# Bilateral Cooperation 2012 -2013

## Stochastic differential equations describing critical fluctuations in a Van der Waals -Maxwell Gas

### 1. General Statement

1.1. **Applicants.**
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**Project responsible:** Prof. Dr. Errico Presutti, Head of group Classical Mechanics, Dipartimento di Matematica, Università di Roma Tor Vergata

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1.2. **Resume.** The aim of the project is to understand all possible fluctuations around the dynamics of a Van der Waals -Maxwell gas at the critical temperature. Different kind of fluctuations can be observed around the microscopic dynamics of gas particles, which depending on time and space scale, might become observable in the limit from microscopic to macroscopic. With respect to the space scale, this means roughly speaking that depending on the strength of the microscope we use, we might observe or not particular fluctuating events around the macroscopic dynamics. Especially at the critical temperature <u>non</u> linear fluctuations become macroscopic observables, if observed in a correct time -space limit. Computing and predicting critical fluctuations around an expected dynamic of a gas is particularly important if critical fluctuations can grow non linearly or even worse, as these can cause unexpected events beside the expected macroscopic deterministic dynamic.

The macroscopic dynamic of a gas is usually described by a deterministic function in space and time solving an equation depending on the temperature $T$. It is obtained in the limit by applying the "law of large numbers" to the microscopic interacting particles.
For the Van der Waals -Maxwell gas the macroscopic dynamic was derived by De Masi -Orlando -Presutti- Triolo [11], and is described by a pseudo -differential equation, i.e. a differential equation with a non local character, being a consequence of the long range interaction of particles described by Van der Waals.
Fluctuations around the deterministic dynamics are described by random processes. De Masi -Orlandi -Presutti -Triolo [11], [12] proved that scaling the same interacting particle dynamical model according to a "central limit theorem" linear fluctuations appear around the deterministic dynamic. In fact at any temperature and any dimension linear fluctuations are described by an Ornstein -Uhlenbeck stochastic process (OU-process), i.e a stochastic process which solves a linear Stochastic Differential Equation (SDE) with Gaussian noise.
We expect also non linear fluctuations to coexist at the critical temperature in a different time and space scaling. This was proven by two of the investigators (in collaboration) for the one -dimensional model in [10] (finite volume) and [14] (finite and infinite volume), while

the 2-dimensional case is still not solved. We observe that in a mean field model a critical mean field temperature $T_m$ exists also for the one dimensional macroscopic model, while a critical temperature $T_c^\epsilon$ described by the Dobrushin-Lanford-Ruelle (DLR) - theory for the corresponding microscopic model exists in dimension 2 and is a perturbation of the mean field critical temperature $T_m^\epsilon$, the perturbation $T_c^\epsilon - T_m$ vanishing in the limit from the microscopic to macroscopic scale, when $\epsilon \to 0$. We intent to prove that critical fluctuations converge at the temperature $T_c^\epsilon$ to the solution of the stochastic quantization equ ation (SQE), having as invariant measure the $\phi_2^4$ Euclidean field. Interesting is that the Wick power appearing in the stochastic quantization equation and related to the $\phi_2^4$ Euclidean field are expected to be generated by the perturbation of the temperature $T_c^\epsilon - T_m$. This was conjectured by one of the investigators in [15]. The reason why we have the necessary tools to solve this interesting problem, will be explained in the next paragraph. Solving this problem gives us the possibility to predict critical fluctuations for the dynamic of a Van der Waals -Maxwell gas and makes a link between the mathematical theory applied to statistical mechanics respectively of the quantum field theory. We plan to finish the project in 12 months.

In a possible continuation of this project we intend to investigate whether also critical fluctuations driven by a non Gaussian Lévy noise do coexist in a different time -space scaling, eventually having non finite moments and hence the characteristic of critical fluctuations discussed in the book by Glimm and Jaffe [3]. We expect non Gaussian Lévy noise driven fluctuations, because the macroscopic dynamic is described by a non local deterministic pseudo differential equation. We recall that there exists also central limit theorem converging to non Gaussian distributions, which is scaled differently with respect to the classical one [1]. Moreover the Kolmogorov equations associated to random processes driven by non Gaussain Lévy noise are non local pseudo differential equations.

**Selected publications related to the project:**

## References

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