eta \in \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ closed pt $(\Rightarrow \mathbb{Q}(x)$ fin $/\mathbb{Q})$

$$\Rightarrow \begin{tikzcd} \bar{\eta} \arrow[rd] & \subset & \operatorname{Spec} A \arrow[d] \\ & & \operatorname{Spec} \mathbb{Z} \end{tikzcd} \quad \Rightarrow \bar{\eta} = \operatorname{Spec} R \text{ with } R \text{ fin} /\mathbb{Z}$$

let $q \in \operatorname{Spec} R$ be s.t. R is unramif at q $/\mathbb{Z}$

i.e. $e(q/p) = 1$

where $p = q \cap \mathbb{Z}$

$\Rightarrow$ $k = R/q$ is a fin field ext of $\mathbb{F}_p$

and $W_2(k) = \dfrac{R}{q^2}$ $\left(\text{note } \dfrac{R}{q^2} = \dfrac{R_q}{q^2 R_q} = \dfrac{R_q}{p^2 R_q}\right)$

$$\Rightarrow \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rd, phantom, "\ulcorner"] & \bar{X} \arrow[d] & X_{0,\bar{\eta}} \arrow[l] \arrow[d] \arrow[ld, phantom, "\urcorner"] & X_0(k)_2 \arrow[l] \arrow[d] \arrow[ld, phantom, "\urcorner"] & X_0(k)_1 \arrow[l, hook'] \arrow[d] \arrow[ld, phantom, "\urcorner"] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} A & \bar{\eta} \arrow[l] & \operatorname{Spec} R/q^2 \arrow[l, hook'] & \operatorname{Spec} R/q \arrow[l, hook'] \end{tikzcd}$$

sm lift $/W_2$ of

- In general, for X sm$/k$ $\exists$ obstruction class

$$ob(X) \in H^2(X, \underbrace{\mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)}_{T_{X/k}})$$

s.t. $ob(X) = 0 \Leftrightarrow X$ has a sm lift $/W_2$

In part any sm curve lifts $/W_2$

any affine scheme — " —

One can also show any ab var lifts $/W_2$

Thm (Deligne-Illusie 1987)

X sm proper $/k$, $\dim X \le p$

Assume X has a smooth lift $/W_2(k)$

Then
$$E_1^{i,j} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{dR}(X/k)$$

degenerates.

$$\left(\Leftrightarrow \quad \dim H^n_{dR}(X/k) = \sum_{i+j=n} \dim_k H^j(X, \Omega^i_X) \right)$$

Comments:

- Supersingular Enriques surfaces do not lift $/W_2$ (see last time)

- For any sm proper surface X $\exists$ $f: X' \to X$ gen fin + sep surj, X' sm prope s.t. X' does not lift $/W_2$ (see example of Mumford last time)

- Alexander Petrov (Feb 2023)
$\exists$ X sm proper of $\underline{\dim X = p+1}$, X has a sm lift$/W(k)$ (in part $/W_2(k)$)

s.t. $\dim_k H^p_{dR}(X_0/k) < \sum_{i+j=p} \dim_k H^j(X_0, \Omega^i_{X_0/k})$

$\Rightarrow$ H-to-dR spectral seq does not deg

We get an algebraic proof that
for X sm proper $/K$, $\mathrm{char}(K) = 0$
the Hodge-to-dR sp seq deg at E_1:

(Faltings also gave an alg proof using p-adic Hodge theory)

let X sm proper $/K$

as in example above we find.

$$\begin{tikzcd}
 & & & & \bar X_1 \arrow[ld] \arrow[d] & \mathrm{sm} \arrow[l] \\
X \arrow[r] \arrow[d, "f"'] & \bar X \arrow[d, "\bar f"] & \bar X_R \arrow[l] \arrow[d] & \bar X_2 \arrow[l] \arrow[d] & \operatorname{Spec}(k) \arrow[ld, hook] & \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} R \arrow[l] & \operatorname{Spec} W_2(k) \arrow[l] & &
\end{tikzcd}$$

$\operatorname{Spec} A$: fin gen$/\mathbb{Z}$

$\operatorname{Spec} R$: fin$/\mathbb{Z}$

$\operatorname{Spec} W_2(k) \to \operatorname{Spec} R$: defined by a closed pt y over some prime $p > \dim X$ s.t. $e(y/p) = 1$ (ramif index)

$\operatorname{Spec}(k) \hookrightarrow \operatorname{Spec} W_2(k)$: def by $W_2(k) \twoheadrightarrow k$

chosen s.t.
- $\bar f$ is sm
- $\operatorname{Spec} A$ is sm.
- $R^j \bar f_* \Omega^i_{\bar X/A}$ is locally free $\forall\, i,j$
- $R^n \bar f_* (\Omega^\bullet_{\bar X/A})$ loc free

can always achieve this after replacing A by $A[1/s]$

$\xrightarrow{\text{cohomology and base change}}$

$$h^{i,j} = \dim_K H^j(X, \Omega^i_{X/K}) = \mathrm{rk}_{\mathcal{O}}\left(R^j \bar f_* \Omega^i_{\bar X/A}\right) = \dim_k H^j(\bar X_1, \Omega^i_{\bar X_1/k})$$

$$h^n = \dim_K H^n_{dR}(X/K) = \mathrm{rk}_{\mathcal{O}}(R^n \bar f_* \Omega^\bullet_{\bar X/A}) = \dim_k H^n_{dR}(\bar X_1/k)$$

Now $\bar{X}_1$ has the sm lift $\bar{X}_2 / W_2$

$\underset{D-I}{\Longrightarrow}$ $h^n = \sum_{i+j} h^{i,j}$ $\Rightarrow$ Hodge-to-de Rham sp seq degs in char 0.

$\square$

<u>Note</u> the above proof does not give the Hodge decomposition.

Before we can discuss the proof of De-Ill we need an

<u>Excursion to derived category:</u>

<u>Recall:</u> $C^\cdot$ cx of $\mathcal{O}_X$-mod

$\Rightarrow H^j(X, C^\cdot) = H^j(\Gamma(X, I^\cdot))$, $C^\cdot \underset{qis}{\xrightarrow{\sim}} \underset{K\text{-inj}}{I^\cdot}$

and $C^\cdot \to D^\cdot$ qis $\Rightarrow H^j(X, C^\cdot) \cong H^j(X, D^\cdot)$

$\leadsto$ suff to work with complexes "up to quasi-isomorphism"

- define $\mathcal{K}(\mathcal{O}_X)$ = homotopy cat of cx'es of $\mathcal{O}_X$-mod

has obj: $C^\cdot$ cx'es of $\mathcal{O}_X$-mod

morph: $\mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}(C^\cdot, D^\cdot) = \mathrm{Hom}_{\mathrm{Comp}(\mathcal{O}_X)}(C^\cdot, D^\cdot) / \sim$

$f \sim g \Leftrightarrow \exists$ homotopy h $f - g = dh + hd$

define $D(\mathcal{O}_X)$ = derived category of $\mathcal{O}_X$-modules
= invert qis in $K(\mathcal{O}_X)$

$obj(D(\mathcal{O}_X)) = obj(K(\mathcal{O}_X)) = obj(Comp(\mathcal{O}_X))$

morph $\varphi : C^\cdot \to D^\cdot$ in $D(\mathcal{O}_X)$ is given by

$$\begin{tikzcd}[column sep=small]
 & C_1^\cdot \arrow[dl, "qis"'] \arrow[dr, "f_1"] & & C_2^\cdot \arrow[dl, "qis"'] \arrow[dr, "f_2"] & & & C_n^\cdot \arrow[dl, "qis"'] \arrow[dr, "f_n"] & \\
C^\cdot & & D_1^\cdot & & D_2^\cdot & \cdots & & D_n^\cdot
\end{tikzcd}$$

zigzag

f_i morph of cx's

Have functor

$Q : Comp(\mathcal{O}_X) \to D(\mathcal{O}_X)$, $C^\cdot \longrightarrow Q(C^\cdot) = C^\cdot$

and $Q(C^\cdot) \cong Q(D^\cdot) \Leftrightarrow \exists$ zigzag of qis' between $C^\cdot$ and $D^\cdot$

Rmks 1) $C^\cdot \xrightarrow{f} D^\cdot$ morph of cx is a qis $\Leftrightarrow$ $Q(f)$ is an isom.

2) Any morph in $D(\mathcal{O}_X)$ can be represented by

$$C^\cdot \xleftarrow[qis]{} E^\cdot \xrightarrow{f} D^\cdot$$

3) $I^\cdot$ K-inj $\Rightarrow$ $Hom_{D(\mathcal{O}_X)}(C^\cdot, I^\cdot) = Hom_{K(\mathcal{O}_X)}(C^\cdot, I^\cdot)$

$= Hom_{Comp(\mathcal{O}_X)}(C^\cdot, I^\cdot)/\sim$

derived functor

a functor $F: \mathrm{Comp}(\mathcal{O}_X) \longrightarrow$ some cat

factors via $D(\mathcal{O}_X)$ iff $F(\text{q-is}) = \text{iso}$

($\rightsquigarrow$ exact functors do)

not true for $\Gamma(X, -) : \mathrm{Comp}(\mathcal{O}_X) \to \mathrm{Comp}(\text{ab gps}) \to D(\text{ab gps})$

$f_* : \mathrm{Comp}(\mathcal{O}_X) \to \mathrm{Comp}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$

$f: X \to Y$

instead choose $\forall\ C^\cdot \in \mathrm{Comp}(\mathcal{O}_X)$ $\quad C^\cdot \xrightarrow[\text{q-is}]{} I^\cdot$ K-inj

and define $R\Gamma(X, C^\cdot) = \Gamma(X, I^\cdot)$

$Rf_* C^\cdot = f_* I^\cdot$

$\to$ defines functors

$R\Gamma : D(\mathcal{O}_X) \to D(\text{ab gps})$, $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$

and $H^i(R\Gamma(X, C^\cdot)) \simeq H^i(X, C^\cdot)$ (as defined in lecture 3)

$H^i(Rf_* C^\cdot) = R^i f_* (C^\cdot)$

. ses $\to$ l.e.s: $\quad 0 \to A^\cdot \xrightarrow{u} B^\cdot \xrightarrow{v} C^\cdot \to$ s.e.s. in $\mathrm{Comp}(\mathcal{O}_X)$

set $\mathrm{cone}(u) = $ cx with
$$\begin{array}{ccc} \mathrm{cone}(u)^n & \longrightarrow & \mathrm{cone}(u)^{n+1} \\ \| & & \| \\ A^{n+1} \oplus B^n & & A^{n+2} \oplus B^{n+1} \\ (a, b) & \longmapsto & (-d_A a,\ u(a) + d_B(b)) \end{array}$$

$\rightarrow$ get

$$\begin{tikzcd} A^\cdot \arrow[r, "u"] \arrow[d, equal] & B^\cdot \arrow[r, "v"] \arrow[d, equal] & C^\cdot \\ A^\cdot \arrow[r, "u"] & B^\cdot \arrow[r] & \mathrm{Cone}(u) \arrow[u, "qis"'] \arrow[r] & A^\cdot[1] \end{tikzcd}$$

$$B^n \longrightarrow A^{n+1} \oplus B^n \qquad A^{n+1}$$

$$b \longmapsto (0, b)$$

$$(a, b) \longmapsto a$$

dist-Δ

$$\Rightarrow \quad R\Gamma(X, A^\cdot) \xrightarrow{u} R\Gamma(X, B^\cdot) \rightarrow R\Gamma(X, C^\cdot)$$
$$\| $$
$$R\Gamma(X, \mathrm{Cone}(u)) \rightarrow R\Gamma(X, A^\cdot)[1]$$

$$\rightsquigarrow \quad H^i(X, A^\cdot) \rightarrow H^i(X, B^\cdot) \rightarrow H^i(X, C^\cdot) \rightarrow H^{i+1}(X, A^\cdot) \rightarrow \cdots$$

l.e.s

Truncation functor:

$$\tau_{\leq n} : \mathrm{Comp}(\mathcal{O}_X) \longrightarrow \mathrm{Comp}(\mathcal{O}_X)$$

$$C^\cdot \longrightarrow \tau_{\leq n} C^\cdot = (\ldots \rightarrow C^{n-2} \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} \ker d^n \rightarrow 0 \rightarrow 0 \ldots)$$

Have $H^j(\tau_{\leq n} C^\cdot) = \begin{cases} H^j(C^\cdot), & j \leq n \\ 0, & j > n \end{cases}$

$\rightsquigarrow$ preserves qis

$\rightarrow$ induces $\tau_{\leq n} : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$

Have $H^j(X, C^\cdot) = H^j(X, \tau_{\leq n} C^\cdot) \quad \forall\ j \leq n$

Thm' (D-I) X sm $/k$

Assume X has a smooth lift $\tilde{X}/W_2$

Then $\exists$ can isom (dep on $\tilde{X}$) in $D(\mathcal{O}_{X^{(p)}})$

$$\varphi_{\tilde{X}} : \underbrace{\underbrace{\bigoplus_{i<p} \Omega^i_{X^{(p)}/k} [-i]}_{\text{sits in degree } i}}_{\text{cx with trivial differential.}} \xrightarrow{\sim} \tau_{\leq p-1} \underset{\substack{\uparrow \\ \text{rel Frob}}}{F_*} \Omega^\bullet_{X/k}$$

s.t.

$$\mathcal{H}^i(\varphi_{\tilde{X}}) : \Omega^i_{X^{(p)}/k} \xrightarrow{\cong} \mathcal{H}^i(\tau_{\leq p-1} F_* \Omega^\bullet_{X/k}) = \mathcal{H}^i(F_*\Omega^\bullet_{X/k})$$

$$\mathcal{H}^i(\varphi_{\tilde{X}}) = C^{-1} \quad \forall i<p$$

inverse Cartier
see lecture 11

Cor: Assume $\dim X < p$ + X proper$/k$ $\Rightarrow$ Hodge-to-dR sp seq deg. at E_1

Pf: $\dim X < p \Rightarrow \tau_{\leq p-1} F_* \Omega^\bullet_{X/k} = F_* \Omega^\bullet_{X/k}$

$$\text{Thm}' \Rightarrow \bigoplus_i \Omega^i_{X^{(p)}/k}[-i] \cong F_* \Omega^\bullet_{X/k} \quad \text{in } D(\mathcal{O}_{X^{(p)}})$$

$$\xRightarrow{H^n(R\Gamma(X,-))} \bigoplus_i H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}/k}) = H^n(X^{(p)}, F_*\Omega^\bullet_{X/k})$$

$$\bigoplus_i H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}/k}) = \bigoplus_i H^{n-i}(X, \Omega^i_{X/k}) \otimes_{k,\text{Frob}} k$$

$$H^n(X^{(p)}, F_*\Omega^\bullet_{X/k}) = H^n(X, \Omega^\bullet_{X/k})$$

$$\Rightarrow h^n = \sum_{i+j=n} h^{i,j} \qquad \square$$

Rmk. There is a trick to deduce from Thm' also that H-dR sp seq deg at E_1 if $\dim X = p$

(uses Grothendieck-Serre duality)

Pf of Thm' next time.