Reciprocity sheaves and their cohomology

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Reciprocity sheaves

(following Kahn-Saito-Yamazaki)

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- projective bundle formula
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- action of proper Chow correspondences
- etc.

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    What a pitty!

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- so far no pole order/ ramification filtration

The idea of reciprocity sheaves is to consider only sheaves whose sections behave in a controlled way at infinity $\rightsquigarrow$

Idea (Kahn 1990's)

Replace $\mathbf{A}^1$-invariance by a modulus condition as the one used by Rosenlicht-Serre to define the generalized Jacobian for curves

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$\rightsquigarrow$ a factors via $U \to \mathrm{Alb}(C, D)$ (dep. on $x \in U(k)$ with a(x)=0)

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- $\Gamma \in \mathsf{Cor}(\mathbf{P}^1 \setminus \{1\}, U)$ prime correspondence, such that

$$\{1\}_{|\Gamma^N} \geq D_{|\Gamma^N}$$

  where $\Gamma^N \to \mathbf{P}^1 \times C$ normalization of closure of Γ
- $\gamma := i_0^* \Gamma - i_\infty^* \Gamma \in \mathsf{Cor}(\operatorname{Spec} k, U)$

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- monoidal structure: $(X, D) \otimes (Y, E) = (X \times Y, p_X^* D + p_Y^* E)$

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- Note: $\mathbb{Z}_{\mathrm{tr}}(\overline{X} \setminus |D|) \twoheadrightarrow \underline{\omega}_! h_0^{\overline{\square}}(\overline{X}, D) \twoheadrightarrow h_0^{\mathbf{A}^1}(X \setminus |D|)$

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- Set $\mathbf{RSC}_{\mathrm{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$ = category of reciprocity sheaves

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- $W_n\Omega^j$ (see second lecture for this and more examples)

Cube invariant sheaves

$F \in \mathsf{RSC} \rightsquigarrow$
$\widetilde{F}(X, D) := \{a \in F(X \setminus |D|) \mid a \text{ has modulus } \underbrace{(\overline{X}, \overline{D} + N \cdot B)}_{\text{comp. of } (X,D)}, N \gg 0\}$

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- (M-reciprocity) $\widetilde{F}(X, D) = \varinjlim_N \widetilde{F}(\overline{X}, \overline{D} + N \cdot B)$
- (semi-purity) $\widetilde{F}(X, D) \subset \widetilde{F}(X \setminus |D|, \emptyset)$

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$\widetilde{F}(X,D) := \{a \in F(X \setminus |D|) \mid a \text{ has modulus } \underbrace{(\overline{X}, \overline{D} + N \cdot B)}_{\text{comp. of } (X,D)}, N \gg 0\}$

Then $\widetilde{F} \in \underline{\mathrm{M}}\mathbf{PST}$ and

- (Cube invariance) $\widetilde{F}(\mathcal{X} \otimes \overline{\square}) = \widetilde{F}(\mathcal{X})$
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$\mathbf{CI}^{\tau} =$ cat of $G \in \underline{\mathrm{M}}\mathbf{PST}$ with cube-invariance and M-reciprocity

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$\rightsquigarrow \underline{\mathrm{M}}\mathrm{NST}$

Note: $\underline{\omega}_!$ restricts to $\underline{\omega}_! : \underline{\mathrm{M}}\mathrm{NST} \to \mathrm{NST}$

Theorem (Kahn-Miyazaki-Saito-Yamazaki)

There exists a sheafification functor $\underline{a}_{\mathrm{Nis}} : \underline{\mathrm{M}}\mathrm{PST} \to \underline{\mathrm{M}}\mathrm{NST}$.

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Question

Does $(*)$ *stabilize for* $G \in \mathbf{CI}^{\tau,sp}$*?*

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$$\underline{a}_{\mathrm{Nis}}(\mathbf{CI}^{\tau,sp}) \subset \mathbf{CI}^{\tau,sp} \cap \underline{\mathbf{M}}\mathbf{NST} =: \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$$

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Proof (Cor):

Thm $\Rightarrow G := \underline{a}_{\mathrm{Nis}}(\widetilde{F}) \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \Rightarrow F_{\mathrm{Nis}} = \underline{\omega}_!(G) \in \underline{\omega}_!(\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}) = \mathbf{RSC}_{\mathrm{Nis}}$ $\square$

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Theorem (S. Saito)

$F \in \mathbf{RSC}_{\mathrm{Nis}} \implies$

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- Injectivity also proved before by Kahn-Saito-Yamazaki

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De Rham-Witt sheaves as reciprocity sheaves

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- $[-] : A \to W_n(A)$, $a \mapsto [a] = (a, 0, \ldots, 0)$ (multiplicative)

satisfying

- $W_1(A) = A$ (as ring)
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Witt vectors

$A = \mathbf{F}_p$-algebra

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- Any commutative unipotent F_p-group scheme $\subset \oplus_{n_i} W_{n_i}$

de Rham-Witt complex (Bl, De-Il, He-Ma)

The de Rham-Witt complex over an $\mathbf{F}_p$-scheme X is a pro-dga

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$W_\bullet\Omega^*_X$ = initial object in the category of pro-dga's as above

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  F lifts the inverse Cartier operator

Descriptions

- Bloch, Kato

$$W_\bullet\Omega^q_R \cong \underbrace{\mathrm{T}}_{p\text{-typical part}}\ \underbrace{\mathrm{S}}_{\text{symbolic part}}\ \mathrm{Ker}\left(K_{q+1}(R[T]/T^\bullet) \xrightarrow{T\mapsto 0} K_{q+1}(R)\right) \quad \text{as pro-object}$$

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- many more… (Hesselholt-Madsen, Cuntz-Deninger, Bhatt-Lurie-Mathew…)

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$$X/k \text{ sm} \qquad u : (X/W_n(k))_{\mathrm{crys}} \to X_{\mathrm{Zar}}$$
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$\rightsquigarrow$ proper pushforward (Gros): for $f : Y \to X$ proper in **Sm**

$$f_* : Rf_*W_n\Omega^j_Y \to W_n\Omega^{j-r}_X[-r], \qquad r = \text{rel-dim}(f)$$

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Remark: F, V, R, d are morphisms in $\mathbf{RSC}_{\mathrm{Nis}}$

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$+$ Voevodsky $\Rightarrow R^i\varepsilon_*(\mathbb{Q}/\mathbb{Z}(j)) \in \mathsf{RSC}_{\mathsf{Nis}}$ (all i, j)
In particular $X \mapsto \mathrm{Br}(X) = H^0(X, R^2\varepsilon_*(\mathbb{Q}/\mathbb{Z}(1)) \in \mathsf{RSC}_{\mathsf{Nis}}$

Computation of the modulus

(Saito-R)

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$\rightsquigarrow$ suffices to understand for all L the filtration

$$F(\mathcal{O}_L) \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) \subset \ldots \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \ldots \subset F(L)$$

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- (Criterion for level n)

  F has level n $\iff$ for all $X \in \mathbf{Sm}$

$$\left\{ a \in F(\mathbf{A}^1_X) \,\middle|\, \begin{array}{l} \rho_x^* a \in \widetilde{F}(\mathcal{O}^h_{\mathbf{P}^1_x,\infty}, \mathfrak{m}_\infty^{-1}), \\ \forall x \in X_{\leq n-1} \end{array} \right\} = F(X)$$

  where $\rho_x : \operatorname{Spec} \operatorname{Frac}(\mathcal{O}^h_{\mathbf{P}^1_x,\infty}) \to \mathbf{A}^1_x \hookrightarrow \mathbf{A}^1_X$

$\mathrm{char}(k) = 0$

Theorem

- $\widetilde{\Omega^j_{/\mathbb{Z}}}$ *has level* $j+1$ *and*

$$\widetilde{\Omega^j_{/\mathbb{Z}}}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \tfrac{1}{t^{n-1}} \cdot \Omega^j_{\mathcal{O}_L/\mathbb{Z}}(\log t), \quad \mathfrak{m}_L = (t)$$

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    - $\mathrm{Conn}^1_{int} \in \mathbf{RSC}_{\mathrm{Nis}}$ *has level 2 (resp. 1)*
    - $\widetilde{\mathrm{Conn}^1_{int}}(\overline{X}, D) =$ *iso-classes of integrable rank* 1 *connections on* X *whose non-log irregularity is bounded by* D

$\mathrm{char}(k) = p > 0$

In order to define the Albanese with modulus in higher dimension Kato-Russell (building on work of Brylinski, Kato, Matsuda) define

$$\mathrm{fil}_r^F W_n(L) := \sum_{j \geq 0} F^j \left(\mathrm{fil}_{r-1}^{\log} W_n(L) + V^{n-s}(\mathrm{fil}_r^{\log} W_s(L)) \right)$$

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In order to define the Albanese with modulus in higher dimension Kato-Russell (building on work of Brylinski, Kato, Matsuda) define

$$\mathrm{fil}^F_r W_n(L) := \sum_{j\geq 0} F^j \left(\mathrm{fil}^{\log}_{r-1} W_n(L) + V^{n-s}(\mathrm{fil}^{\log}_r W_s(L)) \right)$$

- $\mathrm{fil}^{\log}_r W_n(L) = \{(a_0, \ldots, a_{n-1}) \mid p^{n-1-i} v_L(a_i) \geq -r \ \forall i\}$
- $s = \min\{n, \mathrm{ord}_p(r)\}$

Theorem

W_n has level 1 and

$$\widetilde{W_n}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \mathrm{fil}^F_r W_n(L)$$

In particular $\widetilde{\mathbb{G}_a}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \begin{cases} \mathcal{O}_L & r \leq 1 \\ \sum_j F^j(\frac{1}{t^{r-1}}\mathcal{O}_L) & (p, r) = 1 \\ \sum_j F^j(\frac{1}{t^r}\mathcal{O}_L) & p|r \end{cases}$

$\mathrm{char}(k) = p > 0$

Brylinski-Kato-Matsuda filtration on $H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z})$ defined by ($r \geq 1$)

$$\mathrm{fil}_r H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) := \bigoplus_{\ell \neq p} H^1_{\text{ét}}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \oplus \bigcup_n \mathrm{Im}(\mathrm{fil}^F_r W_n(L) \to H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}))$$

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Theorem

$\varepsilon : \mathrm{Sm}_{\text{ét}} \to \mathrm{Sm}_{\mathrm{Nis}} \implies$

$R^1\varepsilon_*\mathbb{Q}/\mathbb{Z} \in \mathrm{RSC}_{\mathrm{Nis}}$ *has level 1 and*

$$\widetilde{R^1\varepsilon_*\mathbb{Q}/}\mathbb{Z}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \mathrm{fil}_r H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z})$$

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Remark: We have $H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(G_L, \mathbb{Q}/\mathbb{Z})$

Yatagawa $\implies$

$$\mathrm{fil}_r H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(G_L/G_L^{r+}, \mathbb{Q}/\mathbb{Z})$$

where $\{G_L^j\}_{j \in \mathbb{Q}_{\geq 0}}$ = Abbes-T. Saito ramification filtration of G_L (decreasing) and $G_L^{r+} = \overline{\bigcup_{s>r} G_L^s}$

in progress, $\mathrm{char}(k) = p > 0$

Voevodsky + Geisser-Levine $\Longrightarrow$

- $R^i\varepsilon_*(\mathbb{Q}/\mathbb{Z}(j)) \in \mathbf{HI}_{\mathbf{Nis}}$ for $i \neq j+1$

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$\rightsquigarrow$ filtration which measures the difference to $\mathbf{A}^1$-invariance

$$H^j(\mathcal{O}_L) \subset \widetilde{H}^j(\mathcal{O}_L, \mathfrak{m}_L^{-1}) \subset \ldots \subset \widetilde{H}^j(\mathcal{O}_L, \mathfrak{m}_L^{-r}) \subset \ldots \subset H^j(L)$$

Above we determined $\widetilde{H}^1(\mathcal{O}_L, \mathfrak{m}_L^{-r})$

candidate for $\widetilde{H}^j(\mathcal{O}_L, \mathfrak{m}_L^{-r})$

- $r, n \geq 1$, set $s = \min\{n, \mathrm{ord}_p(r)\}$

$$\mathrm{fil}_r(W_n\Omega_L^{j-1}/B_\infty) :=$$
$$\mathrm{Im}\left(\mathrm{fil}_{r-1}^{\log} W_n(L) \otimes K_{j-1}^M(L) \oplus V^{n-s}(\mathrm{fil}_r^{\log} W_s(L)) \otimes K_{j-1}^M(\mathcal{O}_L) \to W_n\Omega_L^{j-1}/B_\infty\right)$$

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Proposition

- $\widehat{H}^j \in \mathrm{CI}^{\tau, sp}_{\mathrm{Nis}}$

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Proposition

- $\widehat{H}^j \in \mathbf{Cl}^{\tau,sp}_{\mathrm{Nis}}$
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- $\widehat{H}^j \subset \widetilde{H}^j$

candidate for $\widetilde{H}^j(\mathcal{O}_L, \mathfrak{m}_L^{-r})$

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Remark: Kato defined a filtration $\mathrm{fil}^K_r H^j(L)$, which satisfies

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Trosors under finite groups in char(k) $= p > 0$

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Tensor products and twists

Lax monoidal structure on $\mathrm{RSC}_{\mathrm{Nis}}$

- $F, G \in \mathrm{RSC}_{\mathrm{Nis}} \rightsquigarrow$ define

$$(F, G)_{\mathrm{RSC}_{\mathrm{Nis}}} := \underline{\omega}_!(h_0^{\overline{\square}}(\widetilde{F} \otimes_{\underline{\mathrm{M}}\mathrm{PST}} \widetilde{G}))_{\mathrm{Nis}} \in \mathrm{RSC}_{\mathrm{Nis}}$$

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Theorem (R-Sugiyama-Yamazaki)

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  - $(\mathbb{G}_a, \mathbb{G}_a)_{\mathrm{RSC}_{\mathrm{Nis}}}(X) = H^0(X, \mathcal{O}_{X \times_{\mathbb{Z}} X}/I^2_{\Delta_X})$

Twists

- $G \in \mathsf{CI}^{\tau,sp}_{\mathrm{Nis}}$ define
  - $G(n) := h_0^{\overline{\square}}(G \otimes_{\underline{\mathrm{M}}\mathrm{PST}} \widetilde{K^M_n})^{\mathrm{sp}}_{\mathrm{Nis}} \in \mathsf{CI}^{\tau,sp}_{\mathrm{Nis}}$

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  - $F\langle 1\rangle := (F, \mathbb{G}_m)_{\mathrm{RSC}_{\mathrm{Nis}}}$, $F\langle n\rangle := (F\langle n-1\rangle)\langle 1\rangle \in \mathrm{RSC}_{\mathrm{Nis}}$

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  - $\gamma^n F := \underline{\mathrm{Hom}}_{\mathrm{PST}}(K_n^M, F) \in \mathbf{RSC}_{\mathrm{Nis}}$

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Generalizes part of Voevodsky's cancellation theorem

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Cohomology of reciprocity sheaves

(Binda-R-Saito)

Nice pairs

We say

- $\mathcal{X} = (X, D)$ is an ls modulus pair or write $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ $\iff X \in \mathbf{Sm}$ and $|D|$ SNCD

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- $f : Y \to X$ in $\mathbf{Sm}$ is transversal to D
  $\iff f^{-1}(D_1 \cap \ldots \cap D_r) \hookrightarrow Y$ regular, closed, codim r,
  for all irred cpts $D_1, \ldots, D_r$ of $|D|$.

Projective bundle formula

Theorem

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- $(\mathrm{char}(k) = p > 0, D = \emptyset$, (also Gros))

$$R\pi_* \left(R^{j+1}\varepsilon_*\mathbb{Z}/p^r(j)\right)_P = \bigoplus_{i=0}^{n} \left(R^{j-i+1}\varepsilon_*\mathbb{Z}/p^r(j-i)\right)_X [-i]$$

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$$\begin{tikzcd}
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$\Rightarrow$ exact triangle

$$i_*\gamma^c G_{\mathcal{Z}}[-c] \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} G_{\mathcal{X}} \xrightarrow{\rho^*} R\rho_* G_{(\tilde{X}, D_{|\tilde{X}}+E)} \xrightarrow{\partial} i_*\gamma^c G_{\mathcal{Z}}[-c+1]$$

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$$\widetilde{\mathrm{Lisse}}^1(X, D) = \begin{array}{l}\text{lisse } \bar{\mathbb{Q}}_\ell\text{-sheaves of rank 1 on } X \setminus |D| \\ \text{with Artin conductor} \leq D\end{array}$$

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- pushforward to X (using the propernesss of support over X)

Explanation

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$\rightsquigarrow$ functor $\qquad C_S \to D(S_{\mathsf{Nis}}), \quad (f : X \to S) \mapsto Rf_* F$

Applications (BRS)

Obstructions for existence of zero cycles of degree 1

Theorem

$F \in \mathsf{RSC}_{\mathsf{Nis}}$ $\quad f : X \to S$ *proj, dom in* Sm, $\quad K = k(S)$

Assume $\exists\, \xi \in \mathsf{CH}_0(X_K)^{\deg 1}$

$\Longrightarrow f^* : H^i(S, F_S) \to H^i(X, F_X)$ *is split-injective.*

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and use

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generalized Brauer-Manin obstruction for zero cycles

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Theorem

$$\Psi((\alpha_v)_v) \circ \imath \neq 0 \Longrightarrow \nexists\, \alpha \in \mathrm{CH}_0(X_K) \textit{ with } \alpha \mapsto (\alpha_v)_v$$

Proof: take $\alpha \mapsto (\alpha_v)$, taking $\bar{\alpha} \in \mathrm{CH}_1(X)$ lifting α

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F(X_K) \arrow[r, "\imath"] \arrow[rr, bend left, "0"] & \bigoplus_v H^1_v(S, f_* F_X) \arrow[r] \arrow[d, "\sum_v \Phi(\alpha_v)"'] & H^1(X, F) \arrow[dl, "\bar{\alpha}^*"] \\
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Remark:

If $K = \mathsf{F}_q$ and $F = \mathrm{Br}$ then Ψ becomes (using CFT)

$$\prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\left(\bigoplus_{v \in S_{(0)}} \frac{\mathrm{Br}(X_{K_v})}{\mathrm{Br}(X_{S_v})}, \mathbb{Q}/\mathbb{Z} \right)$$

$\rightsquigarrow$ classical Brauer-Manin obstruction for zero-cycles

(in the function field case)

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- $(f : X \to S), (g : Y \to S) \in C_S, \quad X, Y$ integral

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Proof: pbf+ purity + correspondence action

Birational invariance of cohomology

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$(f : X \to S), (g : Y \to S) \in C_S$ properly birational$/S$.

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$\Rightarrow Z^*$ and $(Z^t)^*$ are inverse to each other $\square$

Example

Assume $\dim X = \dim Y = d \rightsquigarrow$ Theorem applies for $F =$

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- *there is a version of theorem with $F\langle 1\rangle \leftrightarrow \gamma F$ but in this we only get results if resolutions of singularities are available (in $\dim d$)*

Corollary

$S, X, Y \in \mathsf{Sm}$ $\quad X \to S$, $Y \to S$ flat, geom int, proj, gen fiber index 1

($\rightsquigarrow \mathrm{Pic}_{X/S}$, $\mathrm{Pic}_{Y/S}$ representable)

Assume X and Y are stably properly birational$/S$

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Remark: Was known at least for $S = \operatorname{Spec} k$ with k alg closed

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where $\xi \in \mathrm{CH}_0(X)$, $\beta \in \mathrm{CH}_d(Z \times_K X)$ with $\mathrm{codim}(Z, X) \geq 1$

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- this condition was first considered by Bloch-Srinivas (with $\otimes\mathbb{Q}$)

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- Implications of $(*)$ on cohomology yield obstructions for X being retract rational over K

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Thm $\rightsquigarrow$ Yes (if X/k proj)

Indeed in Thm take $S = \operatorname{Spec} k$ $F = R^i\varepsilon_*\mathbb{Z}/p(j)$ and observe $F(k) = 0$

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Example

k alg closed X/k sm proj $\dim X = d$ diagonal of X decomposes

$\Rightarrow H^i(X, R^{d+1}\varepsilon_*\mathbb{Z}/p^n(d)) = 0$ all i

Thank you!