21. October 2015

Exercise sheet 2
Elliptic Curves [1]

Kay Rülling

Recall: Let k be a field and $S = k[X, Y, Z]$ the polynomial ring. Denote by S_d the subgroup of degree d polynomials in S. Then $S = \bigoplus_{d\geq 0} S_d$ is a graded k-algebra. For $n \in \mathbb{Z}$ we denote by $S(n)$ the graded S-module, whose degree d part is given by $S(n)_d := S_{n+d}$. We denote by $\mathcal{O}_{\mathbb{P}^2_k}(n)$ (or just $\mathcal{O}(n)$) the sheaf of $\mathcal{O}_{\mathbb{P}^2_k}$-modules on $\mathbb{P}^2_k = \operatorname{Proj} S$, which is associated to $S(n)$ (see e.g. [Hartshorne, p. 116]). In particular, for $f \in S_d$ we have

$$\mathcal{O}(n)(D_+(f)) = S(n)_{(f)} = \{s/f^r \mid r \in \mathbb{N}_0, s \in S_{n+dr}\}$$

and for $g \in S_e$ the restriction map to $D_+(fg)$ is given by

$$\mathcal{O}(n)(D_+(f)) \to \mathcal{O}(n)(D_+(fg)), \quad s/f^r \mapsto g^r s/(fg)^r.$$

Exercise 2.1. Show that $H^i(\mathbb{P}^2_k, \mathcal{O}(n)) = 0$ for all $i \neq 0, 2$, $n \in \mathbb{Z}$, and

$$\dim_k H^0(\mathbb{P}^2_k, \mathcal{O}(n)) = \begin{cases} 0 & \text{if } n < 0 \\ \frac{(n+1)(n+2)}{2} & \text{else,} \end{cases}$$

and

$$\dim_k H^2(\mathbb{P}^2_k, \mathcal{O}(n)) = \dim_k H^0(\mathbb{P}^2_k, \mathcal{O}(-n-3)), \quad n \in \mathbb{Z}.$$

(*Hint:* Use the Cech complex with respect to the standard open cover of $\mathbb{P}^2_k$.)

Exercise 2.2. Let X be a scheme. We call an $\mathcal{O}_X$-module $\mathcal{L}$ *invertible* if there exists an open cover $X = \cup_i U_i$ and isomorphisms of $\mathcal{O}_{U_i}$-modules $\varphi_i : \mathcal{L}_{|U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i}$. In this case we say that the cover $\{U_i\}$ trivializes the sheaf $\mathcal{L}$ (In the following we write $U_{i,j} = U_i \cap U_j$ etc.)

(1) Given an invertible sheaf $\mathcal{L}$, a cover $X = \cup U_i$ and isomorphisms $(\varphi_i)_i$ as above, show that there are units $u_{i,j} \in O(U_{i,j})^\times$ such that $u_{i,j} = \varphi_{j|U_{i,j}}(\varphi^{-1}_{i|U_{i,j}}(1))$ and for all i, j, k

$$(*) \quad u_{i,j|U_{i,j,k}} u_{j,k|U_{i,j,k}} = u_{i,k|U_{i,j,k}}.$$

[1]This exercise sheet will be discussed on October 28. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `l.zhang@fu-berlin.de`

(2) Let $X = \cup_i U_i$ be an open cover and assume we are given $u_{i,j} \in \mathcal{O}(U_{i,j})$ which satisfy (*) above. Show that

$$U \mapsto \mathcal{L}(U) := \{(s_i) \in \prod_i \mathcal{O}(U \cap U_i) \mid s_{j|U\cap U_{i,j}} = u_{i,j|U\cap U_{i,j}} \cdot s_{i|U\cap U_{i,j}}\}$$

with restriction maps for $V \subset U$ given by $\mathcal{L}(U) \to \mathcal{L}(V)$, $(s_i) \mapsto (s_{i|V\cap U_i})$ defines an invertible sheaf $\mathcal{L}$ on X which is trivialized by the cover $\{U_i\}$.

(3) Let $v_{i,j} \in \mathcal{O}(U_{i,j})^\times$, i, j, be another family of units satisfying (*) and giving rise via (2) to an invertible sheaf $\mathcal{M}$. Show that there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{L} \cong \mathcal{M}$ if and only if there exists units $w_i \in \mathcal{O}(U_i)^\times$ with $u_{i,j}/v_{i,j} = w_{i|U_{i,j}}/w_{j|U_{i,j}}$.

Exercise 2.3. Let $\mathbb{P}^2_k = \operatorname{Proj} k[X_0, X_1, X_2] = U_0 \cup U_1 \cup U_2$ be the standard cover, i.e. $U_i = U_+(X_i)$.

(1) Show that for any $n \in \mathbb{Z}$ the sheaf $\mathcal{O}_{\mathbb{P}^2_k}(n)$ is isomorphic to the invertible sheaf constructed via 2.2, (2) with respect to the units $u_{i,j} = X_i^n/X_j^n \in \mathcal{O}(U_{i,j})^\times$.

(2) Let $F \in k[X_0, X_1, X_2]$ be a homogenous polynomial of degree n defining $C = \operatorname{Proj} k[x_0, X_1, X_2]/(F)$. Show that there is a unique subsheaf $\mathcal{O}(-C)$ of $\mathcal{O}_{\mathbb{P}^2_k}$ which on U_i is given by

$$\mathcal{O}(-C)_{|U_i} = \mathcal{O}_{U_i} \cdot \frac{F}{X_i^n}.$$

(3) Show that there is an exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_{\mathbb{P}^2_k} \to \mathcal{O}_C \to 0,$$

i.e. $\mathcal{O}(-C)$ is the ideal sheaf of the embedding $C \hookrightarrow \mathbb{P}^2_k$.

(4) Show that there is an isomorphism $\mathcal{O}(-C) \cong \mathcal{O}(-n)$.

Exercise 2.4. Let $F \in k[X, Y, Z]$ be a homogenous polynomial of degree $n \geq 1$ and set $C = \operatorname{Proj} k[X, Y, Z]/(F)$. Show that

$$\dim_k H^1(C, \mathcal{O}_C) = \frac{(n-2)(n-1)}{2}.$$

(*Hint:* Use the short exact sequence from 2.3, (3), the associated long exact sequence in cohomology and the computation from exercise 2.1.)