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Exercise sheet 3 for Algebra II

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Exercise 1. (i) Show that $\mathbb{Z}[i] := \mathbb{Z}[x]/ < x^2+1 >$ and $\mathbb{Z}[\sqrt{-5}] := \mathbb{Z}[x]/ < x^2+5 >$ are domains (*Hint:* Use Exercise sheet 2.)

(ii) Show that $\mathbb{Z}[i]$ is an euclidean domain. In particular, it is a UFD.

(iii) Show that if $p \in \mathbb{Z}$ is a prime number, then $p \cdot \mathbb{Z}[i]$ is a prime ideal if and only if -1 is not a square in $\mathbb{Z}/ < p >$.

(iv) Show that $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible but not prime. Hence $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. (*Hint:* To show that 2 is not prime try to factor $6 \in \mathbb{Z}[\sqrt{-5}]$ in two different ways.)

Exercise 2. Let R be an integral domain and $R[X,Y]$ the polynomial ring in two variables with coefficients in R. Let $m, n \in \mathbb{Z}_{\geq 1}$ be positive integers.

(i) Show that the ideal $< X^m - Y^n >$ is prime in $R[X,Y]$ if and only if m and n are coprime, i.e. $< m, n >= \mathbb{Z}$.
(*Hint:* For the "if" direction: Show that the map $\varphi : R[X,Y] \to R[T]$, $f(X,Y) \mapsto f(T^n, T^m)$ is a ring homomorphism, which factors over a ring homomorphism $\bar{\varphi} : R[X,Y]/ < X^m - Y^n > \to R[T]$. Then show that $\bar{\varphi}$ is injective.)

(ii) Let K be a field. Show that $y = \bar{Y} \in K[X,Y]/ < Y^2 - X^3 >$ is irreducible but not prime.

Exercise 3. For a ring R we denote by $\operatorname{Spec} R$ its set of prime ideals. Let $\varphi : R \to R'$ be a ring homomorphism. From the lecture we know that this induces a map (of sets) $\varphi^{-1} : \operatorname{Spec} R' \to \operatorname{Spec} R$, $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$.

(i) Let $R_1, \dots, R_n$ be rings, denote by $R = R_1 \times \dots \times R_n$ their product and by $\pi_i : R \to R_i$, $(a_1, \dots, a_n) \mapsto a_i$, $i = 1 \dots, n$, the projection maps.

Show that $\pi_i^{-1} : \operatorname{Spec} R_i \to \operatorname{Spec} R$ maps bijectively onto $\pi_i^{-1}(\operatorname{Spec} R_i)$ and that we have the following decomposition of $\operatorname{Spec} R$ into disjoint sets

$$\operatorname{Spec} R = \pi_1^{-1}(\operatorname{Spec} R_1) \sqcup \dots \sqcup \pi_n^{-1}(\operatorname{Spec} R_n) \xleftarrow{\text{bij.}} \operatorname{Spec} R_1 \sqcup \dots \sqcup \operatorname{Spec} R_n.$$

(ii) Let $\pi : R \to R_{\mathrm{red}} := R/\mathrm{nil(R)}$ be the canonical surjection. Show that $\pi^{-1} : \operatorname{Spec} R_{\mathrm{red}} \to \operatorname{Spec} R$ is bijective.

Exercise 4. Let R be a reduced ring with minimal prime ideals $\mathfrak{p}_\lambda$, $\lambda \in \Lambda$. Show that the ring homomorphism $R \to \prod_{\lambda\in\Lambda} R/\mathfrak{p}_\lambda$, $x \mapsto (x \bmod \mathfrak{p}_\lambda)$ is injective.