Introduction to Financial Risk Measurement
Einführung in die Bewertung von Finanzrisiken

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Introduction to Financial Risk Measurement

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- Convex Measure
- Coherent Risk Measure
- Acceptance set
- Résumé

Example of Monetary Measure of Risk
- Worst-Case Risk Measure
- Value at Risk

VaR and Regulatory approach
We begin with the introduction of a probabilistic framework for modeling financial risk from the perspective of an investor. For a given financial positions, let \( \Omega = \{ \nu_1, \nu_2, \nu_3, \ldots, \nu_n \} \) be a finite set of possible future value of this positions, the uncertainty about the future can be represent by a probability space \((\Omega, F, P)\). Then the random variables 

\[
X : \Omega \rightarrow \mathbb{R}, \quad \nu \rightarrow X_t(\nu)
\]

denote the position value (pay off) of \( \nu \) at time \( t \) if the scenario \( \nu \in \Omega \) is realized.
In general, we can define a **financial risk** as the change in the position-values between two dates \( t_0 \) and \( t_1 \) (current date is \( t_0 = 0 \)), or as the dispersion of unexpected outcomes due to uncertain events (scenarios \( \nu \)). We can quantify the risk of \( X \) by some number \( \rho(X) \), where \( X \) belongs to a given linear function space \( \mathcal{X} \), that contains the constant function. (note that \( \mathcal{X} \) can be identified with \( \mathbb{R}^n \), where \( n = \text{card}(\Omega) \)).

**Definition: monetary measure**

A function \( \rho : \mathcal{X} \to \mathbb{R} \) is called a monetary measure of risk if it satisfies the following conditions for all \( X, Y \in \mathcal{X} \)

- **Axiom M**: Monotonicity
  
  \[ \text{if } X \leq Y, \text{ then } \rho(X) \geq \rho(Y) \]

- **Axiom T**: Translation invariance
  
  \[ \text{if } m \in \mathbb{R}, \text{ then } \rho(X + m) = \rho(X) - m \]
Interpretation

- **Monotonicity**: The risk of a position is reduced if the payoff profile is increased in each state of the World. ($\rho(X)$ is a decreasing function)

- **Invariance**: The invariance property suggests that adding cash to a position reduces its risk by the amount of cash added. This is motivated by the idea that the risk measure can be used to determine capital requirements.

**Exemple: capital requirement for risky position X**

- if $\rho(X) > 0$ then regulatory authority requests additional capital to make this position acceptable.
- if $\rho(X) < 0$ then the position is is already acceptable.

As a consequence for the invariance of $\rho$ we have

$$\rho(X + \rho(X)) = 0 \text{ and } \rho(m) = \rho(0) - m \quad \forall \ m \in \mathbb{R}.$$
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\]
Remark

m corresponds to the constant function

Definition: convex risk measure

A monetary risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called a convex measure of risk if it satisfies:

\begin{align*}
\rho(\lambda X + (1 - \lambda) Y) & \leq \lambda \rho(X) + (1 - \lambda) \rho(Y), \quad \text{for } 0 \leq \lambda \leq 1.
\end{align*}

Interpretation: The convexity property states that the risk of a portfolio is not greater than the sum of the risks of its constituents, that means diversification in a given portfolio does not increase the risk.
Coherent Risk Measure

Definition: Coherent Risk Measure

A convex measure of risk $\rho$ is called a coherent risk measure if it satisfies:

- **Axiom PH**: Positive Homogeneity
  
  If $\lambda \geq 0$ then $\rho(\lambda X) = \lambda \rho(X)$ $\forall X \in \mathcal{X}$

If a monetary measure of risk $\rho$ is positively homogeneous, then it is **normalized** ($\rho(0) = 0$), Under the assumption of positive homogeneity, convexity is equivalent to

- **Axiom S**: Subadditivity
  
  $\rho(X + Y) \leq \rho(X) + \rho(Y)$
Interpretation:

▶ The subadditivity axiom ensure that the risk of a diversified portfolio is no greater than the corresponding weighted average of the risks of the constituents.

▶ Capital requirement for holding company should never be larger than the sum of Capital requirement of all individual subs.

▶ subadditivity reflects the idea that risk can be reduced by diversification

▶ Subadditivity makes decentralization of risk-management systems possible.
Acceptance set

One can separate the set $\mathcal{X}$ in different subsets:

- $L_+ = \{ X \in \mathcal{X} | X(v) \geq 0 \ \forall \ v \in \Omega \}$ (the set of non-negative elements of $\mathcal{X}$)
- $L_- = \{ X \in \mathcal{X} | \exists \ v \in \Omega \ \text{s.t.} \ X(v) < 0 \}$ (the set of negative elements of $\mathcal{X}$)
- $L_{--} = \{ X \in \mathcal{X} | X(v) < 0 \ \forall \ v \in \Omega \}$

For example in case $n = 2$ ($\Omega = \{v_1, v_2\}$) we can represent $\mathcal{X}$ in a 2-coordinate system. Where the x-axis represents the measurements of $x = X(v_1)$ and the y-axis represents the measures of $y = X(v_2)$. 
Representation of $\mathcal{X}$ for $n = 2$
Acceptance set

Depending on its risk policy, the regulator can separate the set $\mathcal{X}$ into two distinct subset:

1. Acceptable set
2. Uncceptable set

Definition: Acceptance set

The acceptance set $A$ is defined as the set of financial positions that are acceptable (from the regulator’s point of view) without any additional capital requirement.
As in [Artzner99] we state here some axioms that an acceptance set $\mathcal{A}$ must satisfy:

- Axiom A1: The acceptance set $\mathcal{A}$ contains $L_+ \subset \mathcal{X}$
- Axiom A2: The acceptance set $\mathcal{A}$ does not intersect $L_-$
- Axiom A3: The acceptance set $\mathcal{A}$ is convex

The separation of acceptance set $\mathcal{A}$ from unacceptable set can be materialized by the characterization of the boundary of $\mathcal{A}$ (of $\partial \mathcal{A}$).
Characterization of $\partial A$ for $n = 2$. 1
In the case \( n = 2 \), we can deduce from the axioms of the acceptance set some geometrical properties of the boundary of \( A (\partial A) \).

Consider the angle \( \alpha \) in the previous picture, then we get according to the axioms of the acceptance set the following relationships:

- Axiom A1 \( \Rightarrow \alpha \geq 90^\circ \)
- Axiom A2 \( \Rightarrow \alpha \leq 270^\circ \)
Relation to Risk Measures I

Definition: Acceptance set associated to a risk measure

The acceptance set associated to a risk measure $\rho$ is the set denoted by $A_\rho$ and defined by

$$A_\rho = \{X \in \mathcal{X} : \rho(X) \leq 0\} \quad (*)$$

Definition: Risk measure associated to an acceptance set

The risk measure associated to the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{X}$ to $\mathbb{R}$ denoted by $\rho_\mathcal{A}$ and defined by

$$\rho_\mathcal{A}(X) = \inf\{m \in \mathbb{R} | X + m \in \mathcal{A}\} \quad (***)$$

The amount $m$ may be interpreted as the required regulatory capital to cover the position’s risk.
Relation to Risk Measures II

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Résume

Example of Monetary Measure of Risk
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VaR and Regulatory approach
The previous graphic illustrate the relationsheap betwenn the acceptance set $\mathcal{A}$ (namely the boundary $\partial \mathcal{A}$ of $\mathcal{A}$) and measure of the risk in the case $n=2$. In this example we can see that initially $X_1(1, -2.34)$ does not belong to the acceptance set, it is also unacceptable from the perspective of the regulator, but we can make it acceptable, according to the definiton (**) by Adding some cash $m = 1$ (or higher than 1). The new position $X_2 = X(1, -2.34) + m(1, 1) = (2, -1.34)$ is on the boundary $\partial \mathcal{A}$ of the acceptance set and therefore acceptable.
Relation to Risk Measures IV

- $\rho_\mathcal{A}$ is a convex risk measure if and only if $\mathcal{A}$ is convex.
- $\rho_\mathcal{A}$ is positively homogeneous if and only if $\mathcal{A}$ is a cone. In particular, $\rho_\mathcal{A}$ is coherent if and only if $\mathcal{A}$ is a convex cone.
- $\mathcal{A}_\rho$ is clearly convex if $\rho$ is a convex measure of risk.

Assume that $\mathcal{A}_\rho$ is a non-empty subset of $\mathcal{X}$ which satisfies $\rho_\mathcal{A} \geq -\infty$ and $X \in \mathcal{A}, Y \in \mathcal{X}, Y \geq X \Rightarrow Y \in \mathcal{A}$ Then:

- $\rho_\mathcal{A}$ is a monetary measure of risk.
- If $\mathcal{A}_\rho$ is a convex set, then $\rho_\mathcal{A}$ is a convex measure of risk.
- $\rho_\mathcal{A}$ is a coherent measure of risk if $\mathcal{A}_\rho$ is a convex cone.
Let $X$ be ( $X$ is define as a random variable $X : \Omega \rightarrow \mathbb{R}$) a value of a financial position, the set $\mathcal{X}$ of all possible financial position is finite (since $\Omega$ assumed to be finite ) and can be separated into acceptable set and non-acceptable sep (Regulatory perspective).

We want to quantify the amount that, added to a non-acceptable position $X$, make its acceptable to the regulator.
Desirable Properties for Monetary Risk Measure

- Therefore we need an appropriate monetary risk measure for the unacceptable position (downsid risk).
- Such a measure $\rho$ should have the following properties.
  - $\rho$ is a decreasing function of $X$
  - $\rho$ is stated in the same units as $X$
  - $\rho$ is positive if $X$ is non-acceptable
  - $\rho$ is a coherent risk measure
Exemple I : Worst-Case Risk Measure

Definition

The worst-case risk measure $\rho_{\text{max}}$ defined by

$$\rho_{\text{max}} (X) = - \inf_{v \in \Omega} X(v) \ \forall X \in \mathcal{X}$$

- The value of $\rho_{\text{max}}$ (Capital requirements) is the least upper bound for the potential loss which can occur in any scenario.
- $\rho_{\text{max}}$ is a coherent measure of risk
- Any normalized monetary risk measure $\rho$ on $\mathcal{X}$ satisfies

$$\rho(X) \leq \rho_{\text{max}} (X)$$

It is also the most conservative measure of risk.
- PML (Probable maximum loss) could be seen as equivalent in insurance
Example II: Value at Risk

Suppose that we have a probability measure $P$ on $(\Omega, \mathcal{F})$. In this context, a position $X$ is often considered to be acceptable if the **probability of a loss** is bounded by a given level $\lambda \in (0, 1)$ that means:

$$P[X < 0] \leq \lambda.$$ 

The corresponding monetary risk measure (necessary capital) is called **Value at Risk (VaR)**

**Definition: Value at Risk**

$$\text{VaR}_\lambda A(X) = \inf \{ m \in \mathbb{R} | P[m + X < 0] \leq \lambda \}$$

- Value-at-Risk refers to a quantile of the loss distribution
- Value-at-Risk is positively homogeneous, but in general it is not convex
VaR and Regulatory approach to Bank sector

VaR is the standard regulatory MMR for Bank sector for market-risk(MR). The internal model (IM) proposes the following formula to calculate the market-risk-capital at day t,

\[
RC_{IM}^t (MR) = \max \left\{ \text{VaR}_{0.99,10}^t ; \frac{k}{60} \sum_{i=1}^{60} \text{VaR}_{0.99,10}^{t-i+1} \right\} + RC_{SR}^t
\]

where

- \( \text{VaR}_{0.99,10}^t \) stands for a 10-day VaR at the 99% confidence level, calculated on day t.
- \( RC \) = Risk Capital
- \( MR = \) Market Risk and \( SR = \) Specific Risk
- \( k \in [3, 5] \) Stress Factor
VaR Regulatory approach to insurance sector

The Solvency Capital Requirement for every individual risk $i$, $SCR_\alpha(i)$, is defined as the difference between the Value at Risk and expected value (premium income),

$$SCR_\alpha(i) = VaR_\alpha(i) - \mu_i$$
