Differential forms & algebraic de Rham cohomology, WS23/24

Lecture 1 — Introduction: differential forms & de Rham complex on manifolds.

Smooth manifolds & differential forms

more details: Tu: introduction smooth mflds or my lecture notes on mflds & diff forms (in German)

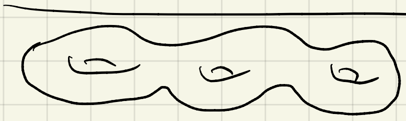
def smooth manifold of dim. n: top. space M + atlas $\{(U_i, \varphi_i : U_i \to \mathbb{R}^n)\}$

s.th. $\bigcup U_i = M$, $\varphi_i : U_i \to \mathbb{R}^n$ is homeo onto open, & $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is C^∞.

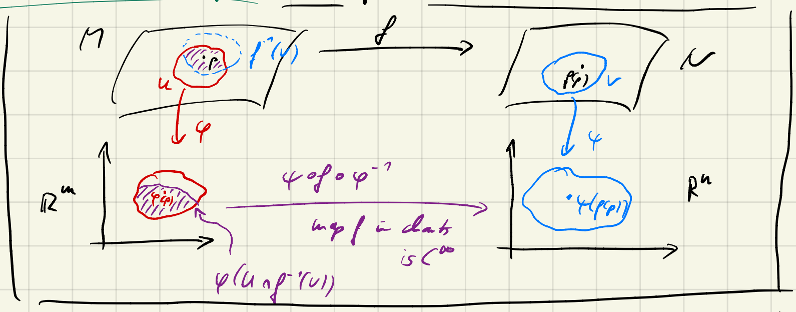
($\to$ locally euclidean, dim well def?)

expl: S^n w/ charts from stereographic projection, $\mathbb{R}P^n$ w/ charts $\{[x_0 : \dots : x_n] \mid x_i \neq 0\} \leftrightarrow \left(\frac{x_0}{x_i}, \dots, \hat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$

Grassmanians $Gr(k,n)$, closed oriented surfaces Σ_g



def smooth maps: $f : M \to N$ of C^∞ in charts

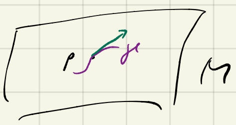


expl:
- polynomially defined maps
- compositions
- Hopf map $\eta : S^3 \to S^2$, $S^3 \subset \mathbb{C}^2 \setminus 0$, $S^2 \cong \mathbb{C}P^1$, $(z,w) \longmapsto [z:w]$

def tangent vectors & bundle: sm mfld M, $p \in M$,

2 equivalent definitions for tangent space T_pM

equivalence classes of smooth curves $\gamma : (a,b) \to M$ through p. (equivalent if define same directional derivative on smooth functions at p)



derivations $D_v : C^\infty_p(M) \to \mathbb{R}$ (stalk of smooth functions at p)

$\mathbb{R}$-linear maps s.th. $D(ab) = aD(b) + D(a)b$

expl for $v \in \mathbb{R}^n$: $D_v : f \longmapsto D_v f = \sum v^i \frac{\partial f}{\partial x^i}(p)$.

for each $p \in M$ have tangent space T_pM
$\rightsquigarrow$ combine into tangent bundle TM

(vector bundle $E \to M$
loc. triv $E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{R}^n$ over U_i for covering $\bigcup U_i = M$
& change of trivialization on $U_i \cap U_j$ linear)

def differential of smooth map

$f : M \to N$ induces $df : T_pM \to T_{f(p)}N$ / bundle maps $df : TM \to TN$

by composition w/ f. (images of tangent vectors or directional derivatives)

Key example: $f : M \to \mathbb{R}$ smooth

differential $df : TM \to T\mathbb{R}$, viewed as section $df : M \to TM^\vee$

$p \longmapsto df = \sum \frac{\partial f}{\partial x^i} dx^i$

$\Omega^k(M)$ & de Rham complex

def differential forms smooth sections of exterior power of cotangent bdl

$$\boxed{\Omega^k(M) = \Gamma(M, \Lambda^k T^\vee M)}$$

(for vb $E \to M$ can perform linear algebra constructions "fiberwise")

(section $\sigma: M \to E$ smooth map s.th. $M \xrightarrow{\sigma} E \xrightarrow{p} M$ is identity)

expl: df as 1-form

local description in charts: $(U, x^1, \dots, x^n)$ chart $\rightsquigarrow$ basis dx^i of cotangent space

$dx^I := dx^{i_1} \wedge \dots \wedge dx^{i_k}$ for $1 \leq i_1 < \dots < i_k \leq n$ basis of $\Lambda^k T^\vee M$

can write $\boxed{\omega = \sum_{I \subseteq \{1, \dots, n\}, \#I = k} a_I \, dx^I}$ w/ a_I smooth functions.

wedge product: $\wedge: \Omega^k(M) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M)$ (product in exterior algebra)

$\rightsquigarrow (\Omega^\bullet(M), \wedge)$ assoc. & graded-commutative $\mathbb{R}$-alg.

$\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$ for $\alpha \in \Omega^i, \beta \in \Omega^j$

exterior derivative: there exists a unique $\mathbb{R}$-linear map

$$\boxed{d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

s.th.:
1. d is graded derivation $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$, $\alpha \in \Omega^k, \beta \in \Omega^\ell$
2. $d^2 = 0$
3. $d: \Omega^0(M) \longrightarrow \Omega^1(M)$ is differential from before.

smooth functions

in chart $(U, x^1, \dots, x^n)$: $\omega = \sum a_I \, dx^I \rightsquigarrow d\omega = \sum da_I \wedge dx^I$

de Rham complex: for smooth mfld M of dim n

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \dots \to \Omega^n(M) \to 0$$

complex of $\mathbb{R}$-vector spaces, i.e. $\boxed{d^2 = 0}$

de Rham cohomology: cohomology of this complex

$$H^i_{dR}(M) = \frac{\ker(d: \Omega^i(M) \to \Omega^{i+1}(M))}{\operatorname{Im}(d: \Omega^{i-1}(M) \to \Omega^i(M))}$$

def $H^i = \ker / \operatorname{Im}$ for cplx in general

terminology: closed form $d\omega = 0$,
exact form $\omega = d\nu$

Rmk: for open $U \subseteq \mathbb{R}^3$, can rewrite de Rham cplx as

$$0 \to C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{rot}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U) \to 0$$

smooth vector fields

encodes all classical vector analysis.

but needs euclidean metric & Hodge star.

Integration, Poincaré lemma & Stokes thm

(✓ Katz history of Stokes' thm.) not due to Stokes

integral for function on open $U \subseteq \mathbb{R}^n$: $f: U \to \mathbb{R} \rightsquigarrow \int_U f\, dx^1 \wedge \dots \wedge dx^n$

orientation of smooth mfd M: better w/ cpt $A \subseteq U$

choice of **volume form**, ie. nowhere vanishing section $vol \in \Omega^{\dim M}(M)$.

($\rightsquigarrow$ for any open $U \in M$, can write n-form uniquely as $C^\infty(U)$-multiple of volume form $\omega = f \cdot vol$, then integrate f as above)

integral of n-form ω w/ cpt support over orientable mfd M of dim n: $\int_M \omega$

piece together integrals in local charts via partition of unity

(volume form removes choices so that everything well-def^d)

Stokes theorem:

M smooth, oriented n-dim mfd w/ boundary

$\omega \in \Omega_c^{n-1}(M)$ sm. $n-1$ form w/ cpt support

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

special case

$$\int_{[a,b]} f\,dx = \int_{[a,b]} dF = \int_{\partial[a,b]} F = F(b) - F(a)$$

reduce general case to this.

Consequence for cpt oriented n-dim mfd w/ $\partial M = \emptyset$

$$\int_M : H^n_{dR}(M) \longrightarrow \mathbb{R} \quad \text{well def}^d\ \mathbb{R}\text{-linear surjection.}$$

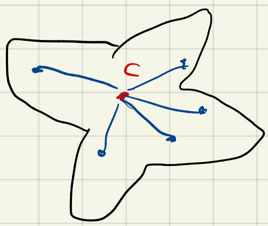
Poincaré duality: perfect pairing $\Omega^k(M) \times \Omega^{n-k}(M) \longrightarrow \mathbb{R}$

$$\omega, \eta \longmapsto \int_M \omega \wedge \eta$$

Poincaré-lemma:

$U \subseteq \mathbb{R}^n$ star-shaped:

then $H^i_{dR}(U) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$



Pf construct **chain homotopy** between id & $\Omega^k(U) \longrightarrow \Omega^k(U)$

$$\begin{array}{ccccc} \Omega^k(U) & & \longrightarrow & & \Omega^k(U) \\ \downarrow & & & & \cup\!\!\mid \\ \Omega^0(U) & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^0(U) \\ & f \longmapsto f(c) & & & \end{array}$$

eval. @ center pt. inclusion of const.

$$\eta : \Omega^p(U) \longrightarrow \Omega^{p-1}(U)$$

s.th. $\eta \circ d + d \circ \eta = id - \chi$.

($\rightsquigarrow$ this implies that id & χ induce the same maps on cohom. & that $\mathbb{R} \xrightarrow{\text{const.}} \Omega^k(U)$ induces isos on cohomology)

take $H: U \times [0,1] \longrightarrow U$ contraction of U to center point.

$$\eta(\omega) = \int_0^1 i_{\partial t} H^*(\omega)\, dt$$

contraction w/ vector field ∂_t on $[0,1]$

pullback of ω along H to $U \times [0,1]$