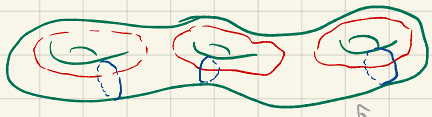
Lecture 10 — Abel–Jacobi maps & geometric applications

today: Hodge structures for geometric applications.

Abel–Jacobi map



counts loops

C alg. curve genus g: $\qquad H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$

sm. proj. cplx.

$H^0(C,\Omega^1) \cong \mathbb{C}^g$ — counts global holomorphic differentials

geom. genus, holomorphic differentials

$\rightsquigarrow$ pairing $H_1(C,\mathbb{Z}) \times H^0(C,\Omega^1) \longrightarrow \mathbb{C} : (\gamma, \omega) \longmapsto \int_\gamma \omega$

embeds $H_1(C,\mathbb{Z}) \hookrightarrow H^0(C,\Omega^1)^\vee$ as lattice

$$\boxed{\operatorname{Jac}(C) = H^0(C,\Omega^1)^\vee / \operatorname{Im} H_1(C,\mathbb{Z})}$$

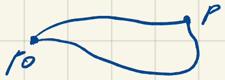
Jacobian of curve C.

choose $p_0 \in C$, basis $\omega_1, \dots, \omega_g \in H^0(C,\Omega^1)$ & def

Abel-Jacobi-map $\mu : C \longrightarrow \operatorname{Jac}(C) : p \longmapsto \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right)$

think of $\int_{p_0}^{p} -$ as linear form $H^0(C,\Omega^1) \to \mathbb{C}$.

different paths



form a loop $\Rightarrow$ well-def$^{\underline{d}}$ up to $\operatorname{Im} H_1(C,\mathbb{Z})$

[depends on p_0, but only up to translation by $[p]-[p_0]$ in $\operatorname{Jac}(C)$]

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aside: Period / elliptic integrals

integrals like $\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$ (from arc-length of ellipse)

or $\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$ (from period of pendulum)

can be rewritten as $\int_{p_0}^{p} \frac{dx}{y}$ for ell. curve $y^2 = x^3 + ax^2 + bx + c$

invariant diff. form on ell. curve.

integral over closed loops = periods = entries of period matrix

$$\left(\int_{\gamma_j} \omega_i \right) \in H^0(C,\Omega^1)^\vee$$

The Abel-Jacobi theorem

C sm. proj. alg. curve, $c_0 \in C$
genus g

extend Abel-Jacobi map $\mu : C \longrightarrow \mathrm{Jac}(C)$

to $AJ : \mathrm{Div}^0(C) \longrightarrow \mathrm{Jac}(C) : \sum n_i([x_i]-[c_0]) \longmapsto \sum n_i \mu(x_i)$

$\ker\left(\bigoplus_{x \in C} \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z} \right)$ degree-0 divisors   by linearity

① Abel's thm

$D = \sum n_i [x_i]$ w/ $\sum n_i = 0$

D is principal, ie. $D = \mathrm{div}(f)$ for meromorphic $f : C \longrightarrow \mathbb{P}^1$

iff $AJ(D) = 0$ in $\mathrm{Jac}(C)$

$\Longleftrightarrow$ AJ descends to injective map $\mathrm{Cl}^0(C) \longrightarrow \mathrm{Jac}(C)$

$\mathrm{Cl}^0(C) = \mathrm{Div}^0(C) / \text{principal div.}$

② Jacobi inversion thm.

every pt in $\mathrm{Jac}(X)$ is of the form $AJ(D)$ for some $D = \sum_{i=1}^{g} n_i([x_i]-[c_0])$

$\hat{=}$ $AJ : \mathrm{Cl}^0(C) \rightarrow \mathrm{Jac}(C)$ is surjective.

Consequence: $\mathrm{Jac}(C)$ parametrizes degree-0 divisors on C

even better: $\mathrm{Jac}(C)$ has algebraic structure (abelian variety of dim g)

and is birational to $\mathrm{Sym}^g(C) = C^{\times g} / \Sigma_g$ g-th symmetric power

example: E elliptic curve

by Abel-Jacobi thm ($+\,\varepsilon$) $E \longrightarrow \mathrm{Jac}(E) : p \longmapsto [p]-[0]$

is an isomorphism of alg. varieties

$\rightsquigarrow$ analytic construction of gp law

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rem: not all complex tori are algebraic,

not all abelian varieties are Jacobians of curves

Θ : theta divisor / polarization of ab. var.

— is the coh. class $\frac{1}{(g-2)!} \underbrace{\Theta \wedge \cdots \wedge \Theta}_{g-2}$ that looks like

image of $\mathrm{Sym}^2(C) \rightarrow \mathrm{Jac}(C)$ actually algebraic ?

Intermediate Jacobians: complex tori from Hodge structures

X sm. proj. cplx. var $\Rightarrow$ Hodge decomposition / filtration

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}}$$

$$F^r H^k(X,\mathbb{C}) = \bigoplus_{p\geq r} H^{p,k-p}(X)$$

for odd-degree cohomology

$$H^{2k-1}(X,\mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}$$

this means $\overline{F^k H^{2k-1}}$ has trivial intersection w/ $H^{2k-1}(X,\mathbb{R})$

$$H^{2k-1}(X,\mathbb{R}) \xrightarrow{\simeq} H^{2k-1}(X,\mathbb{C}) / F^k H^{2k-1}(X,\mathbb{C})$$

ISO of real vector spaces but then $H^{2k-1}(X,\mathbb{R})$ acquires cplx structure.

image of $H^{2k-1}(X,\mathbb{Z})$ is lattice $\left(\operatorname{rk} H^k(\mathbb{Z}) = \dim H^k(\mathbb{R}) \right)$

Def k-th intermediate Jacobian

$$\boxed{\operatorname{Jac}^k(X) := H^{2k-1}(X,\mathbb{C}) / F^k H^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})}$$

complex torus assoc. to sm. proj. var or more generally pure Hodge structure odd wt

Rem: can use Poincaré duality to interpret as $H_{2n-2k-1}(X,\mathbb{Z}) \to F^{n-k+1}H^{2n-2k+1}(X)^\vee$ in terms of integrals.

then recover previous def for $k=1$: $F^1H^1 = H^{1,0} = H^0(X,\Omega^1)$

aside: Cycle class map & higher Abel-Jacobi maps

X smooth variety $/\mathbb{C}$ (detecting homologically trivial cycles)

cycle class maps

$$CH^n(X) \longrightarrow H^{2n}(X(\mathbb{C}),\mathbb{Z}) : Z \subseteq X \longmapsto \text{Poincaré dual of } Z(\mathbb{C}) \subseteq X(\mathbb{C})$$

$\shortparallel$

codim n alg. subvarieties mod rat. equivalence

classes in kernel are homologically trivial cycles.

homologically trivial cycles detected by Abel-Jacobi maps

$$AJ: CH^n_{hom}(X) \longrightarrow \operatorname{Jac}^{2n-1}(X)$$

(expl: $CH^1 \to H^2$ for curve C is degree map, $\Rightarrow$ homologically triv cycles = deg 0 divisors)

aside: cycle class maps: motivic cohomology vs. Hodge theory

Bloch cycle class map $\quad CH^p(X,n) \longrightarrow H_D^{2p-n}(X, \mathbb{Z}(p))$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{R}{H^{2p-n}_{mot}}(X,\mathbb{Z}(p))$

Deligne cohomology: sheaf hypercohom. of

$\mathbb{Z}(p)_D : 0 \to (2\pi i)^p \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0$

$$0 \to \underbrace{\frac{H^{2p-n-1}(X,\mathbb{C})}{F^p H^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Z}(p))}}_{\text{intermediate Jacobian}} \to H_D^{2p-n}(X,\mathbb{Z}(p)) \to \underbrace{H^{2p-n}(X,\mathbb{Z}(p)) \cap F^p}_{\text{integral Hodge classes}} \to 0$$

philosophy: interpret Bloch cycle class map as maps $\mathrm{Ext}^1_{MM} \longrightarrow \mathrm{Ext}^1_{MHS}$

intermediate Jacobians of Hodge structure as $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(\,), H)$

Non-rationality of cubic 3-folds

cubic 3-fold $= V(F) \subseteq \mathbb{P}^4$ for deg 3 homog. polynomial F

e.g. $v^2w + w^2x + x^2y + y^2z + z^2v = 0$.

intermediate Jacobian:

$$\mathrm{Jac}^2(X) = H^3(X,\mathbb{C}) \big/ \left(H^{2,1} \oplus H^3(X,\mathbb{Z})\right)$$

Hodge-diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \end{array}$$

Clemens–Griffiths:

1. $\mathrm{Jac}^2(X) \simeq \mathrm{Jac}^1_{Pic}(S) \simeq \mathrm{Jac}^2_{Alb}(S)$

   for $S = \{ \underbrace{\ell \in Gr(2,5)}_{\text{lines in } \mathbb{P}^4} \mid \ell \subseteq X \}$ — Fano surface of lines on X

2. $\mathrm{Jac}^2(X)$ is not the Jacobian of a curve

3. if $X \sim \mathbb{P}^3$ birational, then $\mathrm{Jac}^2(X)$

   is Jacobian of curve

   (possibly reducible, comes from blowup centers in factorization of birational maps)

$\rightsquigarrow$ X not rational