Non-internal sets with internal initial segments

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It follows from Theorem 1.4.6(ii) that $\mathbf{\Delta}_2^{\mathtt{ss}}$ sets of $*$-ordinals, with all proper initial segments internal, are necessarily internal themselves. However if the internality is replaced by a stronger condition of h-standardness [1] then the result is not true any more. We present here two counterexamples of this kind.

We argue in **HST**, but 1) the arguments are easily accomodable in the **BST** environment, and 2) the arguments are also accomodable to the model-theoretic setting (that is, models in the **ZFC** universe).

Example 1

The following theorem is, essentially, an **HST** version of a result communicated to us by Andreas Blass [2]. As usual, **CH** is the continuum hypothesis $\aleph_1 = 2^{\aleph_0}$.

Theorem 1 (HST). *Suppose that* **CH** *holds in the well-founded universe* $\mathbb{WF}$. *Then for any* $h \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ *there is a set* $X \subseteq {}^*\mathbb{N}$ *such that*

(i) $X \cap [0, a]$ *is* h*-standard for any* h*-standard* $a \in {}^*\mathbb{N}$;

(ii) X *itself is internal, but* <u>*not*</u> h*-standard.*

Some questions can be addressed to this simple result: for instance is **CH** eliminable ?

Proof. Let $F = \mathbb{N}^{\mathbb{N}}$ be the set of all maps $f : \mathbb{N} \to \mathbb{N}$. (Both F and all its elements belong to the well-founded universe $\mathbb{WF}$.) The set $A = \{{}^*\!f(h) : f \in F\}$ of all h-standard elements of ${}^*\mathbb{N}$ is a standard size subset of ${}^*\mathbb{N}$ of cardinality $\le \mathfrak{c}$, hence, in the assumption of **CH**, $\mathtt{card}\, A = \mathfrak{c} = \aleph_1$, and there exists a cofinal strictly increasing sequence $\{a_\xi\}_{\xi<\omega_1}$ of elements $a_\xi = {}^*\!f_\xi(h) \in A$, where $f_\xi \in F$ for all $\xi < \omega_1$. (**Exercise**: prove that A is not countably cofinal and does not have a largest element.) We assume w. l. o. g. that $a_0 = 0$ and accordingly f_0 is the constant 0.

Let, further, $\Phi = \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ be the set of all maps $\varphi : \mathbb{N} \to \mathscr{P}(\mathbb{N})$, still a set in $\mathbb{WF}$ together with all its elements. Note that in our assumptions $\mathtt{card}\, \Phi = \mathfrak{c}^{\aleph_0} = \mathfrak{c} = \aleph_1$, and hence Φ can be presented, in $\mathbb{WF}$, in the form $\Phi = \{\varphi_\xi : \xi < \omega_1\}$. Then the set B of all h-standard sets $b \subseteq {}^*\mathbb{N}$ is equal to $\{{}^*\varphi_\xi(h) : \xi < \omega_1\}$.

To prove the theorem, it suffices to find an internal set $X \subseteq {}^*\mathbb{N}$ satisfying

(iii) $a_\xi \in X \iff a_\xi \notin {}^*\varphi_\xi(h)$ holds for any $\xi < \omega_1$

[1] Recall that a set x is h-standard iff it has the form $x = {}^*\!f(h)$, where $f \in \mathbb{WF}$ is a function and $x \in \mathtt{dom}\, {}^*\!f$.

[2] In the following form (in **ZFC**): *assuming* **CH**, *if* U *is a non-principal ultrafilter over* $\mathbb{N}$ *and* ${}^*\mathbf{V}$ *the corresponding ultrapower of the set universe* $\mathbf{V}$ *then there is a non-internal set* $X \subseteq {}^*\mathbb{N}$ *such that any intersection* $X \cap [0, a)$, $a \in {}^*\mathbb{N}$, *is internal.*

together with (i): indeed, (iii) implies that X is different from any set of the form ${}^*\varphi_\xi(h)$, $\xi < \omega_1$, hence, it is not h-standard, as required.

To find such a set X, we define, by induction on ξ, an h-standard set $X_\xi \subseteq [0, a_\xi]$ satisfying (iii) (as X and for this particular ξ) and

(iv) $X_\xi = X_\eta \cap [0, a_\xi]$ whenever $\xi < \eta$.

If this is accomplished then, by Saturation, there is an internal set $X \subseteq {}^*\mathbb{N}$ such that $X_\xi = X \cap [0, a_\xi]$ for any $\xi < \omega_1$, and obviously X satisfies (i) and (iii).

The construction of X_0 and the step $\xi \to \xi + 1$ are trivial. (For instance, we put $X_{\xi+1} = X_\xi$ or $X_{\xi+1} = X_\xi \cup \{a_\xi\}$ in cases, resp., $a_\xi \in {}^*\varphi_\xi(h)$ and $a_\xi \notin {}^*\varphi_\xi(h)$.) Now suppose that $\lambda < \omega_1$ is a limit ordinal, and $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \ldots$ is a cofinal sequence in λ (in $\mathbb{WF}$). By the inductive hypothesis, for any n there is an an h-standard set $X_{\xi_n} \subseteq [0, a_{\xi_n}]$ satisfying (iii) and $X_{\xi_n} = X_{\xi_m} \cap [0, a_{\xi_n}]$ whenever $n < m < \omega$. Then for any n there is a function $\varphi_n \in \Phi$ such that $X_{\xi_n} = {}^*\varphi_n(h)$ (Standard Size Choice is applied).

Now, arguing in $\mathbb{WF}$ we define $\varphi \in \Phi$ as follows: for any $x \in \mathbb{N}$ and any n, if $f_{\xi_n}(x) \le f_{\xi_{n+1}}(x)$ then

$$\varphi(x) \cap [f_{\xi_n}(x), f_{\xi_{n+1}}(x)] = \varphi_{n+1}(x) \cap [f_{\xi_n}(x), f_{\xi_{n+1}}(x)].$$

and, in addition, $\varphi(x) \cap [M, +\infty) = \varnothing$ whenever $M = \sup_{n \in \mathbb{N}} f_{\xi_n}(x) < +\infty$. **Exercise**: use $*$-Transfer to prove that $X_\lambda = {}^*\varphi(h)$ is as required. □

Example 2

The following theorem is essentially from Blass [Bl 1977] (where it is given in terms of ultrapowers). Let $\Xi = \bigcup_{\alpha < \omega_1} {}^*\alpha$: this is a proper and non-internal initial segment of ${}^*\omega_1$. In the following theorem, $\Sigma_1^{\mathsf{st}}(h)$ means the class of all sets $X \subseteq \mathbb{I}$ that admit a Σ_1^{st} definition in $\mathbb{I}$ with h and standard sets as parameters.

Theorem 2 (HST). *For any $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ there is a $\Sigma_1^{\mathsf{st}}(h)$ set $X \subseteq \Xi$ such that*

(i) *all intersections of the form $X \cap {}^*\alpha$, $\alpha < \omega_1$, are h-standard*;

(ii) *there does not exist a h-standard set $H \subseteq {}^*\omega_1$ such that $X = H \cap \Xi$.* [3]

Proof. Step 1. *Arguing in* $\mathbb{WF}$, let us fix a sequence $\{W_\gamma\}_{\gamma < \omega_1}$ of sets $W_\gamma \subseteq \mathbb{N}$ such that $W_\gamma \triangle W_\delta$ is infinite whenever $\gamma \ne \delta < \omega_1$. We are going to define a set $Y_\alpha(n) \subseteq \alpha = [0, \alpha)$ for any $\alpha < \omega_1$ and $n \in \mathbb{N}$, so that

(1) the set $C_{\alpha\beta} = \{n : Y_\alpha(n) = Y_\beta(n) \cap \alpha\}$ is cofinite whenever $\alpha < \beta < \omega_1$;

[3] In other words, X is not h-standardly extendable. Note that by $\aleph_2$-Saturation there exist internal sets $H \subseteq {}^*\omega_1$ such that $X = H \cap \Xi$, but by the theorem they cannot be h-standard.

(2) for any limit ordinal $\gamma < \omega_1$ there exists a cofinal map $b_\gamma : \mathbb{N} \to \gamma$ [4] such that $(*)$ for any n and any ordinal α, $b_\gamma(n) \le \alpha < \gamma$, of the form $\alpha = \lambda + k$, λ limit and $k \in \mathbb{N}$, we have $\alpha \in Y_\gamma(n)$ iff $k \in W_\gamma$.

To begin with, put $Y_0(n) = \varnothing$ and $Y_{\alpha+1}(n) = Y_\alpha(n)$ for all α and n.

Now let us concentrate on the limit step: let $\gamma < \omega_1$ be a limit ordinal.

Let $\alpha_0 < \alpha_1 < \alpha_2 < \dots$ be a cofinal ω-sequence in γ. It follows from (1) that the sets $D_m = \bigcap_{j<m} C_{\alpha_j,\alpha_{j+1}}$ are cofinite (in $\mathbb{N}$). Then the sets $D'_m = \{n \in D_m : n \ge m\} \subseteq D_m$ are still cofinite and $\bigcap_m D'_m = \varnothing$. Note in addition that $D'_0 = D_0 = \mathbb{N}$ and $D_{m+1} \subseteq D_m$ for all m. Thus the map $b_\gamma(n) = \alpha_m$, where m is the least index m with $n \notin D'_{m+1}$, is cofinal in γ.

Now define, for any n, a set $Y_\gamma(n) \subseteq \gamma$ so that $Y_\gamma(n) \cap \overline{\gamma}(n) = Y_{\overline{\gamma}(n)}(n)$ and (2)$(*)$ holds (for this n). It remains to show that (1) holds, that is, the sets $C_{\alpha\gamma} = \{n : Y_\alpha(n) = Y_\gamma(n) \cap \alpha\}$, $\alpha < \gamma$, are cofinite. Let $\alpha < \gamma$. Then $\alpha \le \alpha_m$ for some m. As obviously $C_{\alpha,\alpha_m} \cap C_{\alpha_m,\gamma} \subseteq C_{\alpha\gamma}$, it remains to show that $C_{\alpha_m,\gamma}$ is cofinite — but this is clear because $D'_m \subseteq C_{\alpha_m,\gamma}$.

Step 2. We consider the $*$-extensions $\{{}^*b_\gamma\}_{\gamma<{}^*\omega_1}$ (then ${}^*b_\gamma : {}^*\mathbb{N} \to \gamma$), $\{{}^*Y_\gamma(n)\}_{\gamma<{}^*\omega_1,\, n\in{}^*\mathbb{N}}$ (${}^*Y_\gamma(n) \subseteq \gamma$) and ${}^*C_{\alpha\beta} = \{n \in {}^*\mathbb{N} : {}^*Y_\alpha(n) = {}^*Y_\beta(n) \cap \alpha\}$ ($\alpha < \beta < {}^*\omega_1$) — all of them standard sequences or matrices (of internal objects). Take any $h \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$. If $\alpha < \beta < \omega_1$ then the set ${}^*C_{{}^*\alpha\,{}^*\beta} = {}^*(C_{\alpha\beta})$ is a standard finite subset of ${}^*\mathbb{N}$, and hence ${}^*C_{{}^*\alpha\,{}^*\beta}$ consists of standard elements, thus $h \notin {}^*C_{{}^*\alpha\,{}^*\beta}$. It follows, by $*$-Transfer, that ${}^*Y_{{}^*\alpha}(h) = {}^*Y_{{}^*\beta}(h) \cap {}^*\alpha$ for any pair of $\alpha < \beta < \omega_1$. Thus, the $\Sigma_1^{\mathsf{st}}(h)$ set $X = \bigcup_{\alpha<\omega_1} {}^*Y_{{}^*\alpha}(h) \subseteq {}^*\omega_1$ satisfies ${}^*Y_{{}^*\alpha}(h) = X \cap {}^*\alpha$ for any $\alpha < \omega_1$, and hence (i) of the theorem holds.

We claim that (ii) *of the theorem also holds.*

Step 3. Assume towards the contrary that $\eta \in \mathbb{WF}$ is a map $\mathbb{N} \to \mathscr{P}(\omega_1)$, and the ($h$-standard) set $H = {}^*\eta(h) \subseteq {}^*\omega_1$ satisfies $X = H \cap \Xi$, that is, $H \cap {}^*\alpha = {}^*Y_{{}^*\alpha}(h)$ for all $\alpha < \omega_1$.

Step 4. *Arguing in* $\mathbb{WF}$, we prove

Lemma 3 (in $\mathbb{WF}$)**.** *For any $Z \subseteq \omega_1$ and n, the set Λ of all limit ordinals $\gamma < \omega_1$, such that $Y_\gamma(n) = Z \cap \gamma$, does not contain any its limit point.*

Proof. Assume on the contrary that $\gamma \in \Lambda$ is a limit ordinal, and $\{\gamma_j\}_{j\in\mathbb{N}}$ is a sequence of (limit) ordinals $\gamma_j \in \Lambda$, cofinal in γ. As $\alpha = b_\gamma(n) < \gamma$, there is an index j with $\alpha < \gamma_{j-1} < \gamma_j < \gamma$. Let ξ be the largest of the ordinals α and $\alpha_j = b_{\gamma_j}(n)$. Then, since both γ and γ_j are limit, the infinite interval $U = [\xi, \xi+\omega)$ satisfies $U \subseteq [\alpha_j, \gamma_j) \cap [\alpha, \gamma)$. Therefore, by (2), the intersection $Y_{\gamma_j}(n) \cap U$ corresponds to W_{γ_j} (or rather to a cofinite "tail" of W_{γ_j} because ξ is not necessarily limit) while the intersection $Y_\gamma(n) \cap U$ corresponds to W_γ, in such a way that surely $Y_{\gamma_j}(n) \cap U \ne Y_\gamma(n) \cap U$ because W_{γ_j} and W_γ have infinitely many differences. But this contradicts to the assumption that both γ_j and γ belong to Λ. □

[4] We mean that for any ordinal $\gamma' < \gamma$ the set $\{n : b_\gamma(n) < \gamma'\}$ is finite.

Coming back to the map $\eta : \mathbb{N} \to \mathscr{P}(\omega_1)$, we define, in $\mathbb{WF}$,

$$\Lambda(n) = \{\gamma < \omega_1 : \gamma \text{ limit and } Y_\gamma(n) = \eta(n) \cap \gamma\}\,.$$

Corollary 4 (in $\mathbb{WF}$)**.** *There exists a closed unbounded set* $E \subseteq \omega_1$ [5] *such that* $E \cap \Lambda(n) = \varnothing$ *for any* n.

Proof. We can assume that all sets $\Lambda(n)$ are unbounded in ω_1. In this assumption, a strictly increasing and continuous (that is, $\xi_\lambda = \mathtt{sup}_{\alpha<\lambda}\,\xi_\alpha$ for all limit ordinals λ) sequence $\{\xi_\alpha\}_{\alpha<\omega_1}$ of ordinals $\xi_\alpha < \omega_1$ can be defined so that any interval of the form $[\xi_{\lambda+m}, \xi_{\lambda+m+1})$, λ limit, contains at least one element of each set $\Lambda(n)$, $n \le m$. Then $\xi_\gamma \notin \Lambda(n)$ for all n and all limit γ by the lemma, and hence $E = \{\xi_\gamma : \gamma \text{ limit}\}$ is as required. □

Step 3: continuation. Using Corollary 4, we accomplish the proof of (ii) of the theorem. The set ${}^*\Lambda(h) = \{\beta < {}^*\omega_1 : {}^*Y_\beta(h) = H \cap \beta\}$ contains all $*$-ordinals of the form ${}^*\alpha$, $\alpha < \omega_1$ by the assumption of Step 3. In particular, ${}^*\varepsilon \in {}^*\Lambda(h)$, where ε is any ordinal in E (in $\mathbb{WF}$). On the other hand, by $*$-Transfer, we have ${}^*\varepsilon \notin {}^*\Lambda(n)$ for any $n \in {}^*\mathbb{N}$, in particular, ${}^*\varepsilon \notin {}^*\Lambda(h)$, which is a contradiction.

□ (*Theorem*)

Remark 5. It remains to note the following. For any $h \in {}^*\mathbb{N}$ the class $\mathbb{S}[h] \subseteq \mathbb{I}$ of all h-standard sets (to be more exact, the structure $\langle \mathbb{S}[h]\,; \in, \mathtt{st}\rangle$) satisfies the axioms of $\mathbf{BST}'_\omega$, a partially saturated version of **BST** with Saturation restricted to countable families and Inner Boundedness accordingly strengthened, see § 3.3a.

The proof of the theorem can be transformed to a formal deduction in $\mathbf{BST}'_\omega$ that there exists an "external subset" of ω_1 whose intersections with all standard ordinals are internal. □

Problem 6. There are two possible generalizations of Theorem 2. First, consider $h \in {}^*X \setminus X$, where $X \in \mathbb{WF}$ is an uncountable cardinal, ω_1 to begin with. Second, consider subsets of ${}^*\kappa$ instead of ${}^*\omega_1$, κ be anything larger.

For instance, given $h \in {}^*\mathbb{N} \setminus \mathbb{N}$, does there exist ? a $\Sigma^{\mathtt{st}}_1(h)$ set $X \subseteq \Xi_2$, where $\Xi_2 = \bigcup_{\alpha<\omega_2} {}^*\alpha$, such that

(i) all intersections of the form $X \cap {}^*\alpha$, $\alpha < \omega_2$, are internal;

(ii) there is no h-standard set $H \subseteq {}^*\omega_2$ such that $X = H \cap \Xi$.

The answer in the negative (for all $h \in {}^*\mathbb{N} \setminus \mathbb{N}$) may give a hint to Problem 3.2.5 (is any locally internal class $K \subseteq \mathbb{I}$ $\in$-definable in $\mathbb{I}$ with parameters). □

References

[Bl 1977] A. Blass, End extensions, conservative extensions, and the Rudin-Frolik ordering. *Trans. Amer. Math. Soc.* 1977, 225, pp. 325 – 340.

[5] In fact only $E \neq \varnothing$ will be used.