Fourier Transform Methods in Option Pricing

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Abstract

In this essay, we presented a very fast and efficient method for pricing option. The method have been introduced by Peter Carr and Dilip Madan in 1999 to compute the option price numerically by using the fast Fourier transform. Based on the Black-Scholes model, we computed the European call option price numerically by modifying the option price function to enforce integrability and we calculated its Fourier transform by using the characteristic function of the underlying asset. By formulating an analytic expression of the Fourier transform of the modified value, we obtained the call price by Fourier inversion. We compared the analytic form of the option value under the Black-Scholes with the numerical one. We do not consider the analytic solution of the Variance Gamma because of the complexity of the algebra involved but we observed graphically that the magnitude of the error introduced by Fourier pricing under the Black-Scholes for which the fair price is known analytically is less than one percent.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Kouemo Tchamga Nicole Flaure, 22 May 2009
# Contents

Abstract i

## 1 Introduction 1

1.1 Notation and Modelling Assumptions 2

1.2 Theory of Arbitrage 3

## 2 Mathematical Models 4

2.1 Basic Concept 4

2.1.1 The Mathematics Elements 4

2.1.2 The Brownian motion 5

2.1.3 Random Walk of Asset Prices 5

2.1.4 Payoff Function 6

2.1.5 Itô’s Lemma 7

2.2 The Black-Scholes Model 7

2.2.1 The Black-Scholes PDE 7

2.2.2 Pricing in the Risk-Neutral World 9

2.2.3 The Distribution of Asset Prices 10

2.2.4 Option Pricing: The Black-Scholes Formula 13

2.3 The Variance Gamma Model 13

2.3.1 The Definition and the Characterisation of the Variance Gamma Model 13

2.3.2 The Gamma Distribution 14

2.3.3 The VG Distribution 14

2.3.4 VG Stock Price Model 17

## 3 The Numerical Valuation Methods with FFT 19

3.1 Review of Fourier Method in Option Pricing 19

3.2 The Characteristic Function 19

3.3 The Fourier Transform Method 20

3.3.1 Valuation Under The Black-Scholes Model 20

3.3.2 Valuation Under The Carr and Madan Method 21

3.3.3 Discrete Approximation 23
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.4</td>
<td>Valuation Under The Peter Tankov Model</td>
<td>24</td>
</tr>
<tr>
<td>3.3.5</td>
<td>Numerical Fourier Inversion</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>Application</td>
<td>27</td>
</tr>
<tr>
<td>4.1</td>
<td>Application to the Black-Scholes Model</td>
<td>27</td>
</tr>
<tr>
<td>4.2</td>
<td>FFT Error Behaviour in the Black-Scholes</td>
<td>28</td>
</tr>
<tr>
<td>4.3</td>
<td>Application to the Variance Gamma Model</td>
<td>28</td>
</tr>
<tr>
<td>5</td>
<td>Conclusion and Future Work</td>
<td>30</td>
</tr>
<tr>
<td>A</td>
<td>Some Python Codes</td>
<td>31</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>36</td>
</tr>
</tbody>
</table>
1. Introduction

Option pricing is one of the most important areas in financial mathematics and incorporates many different disciplines in mathematics. One such example is the use of the Fourier transform.

In finance, an option is a contract between a buyer (holder) and a seller (writer) that gives the buyer the right-but not the obligation-to buy or to sell a particular asset (the underlying asset) on a prescribed day at an agreed price. In return for granting the option, the seller collects a payment (the premium) from the buyer. A call option gives the buyer the right to buy the underlying asset; a put option gives the buyer of the option the right to sell the underlying asset. If the buyer chooses to exercise this right, the seller is obliged to sell or buy the asset at the agreed price. The buyer may choose not to exercise the right and let it expire. According to this definition, the asset in which the transaction may take place is known as the underlying. The prescribe date is called the maturity time or expiry. And the price at which the buyers may purchase or sell the underlying is called the exercise price or strike price.

The history of options dates back to hundreds of years ago. Paul Wilmott [WHd95] mentioned the stories of Thales in Greece who used the ancient type of options to secure a low price for olives in advance of the harvest. In The Netherlands, trading in tulip derivatives blossomed during the early 1600s. Although options were originally traded in the ‘over-the-counter’ (OTC) market, where the terms of the contract were customised or negotiated, option trading really took off when the first option exchange, the Chicago Board Options Exchange (CBOE) was organized in 1973 to trade standardised contracts, which greatly increased the market and liquidity of options. Trading in put options began later in 1977. Nowadays, the option market has grown to be so huge, that the size of the (OTC) market had reached $596 trillion by the end of June 2007 [BfIS09].

Usually, options have two primary usages, speculation and hedging. Speculative investors trade the options to make profit according to their judgments on the trends of the asset prices. For example, such an investor will choose to buy a call option if he thinks the underlying price is going to increase in the following months. If his forecast is correct, he makes money, otherwise he loses money. By contrast to buying the stock share, options can be a cheap way for investment when the stock price is much higher than the cost of the option. Hedging happens when investors want to minimize the risk that they may be subjected to because of the unpredictable events in the financial market, i.e. due to the random walk of the underlying price. In the market, asset holders can choose to buy put options if they think that the asset price is going to fall. If they are correct, the profit they make from the options will reduce the loss due to holding the asset. Therefore, this hedging strategy becomes an insurance against the adverse movements in the underlying.

In the financial market, options are classified into different categories according to the elementary concepts which make up those options. If the option gives the buyer the right to buy the underlying, it is a call option. By contrast, it is a put option if the buyer has the right to sell the underlying at (or by) the expiry. American options enable the buyers to exercise them before the expiry, and European options can only be exercised at the expiry. Call and put options are known as the plain vanilla options because they are basic. There are also some other complicated types of options. Asian options strike prices are prescribed as some form of the average of the underlying prices over a period. Lookback options depend on the maximum or minimum price. Barrier options can either come into existence or become worthless if the underlying asset reaches some prescribed value before expiry. Such complicated options, as Asian, lookback and barrier options are usually called exotic options.

Because options offers their buyers some privileges, they have some value. An option’s value should
be equal to its price when traded in the market, otherwise the arbitrage opportunity will appear. A
correct mathematical valuation of the option value is thus one of the targets in the research on options.
Roughly speaking, an options value is influenced by many factors. For example, a European call option
will have a positive value if the underlying assets price is higher than the strike price, otherwise it may
be profitless. Time to expiry plays also a role. An option with an expiry of 6 months has more value
than one which will expire tomorrow, because the underlying price may have more potential to change
over a longer period. There are also many other factors, such as the options type, the bank interest
rate, dividends and so on, which may influence an options value. The relationship between those factors
and the option value is described differently in various models.

In the theory of option pricing, there are some assumptions for modelling. There is no market friction,
no default risk and no arbitrage. This means: no transaction costs, no bid/ask spread, perfect liquid
markets, no taxes, no margin requirements, no restrictions on short sales, no transaction delays, market
participants act as price takers, market participants prefer more to less. The bank interest rate is
assumed to be fixed in short-term and does not change. The basic theory says, if the interest rate is \( r \)
and an amount of money \( B \) is deposited at time 0, its value will be \( Be^{rt} \) at time \( t > 0 \). Equivalently,
if we borrow \( Be^{-rt} \) currency units, we will have to pay back \( B \) currency units at time \( t \) later.

Many assets, such as equities, pay out dividends. The dividends can be seen as the payments to the
shareholders out of the profits made by the company. The likely dividend stream of a company in the
future is reflected by its current share price. But for the purpose of this works dividends will not be
considered.

This essay is focused on 2 different problems in quantitative finance. The first one is how to apply
the Fourier transform introduced by Carr and Madan [CM99] when computing option prices with the
Fourier transform method based on the Black-Scholes model and how to deduce the option pricing for
the Variance Gamma model. The second problem is to compare the analytic solution to the numerical
one and plot some of the result in order to appreciate the curve of the option price of both models.
Chapter 2 introduces the mathematical models and methods frequently applied in option pricing. In
chapter 3, we present the numerical valuation method a Fast Fourier transform. In chapter 4, the
implementation details and some plots results are presented.

1.1 Notation and Modelling Assumptions

We will introduce some important notations which we use throughout this essay. The option value
is denoted by \( V \). To make the distinction between call and put options value, \( C \) and \( P \) are used
respectively. To show that the option value is a function of the current value of the underlying asset
price \( S \) and time, \( t \), \( V \) is sometimes denoted as \( V(S,t) \). In general, standard option valuation models
depend on the following factors:

- \( S \), the current market price of the underlying asset;
- \( \sigma \), the volatility of the underlying asset or an estimate of the future volatility of the underlying
  asset’s price over the life of the option;
- \( K \), the exercise or the strike price of the option, particularly in relation to the current market price
  of the underlier;
Section 1.2. Theory of Arbitrage

- $T$, the expiry date or the time to expiration together with any restrictions on when exercise may occur;
- $r$, the interest rate;

The mathematical model follows the following assumptions [Ber07]:

1. No market friction: In the real world there are brokers, spread, taxes, margin requirements and so on. We do not consider these factors at all when giving a fair price for an option.
2. No dividends: for example many stocks regularly payout to their stockholders. Though obviously unrealistic, this assumption is fairly easily relaxed. For the purposes of this work, dividend will not be considered.
3. No arbitrage: This assumption is central to the modelling of market price. It is simply says that one cannot make a risk-free profit off the market. If for example, one were able to derive some method for making risk-free profits, it would be impossible for the market to be in equilibrium. We know this is not the case. Mathematically, the no arbitrage assumption provides us with a risk neutral probability measure, generally denoted $\mathbb{Q}$. It is because of the no arbitrage assumption that mathematical option pricing is often termed risk-neutral valuation.
4. Greed: Market participants want more money to less. This is probably the weakest assumption that the reader has encountered to date.

1.2 Theory of Arbitrage

Before moving on the mathematical model, let’s introduce one of the most important theory in option pricing [Gho09b]:

**Definition 1.2.1.** An arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of the capital.

The principle of no arbitrage states that a mathematical model of a financial market should not allow for the arbitrage possibilities.

A Portfolio or Trading Strategy is a pair of units $(a, b) \in \mathbb{R}^2$ such that

- $a$ units of the risky-asset
- $b$ units of the riskless (or risk-free) asset.
2. Mathematical Models

2.1 Basic Concept

2.1.1 The Mathematics Elements

Let’s consider a \((\Theta, F)\) where \(\Theta\) is the sample space (non-empty set) of the random variable, containing all scenario (events) and \(F\) is a \(\sigma\)-field or a collection of subsets of \(\Theta\) and where \(A \in F\) is the event. The following definitions are obtained from [Gho09a].

**Definition 2.1.1.** A \(\sigma\)-field \(F\) is a family of subsets of \(\Theta\) such that:

- The empty set is contained in \(F\); \(\emptyset \in F\)
- For all \(A \in F\), then \(A^c \in F\)
- \(F\) is closed under the operation of countable unions; If \(A_1, A_2, \ldots \in F\) then \(\bigcup_{n>1} A_n \in F\).

**Definition 2.1.2.** A filtration or informations flow on a time interval, say \([0, T]\), denoted \(\{F_t\}_{t \in [0, T]}\) on a probability space \((\Theta, F, P)\) is an increasing sequence of \(\sigma\)-fields containing information on the evolution of the price process up to time \(T\) such that, for all \(0 \leq s \leq t\) then \(F_s \subseteq F_t \subseteq F\).

A probability space \((\Theta, F, P)\) equipped with a filtration is called a filtered probability space and is denoted \((\Theta, F, \{F_t\}_{t \in [0, T]}, P)\).

**Definition 2.1.3.** Given a filtered probability space \((\Theta, F, \{F_t\}_{t \in [0, T]}, P)\) where \(\{F_t\}_{t \in [0, T]}\) is the natural filtration, A stochastic process is a family of \(P\), \(\{F_t\}_{t \geq 0}\)-adapted reals valued functions indexed by time, \(X_t \in [0, T]\) on \(\Theta\). \(\{X_t\}_{t \in [0, T]}\) is adapted to \(\{F_t\}_{t \in [0, T]}\) if \(\{X_t\}_{t \in [0, T]}\) is \(F_t\)-mesurable. For each realisation of randomness \(\omega\), the trajectory \(X(\omega) : t \rightarrow X_t(\omega)\) defines a function of time and is called a sample path. More formally, A stochastic process is a function

\[
X : [0, T] \times \Theta \rightarrow \mathcal{F}.
\]

It follows that stochastic processes are random functions taking values in functions spaces.

**Definition 2.1.4.** Given a filtered probability space \((\Theta, F, \{F_t\}_{t \in [0, T]}, P)\), the sequence of random variables \(\{X_t\}_{t \geq 0}\) is a Martingale with respect to \(P\) and the filtration \(\{F_t\}_{t \geq 0}\) if for all \(0 \leq s \leq t\):

- \(\mathbb{E}[|X_t|] < \infty\) for all \(t\)
- \(\mathbb{E}_P[X_t/X_s] = X_s\)

Otherwise,

\(\{X_t\}_{t \geq 0}\) is a \(P\)-super-martingale if \(\mathbb{E}_P[X_t/X_s] \leq X_s\) and a \(P\)-sub-martingale if \(\mathbb{E}_P[X_t/X_s] \geq X_s\).

**Definition 2.1.5.** A stochastic process \(\{X_t\}_{t \geq 0}\) with its natural filtration \(\{F_t\}_{t \geq 0}\) is a Markov process if for all \(0 \leq s \leq t\):

\[
\mathbb{P}(X_{t+s} \in A/F_t) = \mathbb{P}(X_{t+s} \in A/X_t)\quad \text{for all} \quad A \in F.
\]
2.1.2 The Brownian motion

**Definition 2.1.6.** A stochastic process \( B = \{B_t, t \geq 0 \} \) is a standard Brownian motion on some probability space \((\Theta, F, P)\), if

1. \( B_0 = 0 \) a.s.,
2. \( B \) has independent increments,
3. \( B \) has stationary increments,
4. \( B_{t+s} - B_t \) is normally distributed with mean 0 and variance \( s > 0 \): \( B_{t+s} - B_t \sim \mathcal{N}(0, s) \).

Note that the second item in the definition implies that Brownian motion is a Markov process. Moreover Brownian motion is the basic example of a Lévy process. In the above, we have defined Brownian motion without reference to a filtration. Without other notice, we will always work with the natural filtration \( F = F^B = \{F_t, 0 \leq t \leq T\} \) of \( B \). We have that Brownian motion is adapted with respect to this filtration and that increments \( B_{t+s} - B_t \) are independent of \( F_t \).

One can proof that Brownian motion has continuous paths, i.e. \( B_t \) is a continuous function of \( t \). However the paths of Brownian motion are very erratic. They are for example nowhere differentiable. Moreover, one can prove also that the paths of Brownian motion are of infinite variation, i.e. their variation is infinite on every interval.

In figure 2.1 one can see the realization of the standard Brownian motion.

![Figure 2.1: A sample path of the Standard Brownian motion](image)

2.1.3 Random Walk of Asset Prices

In the research on option pricing, the dynamics of the asset price is usually represented by its relative change, \( dS/S \), called return. The most common model, the geometric Brownian motion model (GBM), says that the return of the asset price is made up of two parts as \([WHd95]\).

\[
\frac{dS}{S} = \mu dt + \sigma dB \tag{2.1}
\]

where \( \mu \), known as the drift, marks the average rate of growth, and \( \sigma \) is called volatility that keeps the information of the standard deviation of the return. The first part \( \mu dt \) reflects a predictable, deterministic
and anticipated return which is similar to the return of investment in banks. The second part $\sigma dB$ simulates the random change in the asset price in response to external effects, such as uncertain events. The quantity $dB$ contains the information of the randomness of the asset price and is known as the Wiener process or Brownian motion. It is a random variable which follows a normal distribution, with mean zero and variance $dt$. This means that $dB$ can be written as $dB = \varphi \sqrt{dt}$. Here $\varphi$ is a random variable with a standardized normal distribution. Its probability density function is given by

$$f(\varphi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \varphi^2}$$

(2.2)

for $-\infty < \varphi < +\infty$. With the definition of the expectation

$$E[F(\cdot)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\varphi) e^{-\frac{1}{2} \varphi^2} d\varphi.$$  

(2.3)

For any function $F$, we have

$$E[\varphi] = 0, \quad E[\varphi^2] = 1.$$  

The stochastic differential equation (2.1) has the unique solution given by

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma B_t\right).$$

This functional of Brownian motion is called geometric Brownian motion. The geometric Brownian motion model, and the log-normal distribution in equation (2.24), are the basis for the Black-Scholes model for stock-price dynamics in continuous time. In figure 2.2 one see the realization of the geometric Brownian motion based on the sample path of the standard Brownian motion.

Figure 2.2: A sample path of the Geometric Brownian motion $S_0 = 100$, $\sigma = 0.4$, $r = 0.05$

### 2.1.4 Payoff Function

The value of an option at its expiry is usually called the payoff function. For a European call option with a strike price $K$, the payoff is $[WHd95]$

$$C(S_T, T) = \begin{cases} S_T - K, & \text{if } S_T > K, \\ 0, & \text{otherwise.} \end{cases}$$

(2.4)

This can also be written more concisely as $\max(S_T - K, 0)$, or $(S_T - K)^+$. In the case of $S_T > K$, the option is called “in the money”. It is said to be “out of the money” if $S_T < K$. If $S_T = K$, it is “at the money”. Similarly, the payoff function is $(K - S_T)^+$ for a European put option.
2.1.5 Itô’s Lemma

In practice, stock prices are discrete values at discrete time points. Changes can be observed only when the exchange is open. Nevertheless, the continuous-variable, continuous-time processes prove to be useful models for many purposes. To value an option, it is necessary to set up the mathematical models in the continuous time limit $dt \to 0$ and it is more efficient to solve the resulting differential equations, rather than to simulate the random walk on a practical time scale. Therefore, it is needed to handle the $dB$ term in equation (2.1) as $dt \to 0$. In [WHd95], Itô’s lemma provides a type of machinery as

$$
\frac{df}{dt} = \sigma S \frac{\partial f}{\partial S} dB + \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt
$$

where $f$ is a function of $S$. Because logarithmic asset prices are widely used, the differentiation of $f(S) = \log(S)$ gives

$$
\frac{\partial f}{\partial S} = \frac{1}{S} \quad \text{and} \quad \frac{\partial^2 f}{\partial S^2} = - \frac{1}{S^2}
$$

which leads to

$$
\frac{df}{dt} = \sigma dB + (\mu - \frac{1}{2}\sigma^2) dt.
$$

Equation (2.6) is a constant coefficient stochastic differential equation, which says that the difference $df$ is normally distributed. Consider $f$ itself: it is the sum of the jumps $df$ (in the limit, the sum becomes an integral). Since a sum of normal variables is also normal, $f - f_0$ has a normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$, where $t$ is the time elapsed between $f$ and $f_0$, and $f_0 = \log(S_0)$ is the initial value of $f$. The probability density function of $f(S)$ is given by

$$
\frac{1}{\sigma \sqrt{2\pi t}} e^{-((f-f_0)-(\mu - \frac{1}{2}\sigma^2)t)^2/2\sigma^2 t}
$$

for $-\infty < f < +\infty$. Therefore, the probability density function of $S$ is given by

$$
\frac{1}{\sigma \sqrt{2\pi tS}} e^{-((\ln S - \ln S_0) - (\mu - \frac{1}{2}\sigma^2)t)^2/2\sigma^2 t}
$$

for $0 < S < +\infty$.

2.2 The Black-Scholes Model

2.2.1 The Black-Scholes PDE

The most famous model in option pricing is the Black-Scholes model. It is based on the GBM (geometric Brownian motion) model of asset prices: $dS/S = \mu dt + \sigma dB_t$ where $\mu$ and $\sigma$ are fixed values during the lifetime of the option, and $dB_t$ is the Brownian process. According to Itô’s lemma and the Arbitrage theory, a partial differential equation can be obtained by means of setting a portfolio and eliminating the random items by hedging.

Consider a market with a share $S_t$ whose price process satisfies the SDE. We have the following stochastic differential equation

$$
\frac{dS_t}{S_t} = \mu dt + \sigma S_t dB_t.
$$
Considering the risk-free interest rate $r$, and the riskless bank account $A_t$, the dynamics of the asset price is defined as following:

$$dA_t = rA_t dt.$$  \hfill (2.10)

Let $V(t, S_t)$ be European style derivative whose value depends on both the share price and the time. Consider a portfolio $\Pi$ which contains $1$ derivative, and $n$ shares, i.e. its value is

$$\Pi_t = V_t + nS_t.$$  \hfill (2.11)

A small amount of time $dt$ later, the share price has changed. The value of the portfolio changes by

$$d\Pi_t = dV_t + ndS_t.$$  \hfill (2.12)

As seen in subsection 2.1.5, Itô’s Lemma implies that,

$$dV_t = \sigma S \frac{\partial V}{\partial S} dB_t + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$  \hfill (2.13)

Therefore,

$$d\Pi_t = \sigma S \left( \frac{\partial V}{\partial S} + n \right) dB_t + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + n\mu S \right) dt.$$  \hfill (2.14)

If we take $n = -\frac{\partial V}{\partial S}$ (i.e. the portfolio is short $-\frac{\partial V}{\partial S}$ shares), then the portfolio is unaffected by the random changes in stock prices:

$$d\Pi_t = \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt.$$  \hfill (2.15)

Thus, for a brief moment, the portfolio is risk-free. By a no-arbitrage argument, it must earn the same return as the risk-free bank account. This means that

$$d\Pi_t = r\Pi_t dt = r \left[ V - S \frac{\partial V}{\partial S} \right] dt.$$  \hfill (2.16)

By putting together equations (2.15) and (2.16), we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$  \hfill (2.17)

This is the famous Black-Scholes PDE. It is a second-order parabolic PDE, i.e. essentially a heat equation. Consider a European call option $C$ on a share $S$ with strike $K$ and maturity $T$. The volatility of the underlying share $S$ is $\sigma$ and the risk-free rate is $r$. To find the value of the call option, we must solve the following boundary value problem:
2.2.2 Pricing in the Risk-Neutral World

To compute the Black-Scholes prices of vanilla European options, we use a slightly subtle probabilistic argument, rather than a brute force to "solve the PDE " approach.

Same as equation (2.17), we deduced the Black-Scholes PDE for a European-style derivative $V$:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$  \hspace{1cm} (2.19)

It is important to remark that the drift $\mu$ does not occur in the Black-Scholes PDE, though the volatility $\sigma$ does appear. Hence, the price of $V$ is independent of $\mu$, i.e. different values of $\mu$ will give the same price. Since we do not need the drift rate $\mu$ of an underlying asset, we may as well simplify our asset price dynamics by assuming that all assets have the same drift. Now the riskless asset (bank account) has drift $r$, and $r$ occurs in the Black-Scholes PDE. We can not change the drift of the risk-free bank account without changing the PDE, and thus the solution to the pricing problem. So, if we want to assume that all assets have the same drift, we have to assume that the drift of all assets is the risk-free rate $r$. Mathematically, this corresponds to a change of measure from the real world, unknowable probability measure $P$ to a knowable, risk-neutral measure $Q$. In the risk-neutral world, the dynamics of $S$ are

$$dS_t = rS_t dt + \sigma S_t dB_t.$$  \hspace{1cm} (2.20)

Thus, we change the drift of the asset from $\mu$ to $r$. Thus, in a world where all investors are risk-neutral, all assets will have the same expected return, i.e. the same expected return as the risk-free bank account. To summarize, prices in the real and risk-neutral world are the same. It is just probabilities that are changed. Now we can calculate option prices in the risk-neutral world, because the asset price dynamics are known, and so is the distribution of future stock prices. Now suppose that we can find a portfolio $\Pi$ of traded assets which exactly hedges the payoff of a European style derivative $V$, so that

$$\Pi_t = V_t$$

at the derivative’s maturity $T$. Such a portfolio is called a replicating portfolio. By the Law of One Price, therefore, we must have $\Pi_0 = V_0$, where $\Pi_0$ and $V_0$ are, respectively the values of the replicating portfolio and the derivative at $t = 0$. Thus:

If a derivative has a replicating portfolio, then the value of the derivative equals the value of the replicating portfolio.

Now in the Black-Scholes model, as in [Ouw08], any European style derivative has a replicating portfolio: A portfolio consisting, at any time, of $\Delta = \frac{\partial V}{\partial S}$ shares will exactly replicate the derivative $V$ (delta hedging). $\Pi_T$ and $V_T$ are random variables. But since they are identical, they must have the same
expectation, in any world. Since the expected return of all traded assets is \( r \) in the risk-neutral world, and since \( \Pi \) consists entirely of traded assets, the expected return of \( \Pi \) is also \( r \):

\[
E_{RN} [\Pi_T] = \Pi_0 e^{rT},
\]

(2.21)

where \( \Pi_0 \) is the value of the portfolio at \( t = 0 \). Since \( \Pi_0 = V_0 \) (by the Law of One Price) and \( \Pi_T = V_T \) (because \( \Pi_0 \) is a replicating portfolio of \( V \)) as in [Ouw08], we see that

\[
V_0 = e^{-rT} E_{RN} [\Pi_T].
\]

(2.22)

The point is that we can’t calculate \( E_{real} [V_T] \), because we do not know the distribution of the underlying \( S_T \) in the real world. However, we can calculate \( E_{RN} \): Since we know the drift of \( S_T \) in the risk-neutral world, we can calculate the distribution of \( S_T \) here. This brings us to our next topic.

### 2.2.3 The Distribution of Asset Prices

Let \( S_t \) be the asset price described in subsection 2.2. Considering the function \( Y_t = f(S_t) = \ln S_t \), we deduce from equation (2.6) that

\[
dY_t = \sigma dB_t + (\mu - \frac{1}{2} \sigma^2) dt,
\]

(2.23)

by using the fact that \((dB_t)^2 = dt\) and \((dt)^2 = dB_t \ast dt = 0\). \( Y_T \) follow a Brownian motion with a drift. By solving equation (2.23), we obtain

\[
Y_T - Y_0 = \sigma B_T + (\mu - \frac{1}{2} \sigma^2) T,
\]

(2.24)

which implies that \( Y_T \) is normally distributed with mean \((\mu - \frac{1}{2} \sigma^2) T\) and variance \( \sigma^2 T\):

\[
Y_T \sim N \left( Y_0 + (\mu - \frac{1}{2} \sigma^2) T, \sigma^2 T \right).
\]

Thus the log of the stock price is normally distributed. We say that stock prices are log-normally distributed (in the Black-Scholes model). Let \( f_Y \) be the density function of \( Y \).

As in [Ouw08], the density function of a log-normal variable is given by

\[
f_Y(y) = \begin{cases} 
\frac{1}{\sqrt{2\pi \sigma^2}} \exp \left[ -\frac{(\ln y - \mu)^2}{2\sigma^2} \right], & \text{if } y > 0 \\
0, & \text{if } y \leq 0.
\end{cases}
\]

(2.25)

where \( Y = \ln X \) and \( X \sim N(\mu_X, \sigma_X^2) \). Moreover, the mean \( \mu_Y \) and variance \( \sigma_Y^2 \) of \( Y \) are given by

\[
\mu_Y = e^{\mu_X + \frac{1}{2} \sigma_X^2}, \quad \sigma_Y^2 = e^{2\mu_X + \sigma_X^2}[e^{\sigma_X^2} - 1].
\]

(2.26)

In order to give the explicit solutions to the Black-Scholes equation for European call and put options, we need to state the following theorem as in [Ouw08].
Theorem 2.2.1. Suppose that $Y$ is log-normally distributed, where $\ln Y \sim N(m, s^2)$. Let $K$ be a positive constant. Then

$$
P(Y \geq K) = \Phi(d_-)$$

$$
\mathbb{E}[\max \{Y - K, 0\}] = \mathbb{E}[Y] \Phi(d_+) - KN(d_-)
$$

where

$$
d_\pm = \frac{\ln \mathbb{E}[Y]/K}{s} \pm \frac{1}{2} s^2
$$

and

$$
\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.
$$

$N(.)$ is the cumulative distribution function for a standardized normal random variable.

Proof. Since $\ln Y \sim N(m, s^2)$, then it is clear that $X = \frac{\ln Y - m}{s}$, and $X \sim N(0, 1)$, i.e. $X$ is a standard normal random variable. Clearly

$$
P(Y \geq K) = P(\ln Y \geq \ln K) = P\left(X \geq \frac{\ln K - m}{s}\right) = 1 - N\left(\frac{\ln K - m}{s}\right) = N\left(\frac{m - \ln K}{s}\right)
$$

where $N(x)$ is the distribution function of a standard normal random variable, and we used the fact that $1 - N(x) = N(-x)$.

But we know that $\mathbb{E}[Y] = e^{m + \frac{1}{2} s^2}$, so that $m = \ln \mathbb{E}[Y] - \frac{1}{2} s^2$. As a result, we obtain

$$
P(Y \geq K) = N\left(\frac{\ln \mathbb{E}[Y] - \ln K - \frac{1}{2} s^2}{s}\right) = N(d_-).
$$

Using the definition of the $\max \{Y - K, 0\}$ as in equation (2.4),

$$
\mathbb{E}[\max \{Y - K, 0\}] = \int_{-\infty}^{\infty} (y - K) f(y) dy
$$

where $f(y)$ is the density function of $Y$. If we take $x = \frac{\ln y - m}{s}$, then $y = e^{sx + m}$ and

$$
\mathbb{E}[\max \{Y - K, 0\}] = \mathbb{E}[\max \{e^{sx + m} - K, 0\}] = \int_{\ln K - m/s}^{\infty} (e^{sx + m} - K) g(x) dx
$$

where $g(x)$ is the density function of a standard normal random variable $X$.

We can split this up into two integrals:

$$
I = \int_{\ln K - m/s}^{\infty} e^{sx + m} g(x) dx
$$

and

$$
J = -K \int_{\ln K - m/s}^{\infty} g(x) dx.
$$
We simplify the integrand of the first integral by completing the square:

\[
e^{sx+m}g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}e^{sx+m}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}e^{m+s^2/2} = \frac{1}{\sqrt{2\pi}}e^{-(x-s)^2}e^{m+s^2/2} = e^{m+s^2/2}g(x-s) = \mathbb{E}[Y]g(x-s).
\]

We use the fact that, \(\mathbb{E}[Y] = e^{m+s^2/2}\), where \(I\) becomes

\[
I = \int_{\ln K-m/s}^{\infty} e^{sx+m}g(x)dx = \mathbb{E}[Y] \int_{\ln K-m/s}^{\infty} g(x-s)dx
\]

and \(\int_{a}^{\infty} g(x-s)dx\) is just the probability that a standard normal random variable is greater than \(a-s\), which is \(N(s-a) = 1 - N(a-s)\). Thus,

\[
I = \int_{(\ln K-m)/s}^{\infty} e^{sx+m}g(x)dx = \mathbb{E}[Y]N\left(\frac{\ln K - m}{s}\right) = \mathbb{E}[Y]N(d_+)
\]

using \(m = \ln \mathbb{E}[Y] - \frac{1}{2}s^2\).

Similarly,

\[
J = -K \int_{(\ln K-m)/s}^{\infty} g(x)dx = -KN(d_-)
\]

Now, let’s consider the result given by equation (2.24). We have

\[
Y_t - Y_0 = \sigma B_t + (\mu - \frac{1}{2}\sigma^2)t, \quad \text{where} \quad B_t \sim N(0, t)
\]

implies that

\[
Y_t \sim \mathcal{N}\left(Y_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right),
\]

which means that

\[
\ln S_t \sim \mathcal{N}\left(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right).
\]

Thus, \(S_t = e^X\) where \(X \sim \mathcal{N}(\mu_X, \sigma_X^2)\) and

\[
\mu_X = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)t
\]

\[
\sigma_X^2 = \sigma^2 t
\]

Therefore, the probability density function of \(S_t\) is given by

\[
f(S) = \frac{1}{\sigma\sqrt{2\pi}tS}e^{-((\ln S - \ln S_0) - (\mu - \frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \quad 0 \leq S < +\infty,
\]

Using equations (2.26) and (2.27), it follows that the expectation of \(S_t\) is given by

\[
\mathbb{E}[S_t] = e^{\mu_X + \frac{1}{2}\sigma_X^2} = S_0e^{\mu t}.
\]

Replacing \(\mu\) with \(r\) will give the density of \(S_t\) in the risk-neutral world.
Option Pricing: The Black-Scholes Formula

Having the density function of the asset price $S_t$ (equation (2.28)) in the risk-neutral world, we can price practically any European claim $V$ with payoff $\phi(S_t)$:

$$V_t = e^{-r(T-t)} \mathbb{E}_{RN}[\phi(S_T)]$$

$$= e^{-r(T-t)} \int_0^{\infty} \phi(S) f(S) dS. \quad (2.30)$$

Now, let us consider a call option $C$ with strike $K$ and maturity $T$. In this case, $\phi(S_T) = \max\{S_T - K, 0\}$. Thus:

$$C_t = e^{-r(T-t)} \mathbb{E}_{RN}[\phi(S_T)]$$

$$= e^{-r(T-t)} \mathbb{E}_{RN}[\max\{S_T - K, 0\}] \quad (2.31)$$

However, in the risk-neutral world, $S_T$ is log-normally distributed, with $\ln S_t \sim N\left(\ln S_0 + (r - \frac{1}{2} \sigma^2)t, \sigma^2t\right)$. According to theorem 2.2.1, it follows that

$$\mathbb{E}_{RN}[\max\{S_T - K, 0\}] = \mathbb{E}_{RN}(S_T) N(d_+) - K N(d_-)$$

where

$$d_\pm = \frac{\ln [\mathbb{E}(S_T)/K] \pm \frac{1}{2} \sigma^2(T - t)}{\sigma \sqrt{T - t}}, \quad (2.32)$$

but, equation (2.29) implies that $\mathbb{E}_{RN}(S_T) = S_0 e^{rT}$. Therefore,

$$C_t = S_t N(d_+) - K e^{-r(T-t)} N(d_-)$$

where,

$$d_\pm = \frac{\ln [S_0 e^{rT}/K] \pm \frac{1}{2} \sigma^2(T - t)}{\sigma \sqrt{T - t}}, \quad (2.33)$$

and $N(x)$ is the distribution function of a standard normal random variable, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

The Variance Gamma Model

2.3.1 The Definition and the Characterisation of the Variance Gamma Model

The Variance Gamma (VG) process is a pure jump model, and their three parameters $\sigma$, $\theta$, $\nu$ take into account the variance, skewness and kurtosis of the price process [Sch08].

We note that the skewness measures the degree to which a distribution is asymmetric and is defined to be the third moment about the mean, divided by the third power of the standard deviation:

$$\frac{E\left[(X - \mu_X)^3\right]}{Var[X]^{3/2}}.$$
For a symmetric distribution (like the $N \sim (\mu, \sigma^2)$), the skewness is zero. Tail behavior and peakedness are measured by kurtosis, which is defined by

$$E \left[ \frac{(X - \mu)^4}{\text{Var}[X]^2} \right].$$

For the Normal distribution (mesokurtic), the kurtosis is 3. If the distribution has a flatter top (platykurtic), the kurtosis is less than 3. If the distribution has a high peak (leptokurtic), the kurtosis is greater than 3.

Specifically, the VG process is obtained as a Brownian motion (BM) with drift evaluated at a random time $\gamma(t)$:

$$X_t = \theta \gamma(t) + \sigma B_{\gamma(t)},$$

(2.34)

where $B_t$ is a standard BM and $\gamma(t)$ a gamma process evaluated at $t$. The BM requires no further explanation, we can see subsection 2.1.2 for more informations. The gamma process is an infinitely divisible one, obtained by adding independent increments which follow a gamma random variable.

### 2.3.2 The Gamma Distribution

The Gamma distribution is a distribution that lives on the positive real numbers and dependents on two parameters $a$ and $p$. The density function of a random gamma variable of parameters $(a, p)$ is given by:

$$f_{\gamma}(x) = \frac{a^p x^{p-1} e^{-ax}}{\Gamma(p)}, \quad x > 0,$$

(2.35)

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function. This implies that, $\gamma(t)$ is a non-decreasing process distributed as a gamma random variable of parameters $a = 1/\nu$, $p = t/\nu$, and may be approximated as a compound Poisson process. It may also be described by means of its characteristic function, univocally obtained by the inverse Fourier transform of the density function given above, see equation (2.35):

$$\phi_{\gamma(t)}(x) = \left( \frac{1}{1 - i \nu \mu / \mu} \right)^{t/\nu}.$$

(2.36)

As an infinitely divisible process, it may also be characterised by means of its Lévy measure:

$$k_{\gamma}(x)dx = \begin{cases} 
\mu^2 e^{-\frac{\mu x}{\nu}}dx, & \text{if } x > 0, \\
0, & \text{if } x \leq 0.
\end{cases}$$

(2.37)

The integral of this function is infinite, so the gamma process has infinite activity. As $k(0)$ is also infinite, the measure is concentrated near the origin. The coefficient of the diffusion term $-u^2/2$ in the Lévy-Khintchine representation of its characteristic function is zero, and the process is a pure jump one. See figure 2.3

### 2.3.3 The VG Distribution

By evaluating a BM with drift at a gamma random time, we obtain the (Variance Gamma) VG process. Its density function is not as simple as in the gamma process, but its characteristic function and Lévy density are.
The VG process $X(t; \sigma, \nu, \theta)$, is defined in terms of the Brownian motion with drift $b(t; \theta, \sigma)$ and the gamma process with unit mean rate, $\gamma(t; 1, \nu)$ as

$$X(t; \sigma, \nu, \theta) = b((t; 1, \nu); \theta, \sigma).$$

The VG process is obtained on evaluating Brownian motion at a time given by the gamma process. The VG process has three parameters: (i) $\sigma$ the volatility of the Brownian motion, (ii) $\nu$ the variance rate of the gamma time change and (iii) $\theta$ the drift in the Brownian motion with drift. The process therefore provides two dimensions of control on the distribution over and above that of the volatility. We observe below that control is attained over the skew via $\theta$ and over kurtosis with $\nu$.

The characteristic function of the VG model may be evaluated by integrating the Brownian characteristic function with respect the variance gamma measure $f_{\gamma(t)}(x)dx$, with $f_{\gamma(t)}(x)$ as given above by equation (2.35). It yields to the simple expression:

$$\phi_X(u) = \left(1 - i\theta u + \frac{(\sigma^2 \nu}{2u^2}\right)^{t/\nu}.$$  \hfill (2.38)

The VG process can be expressed in an alternative form. It consists of expressing it as the difference of two gamma processes. The Variance Gamma VG$(C, G, M)$ distribution on $(-\infty, +\infty)$ can be constructed as the difference of two gamma random variables. Suppose that $X$ is Gamma$(a = C, b = M)$ random variable and that $Y$ is Gamma$(a = C, b = G)$ random variable and that they are independent of each other. Then

$$X - Y \sim VG(C, G, M).$$

To derive the characteristic function, we start with noting that

$$\phi_X(u) = (1 - \frac{iu}{M})^{-C} \quad \text{and} \quad \phi_Y(u) = (1 - \frac{iu}{G})^{-C}.$$

Summing the two independent random variables $X$ and $-Y$, we have

$$\phi_{X-Y}(u) = (1 - \frac{iu}{M})^{-C}(1 + \frac{iu}{G})^{-C} = \left(\frac{GM}{GM + (M - G)iu + u^2}\right)^C.$$
where
\[ C = \frac{1}{\nu}, \]
\[ G = \left( \sqrt{\frac{\theta^2\nu^2}{4} + \frac{2\sigma^2\nu}{2} - \frac{\theta\nu}{2}} \right)^{-1}, \]
\[ M = \left( \sqrt{\frac{\theta^2\nu^2}{4} + \frac{2\sigma^2\nu}{2} + \frac{\theta\nu}{2}} \right)^{-1}. \]

Another way of introducing the Variance Gamma (VG) distribution is by mixing a Normal distribution with a Gamma random variate. The procedure goes as follows: Take a random variate \( G \sim \text{Gamma}(a = 1/\nu, b = 1/\nu) \). Then sample a random variate \( X \sim \text{Normal}(\theta G, \sigma^2 G) \), then \( X \) follows a Variance Gamma distribution. The distribution of \( X \) is denoted VG(\( \sigma, \nu, \theta \)) and thus depends on 3 parameters:

- a real number \( \theta \) (in the mean of the Normal distribution)
- a positive number \( \sigma \) (in the variance of the Normal distribution)
- a positive number \( \nu \) (of the Gamma random variable \( G \)).

One can show using basic probabilistic techniques that under this parameter setting, the characteristic function of the VG \( (\sigma, \nu, \theta) \) law is given by
\[
E[\exp(iuX)] = \phi_{VG}(u; \sigma, \nu, \theta) = (1 - iu\theta\nu + \sigma^2\nu u^2/2)^{1/\nu}.
\]

Going the other way around one can use:
\[ \nu = \frac{1}{C}, \]
\[ \sigma^2 = \frac{2C}{MG}, \]
\[ \theta = \frac{C(G - M)}{MG}. \]

Its density function is given by
\[
f_{VG}(x; C, G, M)(x) = \frac{(GM)^C}{\sqrt{\pi} \Gamma(C)} \exp\left( \frac{(G - M)x}{2} \right) \left( \frac{|x|}{G + M} \right)^{C-1/2} K_{C-1/2} \left( \frac{(G + M)|x|}{2} \right),
\]

where \( K_{\nu}(x) \) denotes the modified Bessel function of the third kind with index \( \nu \) and \( \Gamma(x) \) denotes the gamma function. For more informations, see [Sch08].

The VG has been described and characterised. Now we introduce the statistical and the risk-neutral price dynamics. See figure 2.4 for the VG process path.
2.3.4 VG Stock Price Model

This section describes the statistical and risk neutral dynamics of the stock price in terms of the VG process. See figure 2.5 for VG stock price process. We will not derive the closed forms for the return density and the prices of European options on the stock. The analytic form of the fair price under the VG is prohibitively complicated and will not be considered. This closed form has a disadvantage. The functions involved, expressible as power series, are computationally expensive. Calculating the call price using the analytic form is slower than computing the price numerically. Carr-Madan [CM99] transform the integral in equation (3.15) in such a way that option prices may be performed by FFT. It leads to a much faster solution than using the closed form. The new specification for the statistical stock price dynamics is obtained by replacing the role of Brownian motion in the original Black-Scholes geometric Brownian motion model by the VG process. Let the statistical process for the stock price be given by

\[ S_t = S_0 \exp(rt + \omega t + X_t(\sigma, \theta, \nu)), \]

where \( X_t \) is a VG process, \( r \) is the continuously compounded interest rate under the risk neutral process and \( \omega \) is the convexity correction calculated by evaluating the characteristic function at \(-i\), in this case resulting

\[ \omega = \frac{1}{\nu} \ln(1 - \theta \nu - \frac{1}{2} \sigma^2 \nu). \]

The characteristic function of the VG is reads

\[ \phi_T(u) = \frac{\exp[iu (\ln S_0 + (r + \omega)T)]}{(1 - i \theta \nu u + (\sigma^2 \nu/2) u^2)^T/\nu}. \] (2.39)
Figure 2.5: A sample path of the VG stock price, $S_0 = 100$, $\sigma = 0.1$, $r = 0.04$, $\theta = -0.1$, $\nu = 0.2$
3. The Numerical Valuation Methods with FFT

In this chapter, we describe a numerical approach for pricing options which utilizes the characteristic function of the underlying instrument’s price process. We apply the Fourier transform to the Black-Scholes model. The approach has been introduced by Carr and Madan [CM99] and is based on FFT. The use of FFT is motivated by two reasons. On the one hand, the algorithm offers a speed advantage. This effect is even boosted by the possibility of the pricing algorithm to calculate prices for a whole range of strikes. On the other hand, the characteristic function of the log price is known and has a simple form for many models considered in literature while the density is often not known in the closed form. The approach assumes that the characteristic function of the log-price is given analytically. The basic idea of the method is to develop an analytic expression for the Fourier transform of the option price and to get the price by Fourier inversion.

3.1 Review of Fourier Method in Option Pricing

In this section, we state how most authors e.g. Bakshi and Madan [BM97] and Scott [Sco97] have applied Fourier analysis to determine option prices. Consider a European call option of an underlying asset whose terminal spot price is $S_T$ of some underlying asset. The characteristic function of $s_T = \ln S_T$ is defined by

$$
\phi_T(u) = E[e^{iuy_T}]
$$

where $E$ is the expectation. Assuming the characteristics function was known analytically, Bakshi and Madan [BM97] and Scott [Sco97] calculated the risk neutral probability of finishing in-the-money as

$$
P_r(S_T > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{e^{-iuy_T} \phi_T(u)}{iu} \right) du
$$

where $k = \ln K$ is the log of the strike price. The delta function is numerically obtained as

$$
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{e^{-iuy_T} (\phi_T(u - i)}{iu \phi_T(-i)} \right) du.
$$

Considering a constant riskless rate $r$ and no dividends, the option values is now calculated as

$$
C = S\Pi_1 - Ke^{-rT}\Pi_2.
$$

But in this method, the FFT cannot be applied to evaluate the integral due to the restriction of the integrand to its real part. Further discussion of FFT for option pricing as developed by Carr and Madan [CM99], is presented separately in next section since one of our aims is to use their model in order to price option with FFT algorithm.

3.2 The Characteristic Function

Here we start with the description of a very efficient pricing technique. According to equation (2.30), the solution of the Black-Scholes equation (2.19) has a solution of the form
\[
V(t, S) = e^{-r(T-t)} \mathbb{E}_t^\mathbb{Q} [\phi(T, S)]
\]  
\[ (3.1) \]

where \( T \) is the maturity time and \( \mathbb{Q} \) is the risk-neutral measure. This equation can be written as an integral:

\[
V(x, T) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \phi(x_T)f(x_T|x)dx_T
\]
\[ (3.2) \]

where \( x \) is the logarithmic asset price, \( \phi(x_T) = (\theta(e^{x_T} - K)) + \) is the payoff function, \((\theta = 1 \) (call) or \( \theta = -1 \) (put)) at maturity and \( f(x_T|x) \) is the transition probability density of reaching \( x(T) \) from \( x(t) \). The transition probability density function is usually difficult to be found analytically, whereas its Fourier transform, called the characteristic function, is comparatively easy to be obtained, by means of the moment generating function. The characteristic function reads

\[
f' = \mathcal{F}(f) = \int_{-\infty}^{+\infty} e^{iws} f(s)ds.
\]
\[ (3.3) \]

Therefore, it is convenient to switch the computation to the frequency domain with the help of the characteristic function to solve the option pricing problems. The characteristic function of the logarithmic asset price in the Black-Scholes model [WHd95] is given by

\[
\phi_T(w) = \mathbb{E} \left[ e^{iwx_T} \right],
\]

equivalent formulation yield

\[
\phi_T(w) = e^{iw\mu_1 - \frac{1}{2}\sigma_1^2w^2},
\]
\[ (3.4) \]

where \( \mu_1 = (r - \frac{1}{2}\sigma^2)\Delta t \) and \( \sigma_1 = \sigma \sqrt{\Delta t} \).

### 3.3 The Fourier Transform Method

Equation (3.2) shows a general form of a representation of option prices. Once we know the characteristic function, we can transform the computation from the asset price domain to the frequency domain. The reason why we do it this way is that, characteristic functions are easier to obtain than the density functions themselves.

#### 3.3.1 Valuation Under The Black-Scholes Model

For the Black-Scholes model, the transition probability density is the same as the density probability, where \( f(x_T|x) = f(z) \) with \( z = x_T - x \). As a result, equation (3.2) becomes

\[
V(x, T) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \phi(T, x_T)f(x_T - x)dx_T
\]
\[ = e^{-r(T-t)} \int_{-\infty}^{+\infty} \phi(T, z + x)f(z)dz.\]
\[ (3.5) \]

We then apply the Fourier transform on \( V(t, x) \), with the damping factor \( e^{\alpha x} \) to ensure the existence of the Fourier transform,
We obtain the following result
\[ e^{r(T-t)} V'((w - i\alpha)) = f'(-(w - i\alpha)) \phi'(w - i\alpha), \] (3.12)
where \( \alpha \) is the damping factor, \( f'(:, \cdot) \) is the characteristic function, \( V'(:, \cdot) \) is the Fourier transform of \( V(x, T) \) and \( \phi'(:, \cdot) \) is the Fourier transform of the payoff function. With the transformation, we can compute the right hand side of equation (3.12) and then get the option’s value by performing the inverse Fourier transform.

### 3.3.2 Valuation Under The Carr and Madan Method

In the Carr and Madan model, the technique [CM99] assumes that the characteristic function of the risk-neutral density is known analytically. Given any such characteristic function, we develop a simple analytic expression for the Fourier transform of the option value or its time value. We then use the FFT to numerically solve for the option price or its time value. Further basic ideas on the use of FFT for pricing problems are described by Cerny [Cer04]. Financial engineers use Fourier analysis to identify cyclic patterns in asset price movements. Such processes can either be described in the time domain by \( h(t) \), which is a function of time \( h(t) \), or in the frequency domain where the process is specified by frequency \( f \), that is \( H(f) \), with \(-\infty < f < +\infty\). One goes back and forth between the representations by means of the continuous Fourier transform equation

\[
H(f) = \int_{-\infty}^{+\infty} h(t) e^{2\pi i f t} dt \quad \text{and} \quad h(t) = \int_{-\infty}^{+\infty} e^{-2\pi i f t} H(f) df,
\]

or the discretized form given by

\[
H(f) = \frac{1}{N} \sum_{t=0}^{N-1} h(t) e^{2\pi i ft/N}, \tag{3.13}
\]
\[
h(t) = \frac{1}{N} \sum_{f=0}^{N-1} H(f) e^{-2\pi i ft/N}. \tag{3.14}
\]

Since the call value is a function of the strike price, by approximately mapping call value and strike price to the above equations, we can apply the Fourier transform to the option pricing problem. The characteristic function is given by
\[ \phi_T(u) = E[e^{iuT}] = \int_{-\infty}^{+\infty} e^{ius} q_T(s) ds. \]

Let \( C_T(k) \) be the fair price of a \( T \) maturity call option with strike \( K = e^k \), on an underlying asset \( S_t \). Let \( q_T(s) \) denote the PDF of \( s_T = \log S_T \), and \( \phi_T(u) \) denote the corresponding characteristic function.

Since the fair price \( C_T(k) \) is simply the present value of the expected payoff, we have:

\[ C_T(k) = \exp(-rT) \int_{k}^{+\infty} (e^s - e^k)q_T(s) ds. \]

Let \( \alpha \geq 0 \) and define the modified call price,

\[ c_T(k) = \exp(\alpha k) C_T(k). \]

Let \( \psi_T(v) \) be the fourier transform of \( c_T(k) \),

\[ \psi_T(k) = \int_{-\infty}^{+\infty} e^{ivk} c_T(k) dk. \]

We write the call price function given by Carr and Madan [CM99] as

\[ C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{+\infty} e^{-ivk} \psi_T(v) dv, \quad (3.15) \]

where \( \psi_T(v) \) is the Fourier transform of this call price [CM99],

\[ \psi_T(v) = \int_{-\infty}^{+\infty} e^{ivk} c_T(k) dk, \quad (3.16) \]
\[ = \int_{-\infty}^{+\infty} e^{ivk} e^{-rT} \int_{k}^{+\infty} e^{\alpha k}(e^s - e^k)q_T(s) dsdk \quad (3.17) \]
\[ = \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{s+(\alpha+iv)k} - e^{(1+\alpha+iv)k}) dkds \quad (3.18) \]
\[ = \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \left( \frac{e^{(1+\alpha+iv)s}}{\alpha + iv} - \frac{e^{(1+\alpha+iv)s}}{1 + \alpha + iv} \right) ds, \quad (3.19) \]

which leads to

\[ \psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}, \quad (3.20) \]
\[ = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{(\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v)((\alpha^2 + \alpha - v^2) - i(2\alpha + 1)v)}, \quad (3.21) \]
\[ = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{(\alpha^2 + \alpha - v^2)^2 + (2\alpha + 1)^2 v^2}. \quad (3.22) \]

\( \psi_T(v) \) is odd in its imaginary part and even in its real part. Here \( k \) is the log-strike price \( K \) (\( k = \log K \)) and is identified to \( t \) in (3.14) [TT03]. That is, the price needs to be computed at various strike prices
of the underlying assets in the option contract. Furthermore, \( v \) correspond to \( f \), \( \psi_T(v) \) is the Fourier transform of the call price \( C_T(k) \), and \( \phi_T \) is the Fourier transform of \( q_T(s) \), the risk-neutral density function of the pricing model. The integral on the right hand side of equation (3.15) is a direct Fourier transform and lends itself to the application of the FFT in the form of summation given by equations (3.13) and (3.14).

### 3.3.3 Discrete Approximation

We wish to turn equation (3.15) into a form suitable for an FFT algorithm. That is, both \( C_T(k) \) and the integral must be discretized. This implies that the integral need to be truncated. If \( I = e^{-\alpha k} / \pi \) and \( \omega = e^{-i} \), then

\[
C_T(k) = I \int_{0}^{+\infty} \omega^{vk} \psi_T(v) dv.
\]  

(3.23)

If \( v_j = j\eta \) and the trapezoid rule are applied to the right handside of equation (3.23), then \( C_T(k) \) can be written as

\[
C_T(k) = I \sum_{j=0}^{N-1} \omega^{v_j k} \eta \psi_T(v_j), \quad j = 0, \ldots, N - 1,
\]  

(3.24)

where the effective upper limit of integration is \( N\eta \) and \( v_j \) corresponds to various prices with \( \eta \) spacing. In general, the strikes near the spot price are of interest because such options are traded most frequently. We thus consider an equidistant spacing of the log-strikes around the log spot price \( s_0 \):

\[
k_u = -\frac{1}{2} N\varsigma + \varsigma u + s_0, \quad u = 0, \ldots, N - 1,
\]  

(3.25)

where \( \varsigma > 0 \) denotes the distance between the log strikes. Substituting these log-strikes yields, for \( u = 0, \ldots, N - 1 \):

\[
C_T(k) = \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=0}^{N-1} e^{-i j \varsigma u} e^{i \left(\frac{1}{2} N\varsigma \eta s_0\right)v_j} \eta \psi_T(v_j), \quad j = 0, \ldots, N - 1.
\]  

(3.26)

Now, the FFT can be applied to

\[
x_j = e^{i \left(\frac{1}{2} N\varsigma \eta s_0\right)v_j} \eta \psi_T(v_j), \quad j = 0, \ldots, N - 1
\]  

(3.27)

provided that

\[
\eta \varsigma = \frac{2\pi}{N}.
\]  

(3.28)

This constraint leads, however, to the following trade-off: The parameter \( N \) controls the computation time and thus is often determined by the computational setup. Hence, the right handside may be
regarded as given or fixed. One would like to choose a small $\varsigma$ in order to get many prices for strikes near the spot price. But the constraint implies that a big $\eta$ gives a coarse grid for integration. So we face a trade-off between accuracy and the number of interesting strikes.

In order to obtain an accurate integration with larger values of $\eta$, we incorporate Simpson’s rule weightings into our summation in equation (3.26). With Simpson’s rule weightings and the restriction $\eta = \frac{2\pi}{N}$, in line with Carr and Madan [CM99], the call price is given as the following

$$C_T(k_u) = \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=0}^{N-1} e^{-i\eta j u} e^{i(\frac{1}{2}N\varsigma - s_0)} v_j \psi_T(v_j) \frac{\eta}{3} \left\{ 3 + (-1)^j - \delta_{j-1} \right\},$$

(3.29)

where $\delta_n$ is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The summation in equation (3.29) is an exact application of the FFT. Now that we have a discretized Fourier transform to perform, there are countless FFT algorithms available to us. Python or Scipy for example, has a built in module scipy.fftpack using the command scipy.fftpack.fft(·) or scipy.fftpack.ifft(·) to speed the FFT algorithm.

One needs to make the appropriate choices for $\eta$ and $\alpha$. The next chapter addresses these issues of the choice of our parameter and their application.

### 3.3.4 Valuation Under The Peter Tankov Model

Peter Tankov [Tan] develops a new idea based on the Carr and Madan [MCC98] approach for pricing European call options in models where the characteristic function of the log stock price is known using Fourier transform and, in particular, the Fast Fourier transform algorithm [CT65]. In fact, this method proposes several improvements to the original procedure and gives a rigorous analysis of truncation and discretization errors.

Let $\{X_t\}_{t \geq 0}$ be a stochastic process on $(W, F, P)$ such that $e^{X_t}$ is a martingale. In order to compute the price of a call option

$$C_T(k) = e^{-rT} E \left[ \left( e^{rT + X_T} - e^k \right)^+ \right],$$

(3.30)

we need to express its Fourier transform in log strike in terms of the characteristic function $\phi_T(v)$ of $X_T$ and then find the prices for a range of strikes by Fourier inversion. However, we cannot do this directly because $C_T(k)$ is not integrable (it tends to a positive constant as $k \to -\infty$). The key idea is to instead compute the Fourier transform of the (modified) time value of the option, that is, the function

$$z_T(k) = e^{-rT} E \left[ \left( e^{rT + X_T} - e^k \right)^+ \right] - \left( 1 - e^{k-rT} \right)^+.$$  

(3.31)

Proposition 3.3.1. Let $\{X_t\}_{t \geq 0}$ be a stochastic process on $(W, F, P)$ such that $e^{X_t}$ is a martingale and

$$E \left[ e^{(1+\alpha)X_t} \right] < \infty \quad \forall t,$$

(3.32)

for some $\alpha > 0$. Then the Fourier transform of the time value of a call option is given by:

$$z_T(v) := \int_{-\infty}^{+\infty} e^{ivk} z_T(k) dk = e^{-rT} \frac{i v - \phi_T(v) - 1}{iv(1 + iv)}.$$  

(3.33)
It is important to remark that as \( Rz \to \infty \), \( \phi_T(z) \to 0 \) and \( \varsigma_T(v) \) will behave like \(|v|^{-2}\) at infinity which means that the truncation error in the numerical evaluation of the inverse Fourier transform will be large. The reason of such a slow convergence is that the time value (3.31) is not smooth; therefore its Fourier transform does not decay sufficiently fast at infinity. For most models the convergence can be improved by replacing the time value with a smooth function of strike.

Namely, instead of subtracting the intrinsic value of the option (which is non-differentiable) from its price, we suggest to subtract the Black-Scholes call price with a non-zero volatility (which is a smooth function). The resulting function will be both integrable and smooth. Suppose that the hypothesis of the above proposition is satisfied and denote

\[
\dot{z}_T(k) = e^{-rT}E \left[ \left( e^{rT+X_T} - e^k \right)^{+} \right] - C_{BS}^\Sigma(k),
\]

where \( C_{BS}^\Sigma(k) \) is the Black-Scholes price of a call option with volatility \( \Sigma \) and log-strike \( k \) for the same underlying value and the same interest rate. The above proposition implies that the Fourier transform of \( \dot{z}_T(k) \), denoted by \( \hat{\varsigma}_T(v) \), satisfies

\[
\hat{\varsigma}_T(v) = e^{ivrT} \phi_T(v-i) - \phi_T^\Sigma(v-i) \over iv(1+iv),
\]

where \( \phi_T^\Sigma(v) = \exp\left( -\frac{\Sigma^2T}{2} (v^2 + iv) \right) \). Since for most models found in the literature (except variance gamma) the characteristic function decays faster than every power of its argument at infinity, this means that the expression (3.35) will also decay faster than every power of \( v \) as \( Rv \to \infty \), and the integral in the inverse Fourier transform will converge very fast for every \( \Sigma > 0 \).

**Proof of the proposition 3.3.1.** Since the discounted price process is a martingale, we can write

\[
z_T(k) = e^{-rT} \int_{-\infty}^{+\infty} \mu_T(dx) (e^{rT+x} - e^k)(1_{k\leq x+rT} - 1_{k\leq rT}),
\]

where \( \mu_T \) is the probability distribution of \( X_T \). Condition (3.32) enables us to compute \( \varsigma_T(v) \) by interchanging integrals:

\[
\varsigma_T(v) = e^{-rT} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} \mu_T(dx) e^{ivk} (e^{rT+x} - e^k)(1_{k\leq x+rT} - 1_{k\leq rT})
\]

\[
= e^{-rT} \int_{-\infty}^{+\infty} \mu_T(dx) \left[ \int_{-\infty}^{+\infty} e^{ivk} (e^{rT+x} - e^k)(1_{k\leq x+rT} - 1_{k\leq rT}) dk \right]
\]

\[
= e^{-rT} \int_{-\infty}^{+\infty} \mu_T(dx) \left\{ e^{ivrT} \left( \frac{1-e^{-r}}{iv+1} - \frac{e^{x+ivrT}}{iv(v+1)} + \frac{e^{(iv+1)x+ivrT}}{iv(v+1)} \right) \right\}.
\]

The first term in braces disappears due to the martingale condition and the other two, after computing the integrals, yield (3.35).

**3.3.5 Numerical Fourier Inversion**

Option prices can be computed by evaluating numerically the inverse Fourier transform of \( \hat{\varsigma}_T(v) \) in equation (3.35):

\[
\dot{z}_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \hat{\varsigma}_T(v) dv,
\]
To approximate option prices, we truncate and discretize the integral (3.40) as follows:

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk\hat{\varsigma}_T(v)} dv = \frac{1}{2\pi} \int_{-A/2}^{A/2} e^{-ivk\hat{\varsigma}_T(v)} dv + \varepsilon_T \]  
\[ = \frac{A}{2\pi N} \sum_{m=0}^{N-1} w_m e^{-iv_m k\hat{\varsigma}_T(v_m)} + \varepsilon_T + \varepsilon_D, \]  

(3.41)

(3.42)

where \( \varepsilon_T \) is the truncation error, \( \varepsilon_D \) is the discretization error, \( v_m = -A/2 + m\Delta \), \( \Delta = A/(N-1) \) is the discretization step and \( w_m \) are weights, corresponding to the chosen integration rule (for instance, for the trapezoidal rule \( w_0 = w_{N-1} = 1/2 \) and all other weights are equal to 1). Now, setting \( k_n = \frac{2\pi n}{N\Delta} \) we see that the sum in equation (3.42) becomes a discrete Fourier transform:

\[ \frac{A}{2\pi N} e^{ik_n A/2} \sum_{m=0}^{N-1} w_m f(k_m)e^{-2\pi i mn/N}. \]  

(3.43)

Therefore, the FFT algorithm allows to compute \( \hat{\varsigma}_T(k) \) and therefore option prices for the log strikes \( k_n = \frac{2\pi n}{N\Delta} \). The log strikes are thus equidistant with the step \( d \) satisfying \( d\Delta = \frac{2\pi}{N} \).
4. Application

4.1 Application to the Black-Scholes Model

In this section, we apply the FFT option pricing algorithm of Section 3.3 to the models described in Section 2.2. Besides the speed and accuracy of the FFT algorithm, our aim is to demonstrate the possibility to numerically evaluate the formula of the option price while finding a closed-form solution for the option prices which may require swathes of complicated algebra.

In order to apply the FFT-based algorithm, we need to know the characteristic function of the risk neutral density which has been described in Section 3.2 for the Black-Scholes model. Moreover, we have to decide on the parameters $\alpha$, $N$, $\varsigma$ and $\eta$ of the algorithm. The computation time depends on the parameter $N$ which we set to 512 implying a log strike spacing $\varsigma = 0.015$. As the number of grid points of the numerical integration is also given by $N$, this parameter in addition determines the accuracy of the prices. For parameter $\eta$, which determines the distance of the points of the integration grid, we use 0.75. A limited simulation study showed that the FFT algorithm is not sensitive to the choice of $\eta$, i.e. small changes in $\eta$ gave similar results. For the choice of the dampening coefficient in the call price, we used a value of $\alpha = 1.1$.

For a more detailed analysis, we evaluate the option prices in this case for strikes ranging from 80 to 120 in steps of a dollar, with the spot set at $S_0 = 100$, the interest rate at $r = 0.05$, the variance $\sigma = 0.5$ and the maturity time is set to $T = 1$.

The FFT price approach, however, slightly overestimates the true prices since the call option price is a convex function of the strike, we can see the plotting in figure 4.1. The FFT-based algorithm is fairly efficient as shown in figure 4.2. Moreover, it returns prices for a whole range of strikes at one maturity.

![Figure 4.1: FFT Option pricing in the Black-Scholes model](image)

The differences between the analytical and FFT-based prices come from the fact that the latter method gives the exact values only on the grid. It can be clearly seen that near the spot price $[95, 105]$, the prices obtained by both methods in more accurate, while between $[80, 95]$ the FFT-based algorithm generates higher prices than the analytical solution, and between $[105, 120]$ the analytical price generate higher price than FFT one. We can see the plotting in figure 4.2.
4.2 FFT Error Behaviour in the Black-Scholes

The calculation of option prices by the FFT-based algorithm leads to different errors. The truncation error results from substituting the infinite upper integration limit by a finite number. The sampling error comes from evaluating the integrand only at grid points. Lee [Rog04] gives bounds for these errors and discusses error minimization strategies. Besides the truncation and the sampling error, the implementation of the algorithm often leads to severe roundoff errors because of the complex form of the characteristic function for some models. To avoid this problem, which often occurs for long maturities, it is necessary to transform the characteristic function.

For a more detailed evaluation of the pricing errors, we computed for the imaginary part of our numerical result. In reality, our call option is not complex, so the real part is assumed to be the numerical solution and the imaginary part is the error. In fact, we use the Mean Error (ME) defined as:

\[ ME = \frac{1}{\text{Number of options}} \sum_{\text{options}} (\text{FFT price} - \text{Analytic price}) \]

As we can see in the plot of figure 4.3, the error behaviour shows that the FFT price is more accurate. FFT is substantially more accurate and about twice as fast. Furthermore, the largest error of FFT pricing occurs in the region of \( K = 2.5 \) (which is typically not of interest), and then the error term becomes essentially a constant less than 1 percent in the region of our interest.

The accuracy of the call price and the error depend both of the choice of \( \alpha \). So, one needs to make an optimal choice for each model so that we can obtain more accurate price. The remaining parameters do not affect in reality the call price, but for the value of \( \sigma \) choose between \([0.1, 0.6]\) and \( r \) between \([0.02, 0.06]\) we obtain accurate price.

4.3 Application to the Variance Gamma Model

For the VG model the analytic solution is not considered, we get a feel for the magnitude of error introduced by Fourier pricing from the Black-Scholes for which the fair price is known analytically. In
order to apply the FFT-based algorithm we need to know the characteristic function of the risk neutral density which has been described in Section 2.3 for the VG model. Moreover, the values of the following parameters, \( N, \alpha, \varsigma, r, \sigma, \eta, T \) remain the same as in the Black-Scholes. But, we choose to set the values of \( \theta = 0.1 \) and \( \nu = -0.1 \). As we can see from the plot of figure 4.4, with the parameter as in the Black-Scholes, the VG call option generate higher price with the same range of strike.

![Figure 4.3: The Error behaviour of the FFT price call option in the Black-Scholes](image)

![Figure 4.4: The FFT price call option in the VG model](image)
5. Conclusion and Future Work

In this essay, we have described how one can price very fast and efficiently call options using the theory of characteristic functions and Fast Fourier Transforms. We have developed a solid understanding of the current frameworks for pricing European call options using these techniques, and we have provided the mathematical and practical background necessary to apply and implement the techniques. The Fast Fourier Transform method is particularly interesting in case of advanced equity models, like the Variance Gamma model, its stochastic volatility extension, and many other models like the Heston model, where no closed-form solutions for call option exist.

In fact, we have illustrated how the calculation of the call price via the Carr-Madan formula can be done fast and accurately using the Fast Fourier Transform. Typically, $N$ is a power of 2 (where $N$ is the number of discretisation steps). The number of operations of the FFT algorithm is of the order $O(N \log N)$ and this is in contrast to the straightforward evaluation of the sums which give rise to $O(N^2)$ number of operations.

An important advantage of the method is that the pricer only needs as input the characteristic function of the dynamics of the underlying model. If one wants to switch to another model, only the corresponding characteristic functions needs to be changed and the actual pricing algorithm remains untouched. The methodology cannot only be applied to vanillas, but typically to more general options which depend only on the stock price at maturity.

In practice, it is not always possible to get a characteristic function for the price process being considered (especially when pricing exotic options) and one must resort to Monte-Carlo methods. However, the fact remains that the FFT is the most fast and efficiently method for options price.

More research can be performed in the future. For example, error estimation of this scheme can be reduced considerably. This can be done by performing an optimal choice of the damping parameter $\alpha$ which is the only way to overcome any numerical instabilities and guarantee accurate results for each model.
Appendix A. Some Python Codes

This program computes the analytic, the FFT and the error call option price based on the Black-Scholes model.

```python
import numpy
import scipy
import scipy.fftpack
import math
import Gnuplot

#compute the constants
#########################################################
r=0.05 # risk free rate
sigma=0.5 # variance of the asset price
T=1 # maturity
N=512 # number of point
S0=100 # initial asset price
varsigma=0.015 # the distance between the log strikes
tau=0.75 # the distance of the points of the integration grid
alpha=1.1 # damping coefficient

#compute the expectation and variance of log asset price
SIGMA_1=0.5*(sigma**2)**T # variance
#compute vj corresponds to various log prices with eta spacing and log strike price kj
#--------------------------------------------------------

gp=Gnuplot.Gnuplot(debug=-1)
gp('set data style line')
gp2=Gnuplot.Gnuplot(debug=-1)
gp2('set data style line')
v = numpy.linspace(0, N-1, N)
j=
for j in range(0, len(v), 1):
j.append(v[j])
kj.append(-0.5*N*varsigma+varsigma*v[j]+s0)
d1.append((numpy.log((numpy.exp(s0+r*T))/numpy.exp(kj[j]))-0.5*(sigma**2)*T)/(sigma*numpy.sqrt(T)))
d2.append((numpy.log((numpy.exp(s0+r*T))/numpy.exp(kj[j]))+0.5*(sigma**2)*T)/(sigma*numpy.sqrt(T)))

#compute the characteristic function

#compute the Normal distribution
Normd1=stats.norm.cdf(d1)
Normd2=stats.norm.cdf(d2)

# modified call price option
ModCallOptionPrice=scipy.fftpack.fft(fourierCallPriceA)
#analytic call price option
AnalyticCallOption=scipy.fftpack.fft(AnalyticSolution)

for j in range(0, len(kj), 1):
    callOption.append((kj[j]**(-alpha))/numpy.pi)*ModCallOptionPrice[j]
    AnalyticCallOption.append(s0*Normd2[j]-kj[j]*(numpy.exp(-r*T))*Normd1[j])
    RealCallPriceOption.append(callOption[j].real)
    ImCallPriceOption.append(callOption[j].imag)
    CallOption.append([kj[j]*numpy.exp(4),RealCallPriceOption[j]*numpy.exp(7.1)])
    ErrorOption.append([kj[j]*numpy.exp(0),ImCallPriceOption[j]*numpy.exp(0)])
    AnalyticSolution.append([kj[j]*numpy.exp(4),AnalyticCallOption[j]*numpy.exp(1.7)])
```

This program computes the FPT Call Option price based on the Variance Gamma model.

```python
import numpy
from scipy import *
import scipy.fftpack
import math
import Gnuplot

def compute_call_prices(N, r, sigma, T, S0, varsigma, eta, alpha):
    v = numpy.linspace(0, N-1, N)
    vj = []
    kj = []
    vjcompl = []
    caractfunct = []
    fourierCallPrice = []
    fourierCallPriceA = []
    for j in range(0, len(v), 1):
        vj.append(v[j])
        kj.append(-0.5*N*varsigma+varsigma*v[j]+S0)
        vjcompl.append(complex(1+0.5*(sigma**2)*(vj[j]**2)*nu,-(vj[j]*nu*theta)))
        caractfunct.append((v[j]**(-T/nu))*complex(numpy.cos((s0+(r+omega)*T)*vj[j]),numpy.sin((s0+(r+omega)*T)*vj[j])))
        fourierCallPrice.append((numpy.exp(-r*T)*caractfunct[j])/complex((alpha**2+alpha-(vj[j])**2),(2*alpha+1)*vj[j]))
        fourierCallPriceA.append((eta/3)*(3+(-1**j)-1)*fourierCallPrice[j]*complex(numpy.cos((0.5*N*varsigma-s0)*vj[j]),numpy.sin((0.5*N*varsigma-s0)*vj[j])))
    ModCallOptionPrice = scipy.fftpack.fft(fourierCallPriceA)
    for j in range(0, len(kj), 1):
        CallPriceOption.append((((numpy.log(kj[j]))**(-alpha))/numpy.pi)*ModCallOptionPrice[j])
    RealCallPriceOption = []
    ImCallPriceOption = []
    for j in range(0, len(kj), 1):
        RealCallPriceOption.append(CallPriceOption[j].real)
        ImCallPriceOption.append(CallPriceOption[j].imag)
    return RealCallPriceOption, ImCallPriceOption

# call option price
RealCallPriceOption, ImCallPriceOption = compute_call_prices(N, r, sigma, T, S0, varsigma, eta, alpha)
```
This program computes the Brownian and the Geometric Brownian path:

```python
from __future__ import division
from random import *
from scipy import *
import Gnuplot

g=Gnuplot.Gnuplot(debug=1)
Snul=100
mu=0.05
sigma=0.4
T=1
dt=T/N
tt=arange(0,N,dt)
bmzeros(N)
S=zeros(N)

#tt=[0:dt:T]
S[0]=Snul
for j in range(1,N):
bm[j]=bm[j-1]+sqrt(dt)*gauss(0,1)
S[j] = Snul*exp((mu-(sigma**2/2))*tt[j]+bm[j])

#g('set xrange [0.005,1]')
plot=Gnuplot.PlotItems.Data(bm, with = 'lines lw 1', title='Standard Brownian Motion path')
#plot=Gnuplot.PlotItems.Data(S, with = 'lines lw 0.5', title='Geometric Brownian Motion path')
#g('set size ratio -1')
#g('set xrange [0:10]')
#gp('set xlabel "strike"')
#gp('set ylabel "call option price"')
g.plot(plot)
g.hardcopy(filename='brownian0.eps',eps=True, color=True)
```

This program computes the VG stock price dynamics and the VG process path:

```python
from __future__ import division
from random import *
from scipy import *
import Gnuplot

g=Gnuplot.Gnuplot(debug=1)
Snul=100
T=1
r=0.04
#q=0.03
N=250
nu=0.2
sigma=0.15
theta= -0.10
omega=log(1-(sigma**2)*nu/2-theta*nu)/nu
C=1/nu
G=(sqrt((theta**2)*(nu**2)/4+(sigma**2)*nu/2)-theta*nu/2)**(-1)
M=(sqrt((theta**2)*(nu**2)/4+(sigma**2)*nu/2)+theta*nu/2)**(-1)
dt=T/N
tt=arange(0,N,dt)
vgzeros(N)
S=zeros(N)

#vg[1]=0
for j in range(0,N):
g1= gammavariate(dt*C,1/M)
g2= gammavariate(dt*C,1/G)
vg[j] = vg[j-1] + g1-g2
S[j] = Snul*exp((r+omega)*tt[j]+vg[j])

#g('set xrange [0.005,1]')
plot=Gnuplot.PlotItems.Data(S, with = 'lines lw 2', title='VG stock price')
plot=Gnuplot.PlotItems.Data(vg, with = 'lines lw 2', title='VG process path')
g.plot(plot)
g.hardcopy(filename='vgp.eps',eps=True, color=True)
```

This program computes the Gamma process path:

```python
from __future__ import division
from random import *

g=Gnuplot.Gnuplot(debug=1)
Snul=100
T=1
r=0.04
#q=0.03
N=250
mu=0.2
sigma=0.15
tetah=-0.10
omega=log(1-(sigma**2)*nu/2-theta*nu)/nu
C=1/nu
G=(sqrt((theta**2)*(nu**2)/4+(sigma**2)*nu/2)-theta*nu/2)**(-1)
M=(sqrt((theta**2)*(nu**2)/4+(sigma**2)*nu/2)+theta*nu/2)**(-1)
dt=T/N
tt=arange(0,N,dt)
gamzeros(N)
S=zeros(N)

#vg[1]=0
for j in range(0,N):
g1= gammavariate(dt*C,1/M)
g2= gammavariate(dt*C,1/G)
vg[j] = vg[j-1] + g1-g2
S[j] = Snul*exp((r+omega)*tt[j]+vg[j])

#g('set xrange [0.005,1]')
plot=Gnuplot.PlotItems.Data(S, with = 'lines lw 3', title='VG stock price')
plot=Gnuplot.PlotItems.Data(vg, with = 'lines lw 3', title='VG process path')
g.plot(plot)
g.hardcopy(filename='vgg.eps',eps=True, color=True)
```
from scipy import *
import Gnuplot

g=Gnuplot.Gnuplot(debug=1)

T=1
N=250
dt=T/N
w=10
b=20

tt=arange(0,N,dt)
gm=arange(250)

for j in range(1,N):
gm[j]=gm[j-1]+gammavariate(a*dt,1/b)


g('set yrange [-0.5:0.5]')
plot=Gnuplot.PlotItems.Data(gm, with = 'lines lw 2', title='Gamma process path')
g.plot(plot)
g.hardcopy(filename='gamma1.eps',eps=True, color=True)
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