Lecture 11 de Rham coh in char $p>0$ and the Cartier operator

$k = \bar{k}$ alg cl field, X sm, proj $/k$

Recall Hodge-to-dR sp seq

$$(*) \quad E_1^{i,j} = H^j(X, \Omega_X^i) \Longrightarrow H^n_{dR}(X/k) =: H^n$$

is the following data:

- The Hodge-filtr: $F^i = \mathrm{Im}\left(H^n(X, \Omega_X^{\geq i}) \longrightarrow H^n \right)$

$$\rightsquigarrow \quad 0 = F^{n+1} \subset F^n \subset \dots \subset F^1 \subset F^0 = H^n$$

- k-vsp's $E_r^{i,j}$ with k-lin maps $E_r^{i,j} \xrightarrow{d_r} E_r^{i+r, j-r+1}$
- iso's $$E_{r+1}^{i,j} \cong \frac{\mathrm{Ker}\left(E_r^{i,j} \xrightarrow{d_r} E_r^{i+r, j-r+1} \right)}{d_r\left(E_r^{i-r, j+r-1} \right)}$$

note $E_1^{i,j} = 0 \quad \forall \quad i<0$ or $i > \dim X = N$

$$\Rightarrow \quad E_N^{i,j} = E_{N+1}^{i,j} = \dots = E_\infty^{i,j}$$

- iso's $$E_\infty^{i,n-i} = \frac{F^i}{F^{i+1}} =: \mathrm{gr}_F^i H^n$$

In part $$\dim_k (H^n) = \sum_{i+j=n} \dim E_\infty^{i,j} \leq \sum_{i+j=n} \dim E_1^{i,j}$$

Def: We say $(*)$ <u>degenerates at E_1</u>

$$\iff \quad 0 = d_r : E_r^{i,j} \to E_r^{i+r,\, j-r+1} \quad \underline{\forall\, i,j,r}$$

$$\iff \quad E_1^{i,n-i} = gr_F^i H^n \quad \forall\, i,n$$

$$\iff \quad \dim H^n_{dR}(X/k) = \sum_{i+j=n} \dim H^j(X, \Omega^i_X)$$

Ex: $\quad char(k) = 0 \implies (*)$ deg.

[indeed: wlog $k \subset \mathbb{C}$

$$\implies H^n_{dR}(X/k) \otimes_k \mathbb{C} \overset{GAGA}{=} H^n_{dR}(X(\mathbb{C})^{an}/\mathbb{C})$$

$$\underset{HT}{=} \bigoplus_{i+j=n} H^j(X(\mathbb{C}), \Omega^i_{X(\mathbb{C})/\mathbb{C}})$$

$$\underset{GAGA}{=} \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/k}) \otimes_k \mathbb{C} \quad]$$

Note

$$E_1^{0,j} \xrightarrow{d_1} E_1^{1,j} \xrightarrow{d_1} E_1^{2,j} \to \dots$$

$$= \quad H^j(X, \mathcal{O}_X) \xrightarrow{d} H^j(X, \Omega^1_X) \xrightarrow{d} H^j(X, \Omega^2_X) \to \dots$$

all maps are zero by degeneration

in part $j=0$: $\forall\, \alpha \in H^0(X, \Omega^*_X)$: $d(\alpha) = 0$

(here X sm proj $/k$, $char(k) = 0$)

"Every global differential is closed"

|| In positive characteristic (*) does in gen'l not degenerate!

We see an ex by Mumford:

Let $k = \bar{k}$, $\mathrm{char}(k) = p > 0$

Thm (Mumford)

X sm proj$/k$, $K = k(X)$, $\alpha \in \Omega^1_K$

$\Rightarrow \exists \quad \pi : X' \longrightarrow X$ gen fin + separable

sm proj surf ($\Leftrightarrow$ $k(x')/k(x)$ fin sep ext)

s.t. $\pi^*(\alpha) \in H^0(X', \Omega^1_{X'/k})$

Pf: suff to show

$\forall\, P \in X$ closed $\exists\ \mathcal{U}_P \subset X$ open, $P \in \mathcal{U}$

|| $\exists\ X_P$ proj surf, $X_P' \xrightarrow{\pi_P} X$ gen fin + sep.

s.t. $\pi_P^*(\alpha) \in \Omega^1(\pi_P^{-1}(\mathcal{U}))$

[Then $\exists\ P_1, \dots, P_m$ s.t. $X = \bigcup_i \mathcal{U}_{P_i}$ $\rightsquigarrow$ take X' = resolution of irred cpt of $X'_{P_1} \times_X \dots \times_X X'_{P_m}$ which is dom$/X$]

Take $\mathcal{U} = \mathrm{Spec}\, A \subset X$ open, wlog $\alpha = \frac{a}{b}\, dx$, $x \in K$, $a, b \in A$, $b \neq 0$

Set $K' = \frac{K[z]}{(z^p + b^p \cdot z - x)} \Rightarrow K'/K$ fin sep

and in $\Omega^1_{K'}$: $\alpha = \frac{a}{b}\, d(z^p + b^p \cdot z) = a b^{p-1} dz \in \Omega^1_{A'}$, $A' := \frac{A[z]}{(z^p + b^p z - x)}$

let $X' \xrightarrow{\pi} X$ normaliz of X in K'

$\Rightarrow \quad \pi^*(\alpha) \in H^0(X', \Omega^1_{X'})$

$\square$

Cor: $\exists$ sm proj surf X'/k (char $k = p > 0$!)

s.t. $0 \neq d : H^0(X', \Omega^1_X) \longrightarrow H^0(X', \Omega^2_X)$

in part (*) does not deg.

Pf: in Thm take $X = \mathbb{P}^2_k \supset \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$

and $\alpha = x\,dy \in \Omega^1_{k(x,y)}$

$\underset{\text{Thm}}{\Rightarrow} \quad \exists \quad \pi : X' \to X$ gen fin + sep s.t.

$\pi^*(\alpha) \in H^0(X', \Omega^1_{X'})$

and $d\pi^*(\alpha) = \pi^*(d\alpha) = \pi^*(\underbrace{dx \wedge dy}_{\neq 0}) \underset{\pi \text{ is sep}}{\neq} 0$ $\square$

Rmk X supersing Enriques surface $/k$

(i.e. $\operatorname{char}(k) = p = 2$, $\omega_X = \mathcal{O}_X$, $b_2 = 10$, $H^1(X, \mathcal{O}_X) = k$, Frobenius $= 0$)

Then (Illusie)

$E_1^{\cdot,0}$: $H^0(X, \mathcal{O}_X) \xrightarrow{d=0} H^0(X, \Omega^1_X) \xrightarrow{d=0} H^0(X, \Omega^2_X)$

$E_1^{\cdot,1}$: $\underset{k}{\underset{\shortparallel}{H^1(X, \mathcal{O}_X)}} \xrightarrow{d \neq 0} \underset{k^{12}}{\underset{\shortparallel}{H^1(X, \Omega^1_X)}} \xrightarrow{d \neq 0} \underset{k}{\underset{\shortparallel}{H^1(X, \Omega^2_X)}}$

$\Rightarrow$ (*) does not deg.

To understand the situation in pos char better we need the Cartier op:

<u>Frobenius</u>: k = field, $\mathrm{char}(k) = p > 0$, X = k-scheme

$$F_X : X \longrightarrow X \quad \text{absolute Frob.}$$

$$= \begin{cases} \text{id on top space} \\ F^* : \mathcal{O}_X \longrightarrow \mathcal{O}_X, \quad a \longmapsto a^p \end{cases}$$

Have also $F_k : \mathrm{Spec}\, k \to \mathrm{Spec}\, k$

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r, dashed, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr, bend right] & X^{(p)} := X \otimes_k (k, F_k) \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}\, k \arrow[r, "F_k"] & \mathrm{Spec}\, k \end{tikzcd}$$

relative Frobenius

Have $$F^* : \frac{k[x_1 \dots x_n]}{I} \otimes_{k, \mathrm{Frob}} k \longrightarrow \frac{k[x_1, \dots x_n]}{I}$$

$$\begin{array}{c} \| \wr \\ \dfrac{k[x_1 \dots x_n]}{I} \nearrow \end{array}$$

$\uparrow$ k-linear

$x_i \longmapsto x_i^p$

Lemma Denote by $F_*(\Omega^q_X)$

the $\mathcal{O}_{X^{(p)}}$-mod with Ω^q_X as underlying sheaf of ab gps

and module str given by

$$a \cdot \beta := F^*(a)\,\beta \quad , \quad a \in \mathcal{O}_{X^{(p)}}, \ \beta \in \Omega^q_X$$

Then

$$d : F_*\Omega^q_X \longrightarrow F_*\Omega^{q+1}_X \quad \text{is } \mathcal{O}_{X^{(p)}}\text{-linear.}$$

Pf. to check $d(F^*(a)\beta) = F^*(a)\,d(\beta)$

locally $a = \sum \lambda_i\, x_1^{r_1} \dots x_n^{r_n}$, $F^*(a) = \sum \lambda_i\, x_1^{pr_1} \dots x_n^{pr_n}$

$\Rightarrow d(F^*(a)) = p \cdot (\dots) = 0 \quad \underset{\text{Leibniz rule}}{\Longrightarrow} \text{OK}$ □

Rmk. In particular we see

$$F^*(\mathcal{O}_{X^{(p)}}) \subset \ker(d : \mathcal{O}_X \longrightarrow \Omega^1_X)$$

$\rightsquigarrow$ huge kernel

compare to $\mathbb{C} = \ker(d : \mathcal{O}_{X(\mathbb{C})^{an}} \longrightarrow \Omega^1_{X(\mathbb{C})^{an}})$
↑ P.L.

Thm (Cartier) k field, $\mathrm{char}(k) = p > 0$, X sm$/k$

$\exists!$ isom of $\mathcal{O}_{X^{(p)}}$-mod.

$$C^{-1} : \Omega^i_{X^{(p)}/k} \xrightarrow{\simeq} \mathcal{H}^i\left(F_* \Omega^\bullet_{X/k}\right), \quad i \geq 0$$

such that

(1) for $i=0$: $\underset{\text{rel Frob.}}{C^{-1} = F^*}$

(2) $C^{-1}(\alpha \wedge \beta) = C^{-1}(\alpha) \wedge C^{-1}(\beta)$

(3) $C^{-1} d(\sigma^*(f)) = f^{p-1} df$ in $\mathcal{H}^1(F_* \Omega^\bullet_{X/k})$

Rmk: a) Note

$\alpha \in \Omega^1_{X^{(p)}/k} \Rightarrow \alpha =$ sum of $x\, d\sigma^*(y)$

and $F^*(x\, d\sigma^*(y)) = F^*(x)\, d(y^p) = p\, F^*(x) y^{p-1} dy = 0$

$\Rightarrow$ " $C^{-1} = \frac{1}{p^i} F^*$ " in deg i $\rightsquigarrow$ we have to make sense of this!

b) cf: $\mathcal{H}^i(\Omega^\bullet_{X(\mathbb{C})/\mathbb{C}}) \simeq \begin{cases} 0 & i \neq 0 \\ \mathbb{C} & \end{cases}$ (P.L)

Pf: wlog $X = \mathrm{Spec}\, A$ (by uniqueness it will glue)

$\rightarrow X^{(p)} = \mathrm{Spec}(A^{(p)})$, $A^{(p)} := A \otimes_{k,F} k$

where in $A \otimes_{k,F} k$: $a\lambda \otimes \mu = a \otimes \lambda^p \mu$, $a \in A$, $\lambda, \mu \in k$

Note
$$A \xrightarrow{\sigma^*} A^{(p)} \xrightarrow{F^*} A$$
$$a \longmapsto a \otimes 1$$
$$b \otimes \lambda \longmapsto b^p \lambda$$

in $\Omega^1_{A^{(p)}/k}$: $a\otimes\mu \, d(b\otimes\lambda) = (a\otimes\mu\lambda)\, d(\sigma^*(b))$

$\Rightarrow$ any elt in $\Omega^i_{A^{(p)}/k}$ is sum of elts of the form.

$$\alpha = (a\otimes\lambda)\cdot d(\sigma^* b_1)\wedge\ldots\wedge d(\sigma^* b_i)$$

Assume some C^{-1} as in Thm exists

$$\Rightarrow \quad C^{-1}(\alpha) \underset{(1)+(2)}{=} F^*(a\otimes\lambda)\; C^{-1}(d\sigma^* b_1)\wedge\ldots\wedge C^{-1}(d\sigma^* b_i)$$
$$\underset{(3)}{=} a^p\lambda\; b_1^{p-1}\cdots b_i^{p-1}\, db_1\wedge\ldots\wedge db_i \quad \text{in } H^i(F_*\Omega^\bullet_{A/k})$$

$\Rightarrow$ uniqueness.

<u>Existence</u> suff to construct

$$C^{-1}: \Omega^1_{A^{(p)}/k} \longrightarrow H^1(F_*\Omega^\bullet_{A/k}) \quad A^{(p)}\text{-linear with (3)}$$

$(\Leftrightarrow)$ need to construct derivation
$$\gamma: A^{(p)} \longrightarrow H^1(F_*\Omega^\bullet_{A/k})$$

$$\text{s.t.}\quad \gamma(a\otimes 1) = a^{p-1}\, da$$

$\leadsto$ construct

$$\gamma_0 : A \times R \longrightarrow H^1(F_* \Omega^1_{A/R})$$

with a) γ_0 biadditive

b) $\gamma_0(\lambda a, \mu) = \gamma_0(a, \lambda^p \mu)$ (linearity)

c) $\gamma_0(ab, \lambda) = a^p \gamma_0(b, \lambda) + b^p \gamma_0(a, \lambda)$ (Leibniz rule)

d) $\gamma_0(a, 1) \equiv a^{p-1} da \mod d(A)$ (Property (3))

Set $\gamma_0(a, \lambda) = \lambda a^{p-1} da \mod d(A)$

$\leadsto$ b), c), d) clear.

additivity in λ: ✓

— " — in a:

$$\gamma_0(a+b, \lambda) - \gamma_0(a, \lambda) - \gamma_0(b, \lambda) = \lambda\left((a+b)^{p-1} d(a+b) - a^{p-1} da - b^{p-1} db\right)$$

$$= d\left(\lambda \underbrace{\left(\frac{(a+b)^p - a^p - b^p}{p}\right)}_{\in A}\right)$$

$\leadsto$ unique existence of $C^{-1} : \Omega^i_{A/R} \longrightarrow H^i(F_* \Omega^i_A)$

remains to show: it is an isom if A is sm$/R$

$\leadsto$ after localization can assume

$$\exists \quad R[t_1, \dots, t_n] \longrightarrow A \quad \text{ét}$$

$$\underset{\text{fact}}{\Longrightarrow} \qquad \begin{array}{ccc} A^{(p)} \otimes_{R[t], F_{R[t]}} R[t] & \longrightarrow & A \quad \text{isom.} \\ a \otimes f & \longmapsto & F^*(a) f \end{array}$$

$$\Longrightarrow F_* \Omega^i_{A/R} = A^{(p)} \otimes_{\mathbb{F}_p} \underbrace{\bigoplus_{I, J} \mathbb{F}_p \cdot t^I dt_J}_{K(n)^i}$$

$$I = (i_1, \dots, i_n) \in [0, p-1]^n$$

$$J = (1 \le j_1 < \dots < j_i \le n)$$

$$t^I = t_1^{i_1} \cdots t_n^{i_n}, \quad dt_J = dt_{j_1} \wedge \dots \wedge dt_{j_i}$$

$\to$ get cx $K(n)^\bullet$ s.t.

$$F_* \Omega^\bullet_{A/R} = A^{(p)} \otimes_{\mathbb{F}_p} K(n)^\bullet$$

$$\Rightarrow H^i(F_* \Omega^\bullet_{A/R}) = A^{(p)} \otimes_{\mathbb{F}_p} H^i(K(n)^\bullet)$$

<u>Thus to show:</u>

$$\Omega^i_{A^{(p)}/R} = \bigoplus_J A^{(p)} dt_J \longrightarrow H^i(F_* \Omega^\bullet_{A/R}) \cong A^{(p)} \otimes H^i(K(n)^\bullet)$$

$$a\, dt_J \longmapsto F^*(a)\, t_J^{p-1} dt_J \to a \otimes t_J^{p-1} dt_J.$$

$\Rightarrow$ to show

1) $H^0(K(n)) = \mathbb{F}_p$

2) $H^1(K(n)) = \bigoplus_{j=1}^{n} \mathbb{F}_p \, t_j^{p-1} dt_j$

3) $H^i(K(n)) = \Lambda^i H^1(K(n))$

Using $K(n) = \underbrace{K(1) \otimes_{\mathbb{F}_p} \dots \otimes_{\mathbb{F}_p} K(1)}_{= n\text{-times}}$

and $H^i(K(n)) = \bigoplus_{i_1+\dots+i_n = i} H^{i_1}(K(1)) \otimes_{\mathbb{F}_p} \dots \otimes_{\mathbb{F}_p} H^{i_n}(K(1))$ ("Künneth")

it suff to show:

1') $H^0(K(1)) = \mathbb{F}_p$

2') $H^1(K(1)) = \mathbb{F}_p \, t^{p-1} dt$

$K(1): \bigoplus_{i=1}^{p-1} \mathbb{F}_p t^i \longrightarrow \bigoplus_{i=1}^{p-1} \mathbb{F}_p t^i dt \quad , \quad t^i \longmapsto \begin{cases} 0 & i = 0 \\ i\, t^{i-1} dt & i \geq 1 \end{cases}$

$\Rightarrow$ 1') + 2') $\square$