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Bergische Universität Wuppertal, Due 19.12.2018

Exercise Sheet I -Probability

Notation:

0) $\{\Omega, \mathcal{F}, \mu\}$ denotes a measure space (finite or σ- finite measure), $\{\Omega, \mathcal{F}, P\}$ a probability space

a) $g \in \tau(\{\Omega, \mathcal{F}\})$, if g is a real valued function and $g(s) = \sum_{k=0}^{n-1} g_k \mathbf{1}_{A_k}(s)$, $A_k \in \mathcal{F}$

b) $g \in \Sigma_\infty(\{\Omega, \mathcal{F}\})$, if g is a real valued function and $g(s) = \sum_{k\in\mathbb{N}} g_k \mathbf{1}_{A_k}(s)$, $A_k \in \mathcal{F}$

c) $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ for f real -valued measurable function.

d) λ denotes the Lesbesgues measure , μ_u the uniform distribution.

e) Let $p \geq 1$, $\| \cdot \|_p$ is the norm in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$

f) μ_c denotes the distribution wich distribution function is given by the Cantor function.

Ex. I:
1) Define in two ways the "standard form" of $g \in \tau(\{\Omega, \mathcal{F}\})$ and prove that these are equivalent.

Ex. II:
Let $p \geq 1$ be fixed.

2) Prove that for each $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable function g, with $g \geq 0$, there is $g_n \in \Sigma_\infty(\{\Omega, \mathcal{F}\})$, so that $lim_{n\to\infty}\|g - g_n\|_\infty = 0$, and $g_n \leq g_{n+1}$ for all $n \geq 0$

3) Prove that if $g \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ then g_n ,$n \in \mathbb{N}$, can be chosen so that $g_n \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ and convergence holds in $\| \cdot \|_p$, too.

4) Prove that $\tau(\{\Omega, \mathcal{F}\}))$ is dense in $\Sigma_\infty(\{\Omega, \mathcal{F}\})) \cap \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ w.r.t the norm $\| \cdot \|_p$.

5) Prove that $\tau(\{\Omega, \mathcal{F}\}))$ is dense in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ w.r.t the norm $\| \cdot \|_p$.

Ex. III:
6) Let X_n, $n \in \mathbb{N}$, be real valued random variables on $(\Omega, \mathcal{F}, P)$. Prove that $sup_{n\in\mathbb{N}}(X_n)$ is a random variable.

Ex. IV:
Let $0 \leq p < 1$ be fixed.

7) Write the distribution of $B(n,p)$ as a combination of Delta -Distributions. Write its Distribution function and sketch a picture.

8) Compute $\mathbb{E}[exp(X)]$ for a random Variable X which is $B(n,p)$ distributed (minimizing the effort and passages. Find a smart way!)

Ex. V:

9) Prove: if X is a random variable which distribution function F is strict monoton and continuous, then $F(X)$ is uniformly distributed.

Ex. VI:

10) Let I be an Index set. Let $\mathcal{F}_\alpha$ be a σ -Algebra on a set Ω. Prove that $\cap_{\alpha\in I}\mathcal{F}_\alpha$ is a σ -Algebra on Ω.

Ex. VII:

11) Prove the monotonicity property of a Probability measure P for a sequence of measurable sets $A_n \subset A_{n+1}$, $n \in \mathbb{N}$

12) Use the result in 10) to prove the monotonicity property of a Probability measure P for a sequence of measurable sets $A_n \supset A_{n+1}$, $n \in \mathbb{N}$

Ex. VIII:

13) Let $(\Omega, \mathcal{F})$ be a measurable space. Prove: let $A \in \mathcal{F}$, $A \neq \emptyset$, then $\mathcal{F}|_A := \{C = A \cap B : B \in \mathcal{F}\}$ is a σ -Algebra on A.

14) Let P be a probability measure on $(\Omega, \mathcal{F})$ and $P(A) > 0$. Prove that $P(\cdot/A)|_{\mathcal{F}|_A})$ is a probability measure on $(A, \mathcal{F}|_A)$.

Ex. IX:

15) Find on $([0,1], \mathcal{B}([0,1]), \mu_u)$ a sequence $\{X_n\}_{n\in\mathbb{N}}$ of real valued random variables which each take two different values x_n, y_n and converge in probability to $X = 1_{[0,1]}$, but not in $L^1([0,1], \mathcal{B}([0,1]), \mu_u)$

16) Find on $([0,1], \mathcal{B}([0,1]), \mu_u)$ a sequence $\{X_n\}_{n\in\mathbb{N}}$ of real valued random variables, which each take two different values x_n, y_n, which converge in $L^1([0,1], \mathcal{B}([0,1]), \mu_u)$ but not in $L^2([0,1], \mathcal{B}([0,1]), \mu_u)$

Ex. X:

17) Prove that μ_C has no density

18) Analyze the convergence in probability μ_C of $\{X_n\}_{n\in\mathbb{N}}$, with $X_n := 6^n 1_{[0,\frac{1}{3^n}]}$

Ex. XI:

19) Find an example of two random variables X, Y on $\{\Omega, \mathcal{F}, P\}$, which are uncorrelated , but not stochastic independent

20) Find an example of a sequence of random variables X_n, $n \in \mathbb{N}$, on $\{\Omega, \mathcal{F}, P\}$, which are parewise stochastic independent, but not stochastic independent.

Remark: all results must be motivated and proven