Global solutions for the
Dirac-Klein-Gordon system in two space
dimensions

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Abstract

The Cauchy problem for the classical Dirac-Klein-Gordon system in two space dimensions is globally well-posed for $L^2$ Schrödinger data and wave data in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$. In the case of smooth data there exists a global smooth (classical) solution. The proof uses function spaces of Bourgain type based on Besov spaces – previously applied by Colliander, Kenig and Staffilani for generalized Benjamin-Ono equations and also by Bejenaru, Herr, Holmer and Tataru for the 2D Zakharov system – and the null structure of the system detected by d’Ancona, Foschi and Selberg, and a refined bilinear Strichartz estimate due to Selberg. The global existence proof uses an idea of Colliander, Holmer and Tzirakis for the 1D Zakharov system.

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1 Introduction and main results

Consider the Cauchy problem for the Dirac – Klein – Gordon equations in two space dimensions

\[ i(\partial_t + \alpha \cdot \nabla)\psi + M\beta\psi = -\phi\beta\psi \]
\[ (-\partial_t^2 + \Delta)\phi + m\phi = -\langle \beta\psi, \psi \rangle \]

with (large) initial data

\[ \psi(0) = \psi_0, \phi(0) = \phi_0, \partial_t \phi(0) = \phi_1. \]

Here \( \psi \) is a two-spinor field, i.e. \( \psi : \mathbb{R}^{1+2} \to \mathbb{C}^2 \), and \( \phi \) is a real-valued function, i.e. \( \phi : \mathbb{R}^{1+2} \to \mathbb{R} \), \( m, M \in \mathbb{R} \) and \( \nabla = (\partial_{x_1}, \partial_{x_2}) \), \( \alpha \cdot \nabla = \alpha^1\partial_{x_1} + \alpha^2\partial_{x_2} \). \( \alpha^1, \alpha^2, \beta \) are hermitian \( (2 \times 2) \)-matrices satisfying \( \beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I \), \( \alpha^1\beta + \beta\alpha^1 = 0 \), \( \alpha^i\alpha^k + \alpha^k\alpha^i = 2\delta^{jk}I \).

\( \langle \cdot, \cdot \rangle \) denotes the \( \mathbb{C}^2 \)-scalar product. A particular representation is given by \( \alpha^1 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \), \( \alpha^2 = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \), \( \beta = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \).

We consider Cauchy data in Sobolev spaces: \( \psi_0 \in H^s, \phi_0 \in H^r, \phi_1 \in H^{r-1} \).

The fundamental conservation law is charge conservation \( \|\psi(t)\|_{L^2} = \text{const}. \)

In the \( (1+1) \)-dimensional case global well-posedness for smooth data was already established by Chadam [C] and also for much less regular data by Bournaveas [B], Fang [F], Bournaveas and Gibbeson [BG], Machihara [M], Pecher [P], Selberg [S1], Selberg - Tesfahun [ST] and Tesfahun [T], the last two authors also for data \( \psi_0 \notin L^2 \). In the \( (2+1) \)-dimensional and \( (3+1) \)-dimensional case no global well-posedness results for large data were known so far. In \( (2+1) \)-dimensions local well-posedness was proven by Bournaveas [B1], if \( s > \frac{1}{4} \) and \( r = s + \frac{1}{4} \), which was later improved by d’Anotca, Foschi and Selberg [AFS1] to the case \( s > -\frac{1}{4} \) and \( \max(\frac{1}{4} - \frac{s}{2}, \frac{1}{4} + \frac{s}{2}, s) < r < \min(\frac{3}{4} + 2s, \frac{3}{4} + \frac{3s}{2}, 1 + s) \). Their proof relied on the null structure of the system. This complete null structure was detected by d’Ancona, Foschi and Selberg in their earlier paper [AFS], where it was applied to show an almost optimal local existence result in \( (3+1) \)-dimensions, namely if \( s = \epsilon, r = \frac{1}{2} + \epsilon \) for any \( \epsilon > 0 \).

We now give the first global well-posedness result for large data in two space dimensions. It holds in the case \( s = 0, r = \frac{1}{2} \), and more generally in the case \( s \geq 0, r = s + \frac{1}{2} \), where local well-posedness was known to be true before already (by d’Ancona, Foschi and Selberg [AFS1]). Especially we show the existence of global classical solutions for smooth data. It is necessary to refine the local existence result by replacing Bourgain spaces \( X^{s,b}_\pm \) and \( X^{r, b}_\pm \) for \( b > \frac{1}{2} \) constructed from Sobolev spaces by their analogue constructed from Besov spaces with respect to time, especially \( X^{s, \frac{1}{2} + 1}_\pm \) and \( X^{r, \frac{1}{2} + 1}_\pm \) (see the definition below). Spaces of this type were already successfully used to give a local well-posedness result for the 2D - Zakharov system by Bejenaru, Herr, Holmer and Tataru [BHT] and Colliander, Kenig and Staffilani for generalized Benjamin-Ono equations [CKS]. The precise bound for the existence time then can be combined with the charge conservation to show global well-posedness for our 2D Dirac-Klein-Gordon system. A similar procedure was already used by Colliander, Holmer and Tzirakis for the one-dimensional Zakharov system [CHT]. It turns out that the choice of the regularity parameters \( s \) and \( r \) in our case just allows to estimate both nonlinearities in a unified way. What one also needs are of course the Strichartz estimates for the wave equation, here
also the Besov space version to avoid the endpoint Strichartz estimate in 2D. The Strichartz estimates however are not sufficient for a particularly delicate case where it is essential to use a bilinear refinement which was detected by Selberg [S] and can also be found in Foschi-Klainerman [FK]. This version was already used by d’Ancona, Foschi and Selberg [AFS1] in their local well-posedness result.

We use the following function spaces. Let \( \hat{\varphi} \) denote the Fourier transform with respect to space or time and \( \hat{\varphi} \) the Fourier transform with respect to space and time simultaneously. Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) be a nonnegative function with \( \text{supp} \varphi \subset \{ 1/2 \leq |\xi| \leq 2 \} \) and \( \varphi(\xi) > 0 \), if \( \frac{1}{\sqrt{2}} \leq |\xi| \leq \sqrt{2} \). Setting \( \tilde{\rho}_k(\xi) := \varphi(2^{-k}\xi) \) \((k = 1, 2, \ldots)\) and \( \tilde{\varphi}_0(\xi) := 1 - \sum_{k=1}^\infty \tilde{\varphi}_k(\xi) \) we have \( \text{supp} \tilde{\varphi}_k \subset \{ 2^{k-1} \leq |\xi| \leq 2^{k+1} \} \), \( \text{supp} \tilde{\varphi}_0 \subset \{|\xi| \leq 2\} \) and \( \sum_{k=0}^\infty \tilde{\varphi}_k = 1 \).

The Besov spaces are defined for \( s \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \) as follows:

\[
B^s_{p,q} = \{ f \in \mathcal{S}', \| f \|_{B^s_{p,q}} < \infty \},
\]

where

\[
\| f \|_{B^s_{p,q}} = \left( \sum_{k=0}^\infty (2^{sk}\| \hat{\varphi}_k * f \|_{L^p})^q \right)^{\frac{1}{q}} \text{ if } q < \infty,
\]

\[
\| f \|_{B^s_{p,q}} = \sup_{k \geq 0} 2^{sk}\| \hat{\varphi}_k * f \|_{L^p}
\]

(cf. e.g. Triebel [Tr], Section 2.3.1). Similarly the homogeneous Besov spaces are defined as the set of those \( f \in \mathcal{S}' \), for which \( \| f \|_{B^s_{p,q}} \) is finite, where

\[
\| f \|_{B^s_{p,q}} = \left( \sum_{k=-\infty}^\infty (2^{sk}\| \hat{\varphi}_k' * f \|_{L^p})^q \right)^{\frac{1}{q}}
\]

with the usual modification for \( q = \infty \) and \( \hat{\varphi}_k'(\xi) := \frac{\sum_{j=-\infty}^\infty \tilde{\rho}_j(2^{-j}\xi)}{\sum_{j=-\infty}^\infty \tilde{\rho}_j(2^{-j}\xi)} \) for \( k \in \mathbb{Z} \). We also need the following Bourgain type spaces. The standard spaces belonging to the half waves are defined by the completion of \( \mathcal{S}(\mathbb{R} \times \mathbb{R}^2) \) with respect to

\[
\| f \|_{X^s_{\pm, h}} = \| U_{\pm}(-t)f \|_{H^s_t H^h_x} = \| (\xi)^s(\tau \pm |\xi|)\hat{f}(\tau, \xi) \|_{L^2_{\tau, \xi}}
\]

where

\[
U_{\pm}(t) := e^{\mp it|D|}
\]

and

\[
\| g \|_{H^s_t H^h_x} = \| (\xi)^s(\tau)^h\hat{g}(\xi, \tau) \|_{L^2_{\tau, \xi}}.
\]

We also define \( X^s_{\pm, h, q} \) as the space of all \( u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^2) \), where the following norms are finite:

\[
\| f \|_{X^s_{\pm, h, q}} = \| U_{\pm}(-t)f \|_{B^s_{p,q} H^h_x} = \left( \sum_{k=0}^\infty 2^{sk}\| (\xi)^s\hat{\varphi}_k(\tau \pm |\xi|)\hat{f}(\tau, \xi) \|_{L^2_{\tau, \xi}}^q \right)^{\frac{1}{q}}
\]

for \( 1 \leq q < \infty \), where

\[
\| g \|_{B^s_{p,q} H^h_x} = \left( \sum_{k=0}^\infty 2^{sk}\| (\xi)^s\hat{\varphi}_k(\tau)\hat{f}(\tau, \xi) \|_{L^2_{\tau, \xi}}^q \right)^{\frac{1}{q}}
\]

and

\[
\| f \|_{X^s_{\pm, h, \infty}} = \| U_{\pm}(-t)f \|_{B^s_{p,\infty} H^h_x} = \sup_{k \geq 0} 2^{sk}\| (\xi)^s\hat{\varphi}_k(\tau \pm |\xi|)\hat{f}(\tau, \xi) \|_{L^2_{\tau, \xi}}
\]

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for $q = \infty$, where

$$\|g\|_{B^b_{q,\infty}H^c_x} = \sup_{k \geq 0} 2^{bk} \|\langle \xi \rangle^s \hat{g}(\tau, \xi)\|_{L^2_{\tau,\xi}}.$$  

Note that $U_\pm(t) = e^{\mp it(-\Delta+1)^{1/2}}$ would lead to equivalent norms.

Spaces of type $X^{s,b,q}$ with various phase functions $\phi(\xi)$ instead of $\pm |\xi|$ have been used in the literature before, for example by Colliander, Kenig, and Staffilani in their work on dispersion generalized Benjamin-Ono equations [CKS]. As was observed in [CKS, proof of Lemma 5.1], they can be obtained by real interpolation from the standard $X^{s,b}$-spaces. In fact, by [BL, Theorem 5.6.1] one has for $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $b_0 \neq b_1$ and $b = (1-\theta)b_0 + \theta b_1$, $0 < \theta < 1$, that

$$X^{s,b_0}, X^{s,b_1} \theta, q = X^{s,b,q}.$$  

Using the duality Theorem [BL, Theorem 3.7.1] we see that for $1 \leq q < \infty$

$$(X^{s,b,q})' = X^{-s,-b,q'},$$

where $X$ denotes the space of complex conjugates of elements of $X$ with norm $\|f\|_X = \|\tilde{f}\|_X$. In the proof of the crucial bilinear estimates for local well-posedness we will repeatedly make use of complex interpolation. To justify this we use the corresponding theorem on interpolation of spaces of vector valued sequences [BL, Theorem 5.6.3] and take into account the considerations in [BL, Section 6.4] to see that

$$(X^{s,b_0}, q_0, X^{s_1,b_1,q_1}) = X^{s,b,q},$$

whenever $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $b = (1-\theta)b_0 + \theta b_1$, and $1 \leq q_0, q_1 \leq \infty$ as well as $\frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1}$.

The preceding remarks on duality and interpolation are completely independent of the specific phase function.

For $B \subset S' (\mathbb{R} \times \mathbb{R}^2)$ we denote by $B(T)$ the space of restrictions of distributions in $B$ to the set $(0, T) \times \mathbb{R}^2$ with induced norm.

We use the Strichartz estimates for the homogeneous wave equation in $\mathbb{R}^n \times \mathbb{R}$, which can be found e.g. in Ginibre-Velo [GV], Prop. 2.1.

**Proposition 1.1** Let $\gamma(r) = (n-1)((\frac{1}{q} - \frac{1}{2})$, $\delta(r) = n(\frac{1}{2} - \frac{1}{r})$, $n \geq 2$. Let $\rho, \mu \in \mathbb{R}$, $2 \leq q, r \leq \infty$ satisfy $0 \leq \frac{1}{q} \leq \min(\gamma(r), 1)$, $(\frac{2}{q}, \gamma(r)) \neq (1, 1)$, $\rho + \delta(r) - \frac{1}{q} = \mu$.

Then

$$\|u_0\|_{L^\nu(\mathbb{R}, B^0_{2,2}(\mathbb{R}^n))} \leq c\|u_0\|_{H^{\nu}(\mathbb{R}^n)}.$$  

The same holds with $B^{0}_{2,2}$ replaced by $H^{\nu}$ under the additional assumption $r < \infty$.

The following consequence of estimates of Strichartz type is important for our considerations.

**Proposition 1.2** Let $Y \subset S' (\mathbb{R} \times \mathbb{R}^n)$ be a set of functions of space and time with the property that

$$\|hf\|_Y \leq c\|h\|_{L^\nu\nu} \|f\|_Y$$

for all $h \in L^\infty\nu$ and $f \in Y$. Assume moreover the (Strichartz type) estimate

$$\|U_\pm(t)u_0\|_Y \leq c\|u_0\|_{H^\nu},$$

where $U_\pm(t) = e^{\mp it|D|}$. Then the following estimate holds:

$$\|f\|_Y \leq c\|f\|_{X^{0,\frac{1}{2},1}_\pm}.$$
Proof: We combine Lemma 2.3 in [GTV] with the proof of the embedding $B^{1/2}_{2,1}(\mathbb{R}) \subset C^0(\mathbb{R})$. Let $\psi$ be a $C^\infty_0(\mathbb{R})$ - function with $\psi(\tau) = 1$ for $1/2 \leq |\tau| \leq 2$ and $\hat{\psi}_k(\tau) := \psi(2^{-k}\tau)$ , so that $\hat{\psi}_k(\tau) = 1$ for $2^{k-1} \leq |\tau| \leq 2^{k+1}$. Furthermore we define $\hat{\psi}_0 \in C^\infty_0$ such that $\hat{\psi}_0(\tau) = 1$ for $|\tau| \leq 2$. The functions $\varphi_k$ are those which appear in the definition of the Besov norms (here in the 1-dimensional case). We thus have the property that $\hat{\psi}_k(\tau) = 1$ for $\tau \in \text{supp} \hat{\varphi}_k$ $(k = 0, 1, 2, ...$). We start from

$$f = \int e^{it\tau} U_\pm(t)(\mathcal{F}_i U_\pm(-\cdot)f)(\tau)d\tau.$$  

Then we have with $h = e^{it\tau}$ (for fixed $\tau$):

$$\|f\|_Y \leq \int \|U_\pm(t)(\mathcal{F}_i U_\pm(-\cdot)f)(\tau)\|_Y d\tau \leq c \int \|\mathcal{F}_i U_\pm(-\cdot)f\|_{H^\nu} d\tau$$

$$\leq c \sum_{k=0}^\infty \|\mathcal{F}_i U_\pm(-\cdot)f\|_{L^2_x(H^\nu)} \leq c \sum_{k=0}^\infty \|\mathcal{F}_i U_\pm(-\cdot)f\|_{L^2_x(H^\nu)} \|\hat{\varphi}_k\|_{L^2}$$

$$\leq c \sum_{k=0}^\infty 2^{k/2} \|U_\pm(-\cdot)f\|_{L^2_x(H^\nu)} \|\hat{\varphi}_k\|_{L^2}$$

$$= c \|U_\pm(-\cdot)f\|_{B^{1/2}_{2,1}(H^\nu)} = c \|f\|_{X^{\nu,\frac{1}{2},1}}.$$  

where we used $\|\hat{\varphi}_k\|_{L^2} = 2^{k/2}\|\varphi\|_{L^2}$ , $(k = 1, 2, ...$).

Similarly one can prove a bilinear version:

**Proposition 1.3** Let $Y$ be as in Prop. 1.2. Assume

$$\|U_{\pm1}(t)w_0u_{\pm2}u_1\|_Y \leq c \|w_0\|_{H^\nu} \|u_1\|_{H^\lambda}.$$  

Then

$$\|f_0 f_1\|_Y \leq c \|f_0\|_{X^{\nu,\frac{1}{2},1}} \|f_1\|_{X^{\lambda,\frac{1}{2},1}}.$$  

Here $\pm_1$ and $\pm_2$ denote independent signs.

The main result reads as follows:

**Theorem 1.1** The Cauchy problem for the Dirac - Klein - Gordon system (1), (2), (3) is globally well-posed for data

$$\psi_0 \in L^2(\mathbb{R}^2), \phi_0 \in H^{1/2}(\mathbb{R}^2), \phi_1 \in H^{-1/2}(\mathbb{R}^2).$$  

More precisely there exists a unique global solution $(\psi, \phi)$ such that for all $T > 0$

$$\psi \in X^{0,\frac{1}{2},1}_+(T) + X^{0,\frac{1}{2},1}_-(T), \phi \in X^{0,\frac{1}{2},1}_+(T) + X^{0,\frac{1}{2},1}_-(T),$$

$$\partial_\tau \phi \in X^{-\frac{1}{2},\frac{1}{2},1}_+(T) + X^{-\frac{1}{2},\frac{1}{2},1}_-(T).$$  

This solution has the property

$$\psi \in C^0(\mathbb{R}^+, L^2(\mathbb{R}^2)) , \phi \in C^0(\mathbb{R}^+, H^{\frac{1}{2}}(\mathbb{R}^2)) , \partial_\tau \phi \in C^0(\mathbb{R}^+, H^{-\frac{1}{2}}(\mathbb{R}^2)).$$
Theorem 1.2 Let $s$ be an arbitrary nonnegative number. If
\[ \psi_0 \in H^s(\mathbb{R}^2), \phi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^2), \phi_1 \in H^{s-\frac{1}{2}}(\mathbb{R}^2), \]
the global solution of Theorem 1.1 has the properties: For every $T > 0$
\[ \psi \in X^s_{+} (T) + X^s_{-} (T), \phi \in X^{s+\frac{1}{2}}_{+} (T) + X^{s+\frac{1}{2}}_{-} (T), \]
\[ \partial_t \phi \in X^{s-\frac{1}{2}}_{+} (T) + X^{s-\frac{1}{2}}_{-} (T) \]
and
\[ \psi \in C^0(\mathbb{R}^+, H^s(\mathbb{R}^2)), \phi \in C^0(\mathbb{R}^+, H^{s+\frac{1}{2}}(\mathbb{R}^2)), \partial_t \phi \in C^0(\mathbb{R}^+, H^{s-\frac{1}{2}}(\mathbb{R}^2)). \]

If $s > \frac{5}{2}$, this is a classical solution, i.e.
\[ \psi \in C^1(\mathbb{R}^+ \times \mathbb{R}^2), \phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^2). \]

2 Proof of the Theorems

It is possible to simplify the system (1),(2),(3) by considering the projections onto
the one-dimensional eigenspaces of the operator $-i \alpha \cdot \nabla$ belonging to the eigenvalues $\pm |\xi|$. These projections are given by $\Pi_\pm (D)$, where $D = \frac{\xi}{|\xi|}$ and $\Pi_\pm (\xi) = \frac{1}{2} (I \pm \frac{\xi}{|\xi|} \alpha)$. Then $-i\alpha \cdot \nabla = |D| \Pi_+ (D) - |D| \Pi_- (D)$ and $\Pi_\pm (\xi) \beta = \beta \Pi_\mp (\xi)$. Defining $\psi_\pm := \Pi_\pm (D) \psi$ and splitting the function $\phi$ into the sum $\phi = \frac{1}{2} (\phi_+ + \phi_-)$,
where $\phi_\pm := \phi \pm i A^{-1/2} \partial_t \phi$, $A := -\Delta + 1$, the Dirac - Klein - Gordon system can be rewritten as
\[
\begin{align*}
(-i \partial_t \pm |D|) \psi_\pm &= -M \beta \phi_\mp + \Pi_\pm (\phi \beta (\psi_+ + \psi_-)) \tag{4} \\
(i \partial_t \mp A^{1/2}) \phi_\pm &= \mp A^{-1/2} (\beta (\psi_+ + \psi_-), \psi_+ + \psi_-) + A^{-1/2} (m + 1) (\phi_+ + \phi_-). \tag{5}
\end{align*}
\]

The initial conditions are transformed into
\[ \psi_\pm (0) = \Pi_\pm (D) \psi_0, \phi_\pm (0) = \phi_0 \pm i A^{-1/2} \phi_1 \tag{6} \]

In the following we consider the system of integral equations belonging to the Cauchy problem (4),(5),(6):
\[
\begin{align*}
\psi_\pm (t) &= e^{\mp i t |D|} \psi_\pm (0) - i \int_0^t e^{\mp i (t-s) |D|} \Pi_\pm (D) \left( \frac{1}{2} (\phi_+(s) + \phi_-(s)) \beta (\Pi_+(D) \psi_+(s) + \Pi_-(D) \psi_-(s)) \right) ds + i M \int_0^t e^{\mp i (t-s) |D|} \beta \psi_\mp (s) ds \tag{7} \\
\phi_\pm (t) &= e^{\mp i t A^{1/2}} \phi_\pm (0) + i \int_0^t e^{\mp i (t-s) A^{1/2}} A^{-1/2} (\beta (\Pi_+(D) \psi_+(s) + \Pi_-(D) \psi_-(s)), \\
& \quad \Pi_+(D) \psi_+(s) + \Pi_-(D) \psi_-(s)) ds \tag{8} \\
& \quad + i (m + 1) \int_0^t e^{\mp i (t-s) A^{1/2}} A^{-1/2} (\phi_+(s) + \phi_-(s)) ds 
\end{align*}
\]
We remark that any solution of this system automatically fulfills $\Pi_{\pm}(D)\psi_{\pm} = \psi_{\pm}$, because applying $\Pi_{\pm}(D)$ to the right hand side of (7) gives $\Pi_{\pm}(D)\psi_{\pm}(0) = \psi_{\pm}(0)$ and the integral terms also remain unchanged, because $\Pi_{\pm}(D)^2 = \Pi_{\pm}(D)$ and $\Pi_{\pm}(D)\beta\psi_{\mp}(s) = \beta\Pi_{\mp}(D)\psi_{\mp}(s) = \beta\psi_{\pm}(s)$. Thus $\Pi_{\pm}(D)\psi_{\pm}$ can be replaced by $\psi_{\pm}$, thus the system of integral equations reduces exactly to the one belonging to our Cauchy problem (4), (5), (6).

In order to construct solutions to this system of integral equations we use the following facts for the linear problem which are independent of the specific phase function. The following Proposition is closely related to the exposition in [BHHT][Section 5], where slightly different function spaces are considered. For the moment let $\psi$ denote a smooth time cut-off function and set $\psi_T(t) = \psi(T_t)$, where $0 < T \leq 1$. The solution of the inhomogeneous linear equation
\[
\partial_t v - i\phi(D)v = F \quad v(0) = 0
\]
is denoted by $U_{sR}F$, defined by
\[
U_{sR}F(t) = \int_0^t U(t - t') F(t') dt',
\]
where $U(t)u_0 = e^{it\phi(D)}u_0$ solves the corresponding homogeneous equation with initial datum $u_0$ (cf. [GTV, Section 2]).

Proposition 2.1 Let $0 < T \leq 1$, $-\frac{1}{2} < b' < 0 < b \leq \frac{1}{2}$, $s \in \mathbb{R}$, $u_0 \in H^s$ and $F \in L^1_{t}(I, H^{s'})$ for a time interval $I \subset \mathbb{R}$. Then
\begin{itemize}
  \item[i)] $\|\psi_T U_0\|_{X^s_{\sigma, 1}} \leq c\|u_0\|_{H^s}$,
  \item[ii)] $\|\psi_T U_{sR}F\|_{X^{s, b, 1}} \leq cT^{\frac{1}{2} + b'}\|F\|_{X^{s', s}, \infty}$,
  \item[iii)] $\|\psi_T u\|_{X^{s, b, 1}} \leq cT^{\frac{1}{2} - b}\|u\|_{X^{s', 1, 1}}$.
\end{itemize}
Moreover $X^{s, \frac{1}{2}, 1} \subset C^0(\mathbb{R}, H^s)$ with a continuous embedding.

Proof: Without loss of generality we may assume $s = 0$. Then we consider the scaling transformations $S_T$ and $S^T$ defined by
\[
S_T f(t, x) = f\left(\frac{t}{T}, x\right), \quad S^T f(t, x) = T f(Tt, x),
\]
which are formally adjoint to each other with respect to the inner product in $L^2_{tx}$ (or merely in $L^2_t$, if $f$ does not depend on $x$). An elementary calculation shows that for $b > 0$
\[
\|S_T f\|_{B^b_{2, q} L^{2}} \leq cT^{\frac{1}{2} - b}\|f\|_{B^b_{2, q} L^{2}},
\]
which is still true without the additional $L^2_t$-part of the norm. For $b = \frac{1}{2}$ and $q = 1$ we especially obtain i), when replacing $f$ by $\psi u_0$. To see ii), we write $Kf(t) = \int_0^t f(t') dt'$. Then $\psi_T K f(T t) = \psi K S^T f(t)$ (cf. [BHHT, p. 20]), hence $\psi_T K f = S_T (\psi K S^T f)$ and thus
\[
\|\psi_T K f\|_{B^b_{2, 1} L^{2}} = \|S_T (\psi K S^T f)\|_{B^b_{2, 1} L^{2}} \leq c\|\psi K S^T f\|_{B^b_{2, 1} L^{2}} \\
\leq c \|\psi K S^T f\|_{H^{\frac{1}{2}} + L^{2}} \leq c\|K S^T f\|_{H^{\frac{1}{2}} + L^{2}} \\
\leq c \|S^T f\|_{H^{-\frac{1}{2}} + L^{2}} \leq c\|S^T f\|_{B^b_{2, \infty} L^{2}},
\]
where $b' > -\frac{1}{2}$. Here we used (9), $H^\frac{1}{2} \subset B^\frac{1}{2}_{2,1}$, the fact that $H^\frac{1}{2}$ forms an algebra, Lemma 2.1 from [GTV], and $B^0_{2,\infty} \subset H^{b'}$. Dualizing (9) we see that the latter is bounded by $T^\frac{1}{2} + b' \|f\|_{B^0_{2,\infty}L^2}$, which gives ii), when replacing $f$ by $U(-)F$. Part iii) is a consequence of (9) and the multiplication law for 1-Besov-spaces below.

The additional statement follows from the well-known embedding $B^\frac{1}{2}_{2,1} \subset C^0$.

**Lemma 2.1** (One-dimensional Besov-multiplication-law) For $0 < b \leq \frac{1}{2}$ we have

$$\|\psi u\|_{B^b_{2,1}L^2} \leq c\|\psi\|_{B^b_{2,1}}\|u\|_{B^b_{2,1}L^2}.$$

**Proof:** Let $P_k u = \varphi_k * u$, where $\varphi_k$ are the defining functions of the Besov spaces, and $\bar{P}_k = P_{k-1} + P_k + P_{k+1}$. Then

$$\|\psi u\|_{B^b_{2,1}L^2} = \sum_{k \geq 0} 2^{kb} \|P_k(\psi u)\|_{L^2} \leq \sum_{k,l \geq 0} 2^{kb} \|P_l((P_k \psi) u)\|_{L^2} \leq c \sum_{k \geq 0} 2^{kb} \|P_k \psi\|_{L^2} \|u\|_{L^2} \leq c \|\psi\|_{B^b_{2,1}} \|u\|_{B^b_{2,1}L^2},$$

since $B^\frac{1}{2}_{2,1} \subset L^\infty$. To estimate $\sum_2$ we choose $\frac{1}{p} = \frac{1}{2} - b$, $\frac{1}{q} = b$ so that

$$\sum_2 \leq c \sum_{k \geq 0} \|P_k \psi\|_{L^p} \|u\|_{L^q} \leq c \|\varphi\|_{L^p} \|\bar{P}_k \psi\|_{L^q} \|u\|_{L^2} \leq c \|\varphi\|_{B^b_{2,1}} \|\bar{P}_k \psi\|_{L^q} \|u\|_{L^2},$$

where we used Young’s inequality. Since $\|\varphi\|_{L^q} \sim 2^\frac{b}{2}$, we obtain the desired bound.

**Proposition 2.2** For $0 \leq b' < 1/2$ and $0 < T \leq 1$ we have

$$\|f\|_{L^2(R^2 \times [0,T])} \leq cT^{b'} \|f\|_{X^0,b'}$$

and

$$\|f\|_{X^0,-b'} \leq cT^{b'} \|f\|_{L^2(R^2 \times [0,T])}.$$

**Proof:** By the embedding $H^{b'} \subset L^{\frac{2}{1+b'}}$ ($0 \leq b' < 1/2$) we get

$$\|\psi_\tau g\|_{L^2(0,T)} \leq \|\psi_\tau\|_{L^{\frac{2}{1+b'}}} \|g\|_{L^{\frac{2}{1+b'}}} \leq cT^{b'} \|\psi\|_{L^{\frac{2}{1+b'}}} \|g\|_{H^{b'}}.$$

From this we get:

$$\|\psi_\tau f\|_{L^2} = \|U(-)\psi_\tau f\|_{L^2} = \|\psi_\tau U(-) f\|_{L^2} \leq cT^{b'} \|U(-) f\|_{H^{b'} L^2} = cT^{b'} \|f\|_{X^0,b'}. $$

The second claim follows by duality.

Concerning the nonlinearities we shall prove the following estimates in Chapter 3 below. Here and in the sequel the letter $\psi$ is used again to denote the spinor field.
Proposition 2.3  The following estimates are true:

$$\| (\beta \Pi_{\pm 1}(D) \psi, \Pi_{\pm 2}(D) \psi') \|_{X_{\pm 2}^{0, \frac{1}{2}}} \leq c \| \psi \|_{X_{\pm 2}^{0, \frac{1}{2}}} \| \psi' \|_{X_{\pm 2}^{0, \frac{1}{2}}}$$  \hspace{1cm} (10)

and

$$\| \Pi_{\pm 2}(D)(\phi \beta \Pi_{\pm 1}(D) \psi) \|_{X_{\pm 2}^{0, -\frac{1}{2}}} \leq c \| \phi \|_{X_{\pm 2}^{0, \frac{1}{2}}} \| \psi \|_{X_{\pm 2}^{0, \frac{1}{2}}}.$$  \hspace{1cm} (11)

Here and in the following $\pm_1, \pm_2, \pm_3$ denote independent signs.

The following local existence result now is a consequence of these estimates.

Proposition 2.4 Let $\psi_\pm(0) \in L^2(\mathbb{R}^2)$, $\phi_\pm(0) \in H^\frac{1}{2}(\mathbb{R}^2)$. Then there exists $1 \geq T > 0$ such that the system of integral equations (7), (8) has a unique solution

$$\psi_\pm \in X_{\pm}^{0, \frac{1}{2}, 1}(T), \phi_\pm \in X_{\pm}^{1, \frac{1}{2}, 1}(T).$$

This solution has the following properties:

$$\psi_\pm \in C^0([0, T], L^2(\mathbb{R}^2)), \phi_\pm \in C^0([0, T], H^\frac{1}{2}(\mathbb{R}^2)).$$

$\phi_\pm$ fulfills

$$\| \phi_+(t) \|_{H^\frac{1}{2}} + \| \phi_-(t) \|_{H^\frac{1}{2}} \leq \left( \| \phi_+(0) \|_{H^\frac{1}{2}} + \| \phi_-(0) \|_{H^\frac{1}{2}} \right) + cT^\frac{1}{2} \left( \| \psi_+(0) \|_{L^2}^2 + \| \psi_-(0) \|_{L^2}^2 \right) + c_0 T \frac{1}{2},$$  \hspace{1cm} (12)

for $0 \leq t \leq T$, where $c_0$ is a fixed constant. $T$ can be chosen such that

$$T^\frac{1}{2}(\| \psi_+(0) \|_{L^2} + \| \psi_-(0) \|_{L^2}) \leq c, \hspace{1cm} (13)$$

$$T^\frac{1}{2}(\| \phi_+(0) \|_{H^\frac{1}{2}} + \| \phi_-(0) \|_{H^\frac{1}{2}}) \leq c, \hspace{1cm} (14)$$

$$T^\frac{1}{2}(\| \psi_+(0) \|_{H^\frac{1}{2}}^2 + \| \psi_-(0) \|_{H^\frac{1}{2}}^2) \leq c(\| \phi_+(0) \|_{H^\frac{1}{2}} + \| \phi_-(0) \|_{H^\frac{1}{2}}). \hspace{1cm} (15)$$

In addition, if $T$ fulfills only (13) and (14) we get the same result except estimate (12).

Proof: Consider the transformation mapping the left hand side of our integral equations (7), (8) into the right hand sides. We construct a fixed point of it by the contraction mapping principle in the following set

$$M_T := \{ \psi_\pm \in X_{\pm}^{0, \frac{1}{2}, 1}(T), \phi_\pm \in X_{\pm}^{1, \frac{1}{2}, 1}(T) :$$

$$\| \psi_+ \|_{X_{\pm}^{0, \frac{1}{2}}} + \| \psi_- \|_{X_{\pm}^{0, \frac{1}{2}}} \leq cT^\frac{1}{2}(\| \psi_+(0) \|_{L^2} + \| \psi_-(0) \|_{L^2}) \}.$$

Taking an element $(\bar{\psi}_\pm, \bar{\phi}_\pm) \in M_T$, the nonlinear term on the right hand side of (7) is estimated in the $X_{\pm}^{0, \frac{1}{2}, 1}(T)$ - norm by use of Propositions 2.1 and 2.3 (we
omit $T$ here and in the following)

$$\| \int_0^t e^{\tau(t-s)} [\Pi_+ (D) (\frac{1}{2} (\phi_+(s) + \phi_-(s)) \beta (\Pi_+ (D) \psi_+(s) \\
+ \Pi_- (D) \psi_-(s))] ds \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| \int_0^t e^{\tau(t-s)} [\Pi_+ (D) (\frac{1}{2} (\phi_+(s) + \phi_-(s)) \beta (\Pi_+ (D) \psi_+(s) + \\
+ \Pi_- (D) \psi_-(s))] ds \|_{X^{-\frac{1}{2}, 1}}$$

where in the last line we used (14).

Next we consider the right hand side of (8).

$$\| e^{\tau(t)} [\Pi_\pm (D) \psi_\pm (0)] \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| e^{\tau(t)} [\Pi_\pm (D) \psi_\pm (0)] \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| \psi_\pm (0) \|_{L^2}$$

and by Prop. 2.1 and Prop. 2.2:

$$\| \int_0^t e^{\tau(t-s)} [\beta \psi_\pm (s)] ds \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| \int_0^t e^{\tau(t-s)} [\beta \psi_\pm (s)] ds \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| \psi_\pm \|_{L^2([0,T], L^2)}$$

Next we consider the right hand side of (8).

$$\| \int_0^t e^{\tau(t-s)} A^\frac{1}{2} (\beta (\Pi_+ (D) \psi_+(s) + \Pi_- (D) \psi_-(s)), \\
+ \Pi_+ (D) \psi_+(s) + \Pi_- (D) \psi_-(s))] ds \|_{X^{-\frac{1}{2}, 1}} \leq cT^\frac{3}{2} \| \int_0^t e^{\tau(t-s)} A^\frac{1}{2} (\beta (\Pi_+ (D) \psi_+(s) + \\
+ \Pi_- (D) \psi_-(s)), + \Pi_+ (D) \psi_+(s) + \Pi_- (D) \psi_-(s))] ds \|_{X^{-\frac{1}{2}, 1}}$$

where we used (10) and also (15) in the last line.
The linear terms on the right hand side of (8) are handled as follows:

\[ \| e^{\mp it A^\frac{1}{2}} \phi_\pm(0) \|_{X^\frac{1}{2}, \frac{1}{4}} \leq c T^{\frac{1}{2}} \| e^{\mp it A^\frac{1}{2}} \phi_\pm(0) \|_{X^\frac{1}{2}, \frac{1}{4}} \leq c T^{\frac{1}{2}} \| \phi_\pm(0) \|_{H^\frac{1}{2}} \]

and

\[
\begin{align*}
&\| \int_0^t e^{\mp i(t-s) A^\frac{1}{2}} A^{-\frac{1}{2}}(\phi_+(s) + \phi_-(s)) ds \|_{X^\frac{1}{2}, \frac{1}{4}} \\
&\leq c T^{\frac{1}{2}} \| \int_0^t e^{\mp i(t-s) A^\frac{1}{2}} A^{-\frac{1}{2}}(\phi_+(s) + \phi_-(s)) ds \|_{X^\frac{1}{2}, \frac{1}{4}} \\
&\leq c T^{\frac{1}{2}} (\| \phi_+(0) \|_{H^\frac{1}{2}} + \| \phi_-(0) \|_{H^\frac{1}{2}} ).
\end{align*}
\]

Here we used the following estimate

\[
\begin{align*}
&\| \int_0^t e^{\mp i(t-s) A^\frac{1}{2}} A^{-\frac{1}{2}}(\phi_+(s) + \phi_-(s)) ds \|_{X^\frac{1}{2}, \frac{1}{4}} \\
&\leq c T^{\frac{1}{2}} (\| \phi_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \phi_- \|_{X^\frac{1}{2}, \frac{1}{4}} ) \\
&\leq c T^{\frac{1}{2}} (\| \phi_+ \|_{L^2_{t,x}} + \| \phi_- \|_{L^2_{t,x}} ) \\
&\leq c T^{\frac{1}{2}} (\| \phi_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \phi_- \|_{X^\frac{1}{2}, \frac{1}{4}} ) \\
&\leq c T (\| \phi_+ \|_{H^\frac{1}{2}} + \| \phi_- \|_{H^\frac{1}{2}} ) .
\end{align*}
\]

(16)

Altogether we have shown that the set $M_T$ is mapped into itself. Concerning the contraction property we get similarly for the difference of the right hand sides of (7) applied to functions $(\psi_\pm, \phi_\pm) \in M_T$ and $(\tilde{\psi}_\pm, \tilde{\phi}_\pm) \in M_T$ in the $X^0, \frac{1}{2}, \frac{1}{4}$ - norm an estimate by

\[
\begin{align*}
c T^{\frac{1}{2}} &\left( \| \phi_+ - \tilde{\phi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \phi_- - \tilde{\phi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right) \left( \| \psi_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \psi_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right) \\
&\quad + \| \tilde{\psi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \tilde{\psi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right) \left( \| \psi_+ - \tilde{\psi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \psi_- - \tilde{\psi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right).
\end{align*}
\]

\[
\begin{align*}
&\leq c T^{\frac{1}{2}} \left( \| \phi_+ - \tilde{\phi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \phi_- - \tilde{\phi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right) \\
&\quad + \| \psi_+ \|_{L^2} + \| \tilde{\psi}_+ \|_{L^2} + \| \psi_- \|_{L^2} + \| \tilde{\psi}_- \|_{L^2} \\
&\quad + \| \psi_+ - \tilde{\psi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \psi_- - \tilde{\psi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right) \\
&\quad + \| \psi_+ \|_{H^\frac{1}{2}} + \| \tilde{\psi}_+ \|_{H^\frac{1}{2}} + \| \psi_- \|_{H^\frac{1}{2}} + \| \tilde{\psi}_- \|_{H^\frac{1}{2}} \\
&\leq \frac{1}{2} \left( \| \phi_+ - \tilde{\phi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \phi_- - \tilde{\phi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \\
&\quad + \| \psi_+ - \tilde{\psi}_+ \|_{X^\frac{1}{2}, \frac{1}{4}} + \| \psi_- - \tilde{\psi}_- \|_{X^\frac{1}{2}, \frac{1}{4}} \right),
\end{align*}
\]

using (13) and (14) in the last line. The linear integral term in (7) is treated easily, and the right hand side of (8) can also be estimated similarly. Thus the contraction property is proved leading to a unique (local) solution.
We now show that our local solution belongs to $C^0_t(L^2_x)$. From our integral equation we get
\[ \|\psi\|_{C^0_t(L^2_x)} \leq c(\|\psi\|_{X^0_{-\frac{1}{2},1}}) \]
\[ \leq c(\|\psi(0)\|_{L^2_x} + \|\int_0^t e^{\mp(t-s)D}[\Pi_+(D)(\phi\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-))(s)ds\|_{X^0_{-\frac{1}{2},1}} \]
\[ + |M|\|\int_0^t e^{\mp(t-s)D}[\beta\psi^s(s)ds\|_{X^0_{-\frac{1}{2},1}} \]
\[ \leq c(\|\psi(0)\|_{L^2_x} + T^\frac{1}{2}[\|\Pi_+(D)(\phi\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-))(s)\|_{C^0_t(L^2_x)} \]
\[ + |M|T^\frac{1}{2}[\|\psi\|_{X^0_{-\frac{1}{2},1}} + \|\psi\|_{X^0_{-\frac{1}{2},1}}]) \]
\[ \leq c(\|\psi(0)\|_{L^2_x} + T^\frac{1}{2}[\|\psi\|_{X^0_{-\frac{1}{2},1}} + \|\psi\|_{X^0_{-\frac{1}{2},1}}] \]
\[ + |M|T^\frac{1}{2}[\|\psi\|_{X^0_{-\frac{1}{2},1}} + \|\psi\|_{X^0_{-\frac{1}{2},1}}]) \]

Here we used the estimate
\[ \|\int_0^t e^{\mp(t-s)D}[\beta\psi^s(s)ds\|_{X^0_{-\frac{1}{2},1}} \leq cT^\frac{1}{2}[\|\psi\|_{L^2_x} + \|\psi\|_{L^2_x}] \]
\[ \leq cT^\frac{1}{2}[\|\psi\|_{X^0_{-\frac{1}{2},1}} + \|\psi\|_{X^0_{-\frac{1}{2},1}}] \]

by Prop. 2.1 and Prop. 2.2. We have shown that $\psi \in C^0_t(L^2_x)$.

Next we estimate $\|\phi\|_{H^\frac{1}{2}_x}$ for $0 \leq t \leq T$ by our integral equation (8).
\[ \|\phi\|_{H^\frac{1}{2}_x} \leq \|\phi(0)\|_{H^\frac{1}{2}_x} + c\|\int_0^t e^{\mp(t-s)A^\frac{1}{2}} [\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s), \Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s))]ds\|_{X^\frac{1}{2}_{-\frac{1}{2},1}} \]
\[ + c|m| + 1\|\int_0^t e^{\mp(t-s)A^\frac{1}{2}} [\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s)]ds\|_{X^\frac{1}{2}_{-\frac{1}{2},1}} \]
\[ \leq \|\phi(0)\|_{H^\frac{1}{2}_x} + cT^\frac{1}{2}[\|\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s), \Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)) \]
\[ + c|m| + 1\|\int_0^t e^{\mp(t-s)A^\frac{1}{2}} [\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s)]ds\|_{X^\frac{1}{2}_{-\frac{1}{2},1}} \]
\[ \leq \|\phi\|_{H^\frac{1}{2}_x} + cT^\frac{1}{2}[\|\psi\|^2_{L^2_x} + \|\psi\|^2_{L^2_x}] \]
\[ + c|m| + 1\|\int_0^t e^{\mp(t-s)A^\frac{1}{2}} [\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s)]ds\|_{X^\frac{1}{2}_{-\frac{1}{2},1}} \]
\[ \leq \|\phi\|_{H^\frac{1}{2}_x} + cT^\frac{1}{2}[\|\psi\|^2_{L^2_x} + \|\psi\|^2_{L^2_x}] \]
\[ + c|m| + 1\|\int_0^t e^{\mp(t-s)A^\frac{1}{2}} [\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-)(s)]ds\|_{X^\frac{1}{2}_{-\frac{1}{2},1}} \]

(17)

Here we used (16). By our choice (14) of $T$ we arrive at (12). This proves that $\phi \in C^0_t([0,T], H^\frac{1}{2}_x)$. For our global result it is important to notice that no implicit constant appears in front of the first term on the right hand side.
The additional claim of the proposition is easily proven by the contraction mapping principle in a similar way so that the proof of Prop. 2.4 is complete.

The proof of Theorem 1.1 can now be given along the lines of the paper of Colliander-Holmer-Tzirakis [CHT] for the 1D Zakharov system.

**Proof of Theorem 1.1.** We start by using the addition in Prop. 2.4 leading to a local solution with the required regularity properties. Because \( \| \psi(t) \|_{L^2} \) is conserved we get by iteration a global solution if also \( \| \phi(t) \|_{H^{\frac{1}{2}}} \) remains bounded. Otherwise we use our Prop. 2.4 and remark first that

\[
\| \psi(t) \|^2_{L^2} = \| \psi^+ (t) \|^2_{L^2} + \| \psi^- (t) \|^2_{L^2}
\]

is still conserved. Thus conservation law can be applied because \( \psi \in C^0([0, T], L^2_z) \).

Without loss of generality we can now suppose that at some time \( t \) we have

\[
\| \phi^+(t) \|_{H^{\frac{1}{2}}} + \| \phi^-(t) \|_{H^{\frac{1}{2}}} \gg \| \psi^+(t) \|^2_{L^2} + \| \psi^-(t) \|^2_{L^2}.
\]

Take this time \( t \) as initial time \( t = 0 \) so that

\[
\| \phi^+(0) \|_{H^{\frac{1}{2}}} + \| \phi^-(0) \|_{H^{\frac{1}{2}}} \gg \| \psi^+(0) \|^2_{L^2} + \| \psi^-(0) \|^2_{L^2}.
\]  

(18)

Then (15) is automatically satisfied. We define

\[
T \sim \frac{1}{\left( \| \phi^+(0) \|_{H^{\frac{1}{2}}} + \| \phi^-(0) \|_{H^{\frac{1}{2}}} \right)^T},
\]

so that (13) and (14) are fulfilled. From our estimate (12) we conclude that it is possible to use the local existence result \( l \) times with time intervals of length \( T \), before the quantity \( \| \phi^+(t) \|_{H^{\frac{1}{2}}} + \| \phi^-(t) \|_{H^{\frac{1}{2}}} \) doubles. Here we have

\[
l \sim \frac{\| \phi^+(0) \|_{H^{\frac{1}{2}}} + \| \phi^-(0) \|_{H^{\frac{1}{2}}}}{T^2 (\| \psi^+(0) \|^2_{L^2} + \| \psi^-(0) \|^2_{L^2} + 1) }.
\]

After these \( l \) iterations we arrive at the time

\[
lT \sim \frac{\| \phi^+(0) \|_{H^{\frac{1}{2}}} + \| \phi^-(0) \|_{H^{\frac{1}{2}}}}{\| \psi^+(0) \|^2_{L^2} + \| \psi^-(0) \|^2_{L^2} + 1} \sim \frac{1}{\| \psi^+(0) \|^2_{L^2} + \| \psi^-(0) \|^2_{L^2} + 1}.
\]

This quantity is independent of \( \| \phi^+(0) \|_{H^{\frac{1}{2}}} + \| \phi^-(0) \|_{H^{\frac{1}{2}}} \). Using conservation of \( \| \psi^+(t) \|^2_{L^2} + \| \psi^-(t) \|^2_{L^2} \) it is thus possible to repeat the whole procedere with time steps of equal length. This proves the global existence result.

**Proof of Theorem 1.2.** By the Leibniz rule for fractional derivatives from (10),(11) one easily gets the following estimates for the nonlinearities for arbitrary \( s \geq 0 \):

\[
\| \langle \beta \Pi_{\pm 1}(D) \psi, \Pi_{\pm 2}(D) \phi \rangle \|_{X^{s,-\frac{1}{2}}} \leq c (\| \psi \|_{X^{s+\frac{1}{4}, \frac{1}{4}}}, \| \phi \|_{X^{s-\frac{1}{4}, \frac{1}{4}}}, \| \phi \|_{X^{s, \frac{1}{2}}} \| \psi \|_{X^{s, \frac{1}{2}}}).
\]

and

\[
\| \Pi_{\pm 2}(D) (\phi \beta \Pi_{\pm 1}(D) \psi) \|_{X^{s,-\frac{1}{4}}} \leq c (\| \phi \|_{X^{s+\frac{1}{4}, \frac{1}{4}}}, \| \psi \|_{X^{s+\frac{1}{4}, \frac{1}{4}}} + \| \phi \|_{X^{s-\frac{1}{4}, \frac{1}{4}}}, \| \psi \|_{X^{s-\frac{1}{4}, \frac{1}{4}}}).
\]

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These estimates allow to construct a local solution with the required properties by the contraction mapping principle with an existence time

\[ T \sim \frac{1}{\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2} + \|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}} } \]

similarly as in the proof of Prop. 2.4. By uniqueness the global solution of Theorem 1.1 coincides locally with this solution. Thus the claim of Theorem 1.2 follows.

3 The estimates for the nonlinearities

In this section we give the proof of Prop. 2.3. We first show that the estimates (10) and (11) are completely equivalent to each other. By duality (11) is equivalent to the estimate

\[ \left| \int \langle \beta \Pi_{1,2}^\pm(D) \psi, \Pi_{1,2}^\pm(D) \psi' \rangle dx dt \right| \leq c \| \phi \|_{X^{\frac{1}{2},\frac{2}{3}}_{1,1,1}} \| \psi \|_{X^{\frac{1}{2},\frac{2}{3}}_{1,1,1}} \| \psi' \|_{X^{\frac{1}{2},\frac{2}{3}}_{1,1,1}} \cdot (19) \]

The left hand side equals

\[ \left| \int \phi(\beta \Pi_{1}^\pm(D) \psi, \Pi_{1,2}^\pm(D) \psi') dx dt \right| . \quad (20) \]

Thus (19) is equivalent to (10).

The complete null structure of the system detected by d’Ancona, Foschi and Selberg has the following consequences (cf. [AFS]). Denoting

\[ \sigma_{1,2}(\eta, \zeta) := \Pi_{1,2}^\pm(\zeta) \beta \Pi_{1}^\pm(\eta) = \beta \Pi_{1,2}(\zeta) \Pi_{1}^\pm(\eta), \]

we remark that by orthogonality this quantity vanishes if \( \pm \eta \) and \( \pm \zeta \) line up in the same direction whereas in general (cf. [AFS1], Lemma 1):

Lemma 3.1

\[ \sigma_{1,2}(\eta, \zeta) = O(\zeta(\pm \eta, \pm \zeta)), \]

where \( \zeta(\eta, \zeta) \) denotes the angle between the vectors \( \eta \) and \( \zeta \).

Consequently we get

\[ |\langle \beta \Pi_{1}^\pm(D) \psi, \Pi_{1,2}^\pm(D) \psi' \rangle(\tau, \xi)| \]

\[ \leq \int |\langle \beta \Pi_{1}^\pm(\eta) \tilde{\psi}(\lambda, \eta), \Pi_{1,2}(\eta - \xi) \tilde{\psi}'(\lambda - \tau, \eta - \xi) \rangle| d\lambda d\eta \]

\[ = \int |\langle \Pi_{1,2}(\eta - \xi) \beta \Pi_{1}^\pm(\eta) \tilde{\psi}(\lambda, \eta), \tilde{\psi}'(\lambda - \tau, \eta - \xi) \rangle| d\lambda d\eta \]

\[ \leq c \int \Theta_{1,2} \| \tilde{\psi}(\lambda, \eta) \| \| \tilde{\psi}'(\lambda - \tau, \eta - \xi) \| d\lambda d\eta, \]

where \( \Theta_{1,2} = \zeta(\pm \eta, \pm 2(\eta - \xi)). \)

We also need the following elementary estimates which can be found in [AFS], section 5.1.

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Lemma 3.2 Denoting

\[ A_{\pm} = \tau \pm_1 |\xi|, \quad B = \lambda + |\eta|, \quad C_{\pm} = \lambda - \tau \pm_1 |\eta - \xi|, \quad \Theta_{\pm} = \zeta(\eta, \pm(\eta - \xi)) \]

and

\[ \rho_+ = |\xi| - |\eta| - |\eta - \xi|, \quad \rho_- = |\eta| + |\eta - \xi| - |\xi| \]

the following estimates hold:

\[ \Theta^2_+ \sim \frac{|\xi|\rho_+}{|\eta| |\eta - \xi|}, \quad \Theta^2_- \sim \frac{|\eta| + |\eta - \xi|\rho_-}{|\eta||\eta - \xi|} \sim \frac{\rho_-}{\min(|\eta|, |\eta - \xi|)} \]

as well as

\[ \rho_{\pm} \leq 2\min(|\eta|, |\eta - \xi|) \]

and

\[ \rho_{\pm} \leq |A_{\pm}| + |B| + |C_{\pm}|. \]

Proof: We only prove the last estimate. We have

\[ \rho_+ \leq |\xi| + |\eta| + |\eta - \xi| = |\xi| + \tau \pm \lambda \pm |\eta| \pm \tau \mp |\eta| - |\eta - \xi| \]

\[ \leq |\xi| + \tau + |\lambda + |\eta|| + |\tau - |\eta - \xi|| \]

\[ \leq |A_{\mp}| + |B| + |C_+| \]

and

\[ \rho_- = (\lambda + |\eta|) + (\tau - |\lambda + |\eta - \xi||) - (\tau + |\xi|) \]

\[ \leq |\lambda + |\eta|| + |\lambda - |\eta - \xi|| + |\tau + |\xi|| \]

\[ = |B| + |C_-| + |A_+| \]

as well as for \( \tau \geq 0 \):

\[ \rho_- \leq \lambda + |\eta| + |\tau - |\lambda + |\eta - \xi|| \leq |B| + |C_-| \]

and for \( \tau \leq 0 \):

\[ \rho_- \leq |\lambda + |\eta|| + |\tau - |\lambda + |\eta - \xi|| + |\tau + |\xi|| \leq |B| + |C_-| + |A_-|. \]

Proof of Prop. 2.3: In order to prove (19) first for the signs \( \pm_1 = + \) and \( \pm_2 = \pm \) and taking into account (20) and (21) we have to show (recalling \( \Theta_{\pm} := \zeta(\eta, \pm(\eta - \xi)) \)):

\[ I_{\pm} := \int \int \Theta_{\pm} \tilde{\psi}(\lambda, \eta) \tilde{\psi}(\lambda - \tau, \eta - \xi) d\lambda d\eta \tilde{\phi}(\tau, \xi) d\tau d\xi \]

\[ \leq c\|\psi\|_{X^{*, \frac{1}{2}}} \|\psi\|_{X^{*, \frac{1}{2}}} \|\phi\|_{X^{0, \frac{1}{2}}} \cdot (22) \]

We may assume here without loss of generality that the Fourier transforms are nonnegative. Defining

\[ \tilde{\mathcal{F}}(\lambda, \eta) := (\lambda + |\eta|)^{\frac{1}{2}} \tilde{\psi}(\lambda, \eta) \]

\[ \tilde{\mathcal{G}}_{\pm}(\lambda, \eta) := (\lambda \pm |\eta|)^{\frac{1}{2}} \tilde{\psi}(\lambda, \eta) \]

\[ \tilde{\mathcal{H}}_{\pm}(\tau, \xi) := (\tau \pm |\xi|)^{\frac{1}{2}} \tilde{\phi}(\tau, \xi) \]

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we thus have to show

\[
J_{\pm} := \left| \int \int \Theta_{\pm} \tilde{F}(\lambda, \eta) \tilde{G}_{\pm}(\lambda - \tau, \eta - \xi) \frac{\tilde{H}_{\pm 1}(\tau, \xi)}{(B)^{\frac{1}{2}}} \frac{\tilde{H}_{\pm 1}(\tau, \xi)}{(C_{\pm})^{\frac{1}{2}}} d\lambda d\eta d\tau d\xi \right| \\
\leq c\|F\|_{X_{\pm}^{0, 0, 1}} \|G_{\pm}\|_{X_{\pm}^{0, 0, 1}} \|H_{\pm 1}\|_{X_{\pm}^{0, 0, 1}}.
\]

Let us first consider the low-frequency case, where \(\min(|\eta|, |\eta - \xi|) \leq 1\). Assuming without loss of generality (by symmetry) \(|\eta| \leq 1\) we estimate

\[
I_{\pm} \leq \|\psi\|_{L_{t}^{4}(L_{x}^{\infty})} \|\psi\|_{L_{t}^{4}(L_{x}^{\infty})} \|\mu\|_{L_{t}^{4}(L_{x}^{\infty})} \\
\leq \|\psi\|_{X_{\pm}^{0, \frac{1}{4} + \epsilon}} \|\psi\|_{X_{\pm}^{0, \frac{1}{4} + \epsilon}} \|\mu\|_{X_{\pm}^{0, \frac{1}{2} + \epsilon}},
\]

which implies the desired estimate. From now on we assume \(|\eta|, |\eta - \xi| \geq 1\).

**Estimate for \(J_{+}\):** We use

\[
\Theta_{+} \leq \frac{|\xi|^{\frac{1}{2}} \rho_{+}^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}}} \leq c \frac{\langle \xi \rangle^{\frac{1}{2}} \rho_{+}^{\frac{1}{2}}}{\langle \eta \rangle^{\frac{1}{2}} (\eta - \xi)^{\frac{1}{2}}} (\langle A_{\pm} \rangle^{\frac{1}{2}} + \langle B \rangle^{\frac{1}{2}} + \langle C_{\pm} \rangle^{\frac{1}{2}})
\]

and also

\[
\rho_{+} \leq 2 \min(|\eta|, |\eta - \xi|).
\]

We thus get

\[
J_{+} \leq c(I_{1}^{+} + I_{2}^{+} + I_{3}^{+}),
\]

where

\[
I_{1}^{+} = \int \tilde{F}(\lambda, \eta) \tilde{G}_{+}(\lambda - \tau, \eta - \xi) \tilde{H}_{\pm 1}(\tau, \xi) d\lambda d\eta d\tau d\xi,
\]

\[
I_{2}^{+} = \int \tilde{F}(\lambda, \eta) \tilde{G}_{+}(\lambda - \tau, \eta - \xi) \tilde{H}_{\pm 1}(\tau, \xi) d\lambda d\eta d\tau d\xi,
\]

\[
I_{3}^{+} = \int \tilde{F}(\lambda, \eta) \tilde{G}_{+}(\lambda - \tau, \eta - \xi) \tilde{H}_{\pm 1}(\tau, \xi) d\lambda d\eta d\tau d\xi.
\]

We only consider \(I_{1}^{+}\) and \(I_{2}^{+}\), because \(I_{3}^{+}\) is similar to \(I_{2}^{+}\).

**Estimate for \(I_{1}^{+}\):** Hölder’s inequality and Parseval’s identity give

\[
I_{1}^{+} \leq c\|H_{\pm 1}\|_{L_{t}^{2}} \left\| F^{-1} \left( \frac{\tilde{F}(\lambda, \eta)}{(\eta)^{\frac{1}{4}}(B)^{\frac{1}{4}}} \right) \right\|_{L_{t}^{4}} \left\| F^{-1} \left( \frac{\tilde{G}_{+}(\lambda - \tau, \eta - \xi)}{(\eta - \xi)^{\frac{1}{4}}(C_{\pm})^{\frac{1}{4}}} \right) \right\|_{L_{t}^{4}}.
\]

Concerning the last two factors we use Strichartz’ inequality for the wave equation which gives for \(U(t) = e^{tD}\):

\[
\|U(t)u_{0}\|_{L_{t}^{2}(H_{x}^{-\frac{1}{2}, s})} \leq c\|u_{0}\|_{L_{x}^{2}}.
\]

This implies by Prop. 1.2:

\[
\|f\|_{L_{t}^{2}(H_{x}^{-\frac{1}{2}, s})} \leq c\|U(-t)f\|_{H_{x}^{\frac{3}{2}, 1}} = c\|f\|_{X_{+}^{0, \frac{1}{2}, 1}}.
\]
Moreover we have
\[ \|f\|_{L^1_t L^2_x} = \|U(-t)f\|_{L^1_t L^2_x} \leq c\|U(-t)f\|_{B^0_{2,1}} = c\|f\|_{X^{0,0.1}_+}. \]

Complex interpolation gives by [BL], Thm. 6.4.5:
\[ \|f\|_{L^1_t(B^0_{2,1})} \leq c\|U(-t)f\|_{B^0_{2,1} L^2_x} = c\|f\|_{X^{0,0.1}_+}. \]

This is equivalent to
\[ \|f\|_{L^1_t L^2_x} \leq c\|U(-t)f\|_{B^0_{2,1} H^{0.4}_+} = c\|f\|_{X^{0,0.4}_+}. \]

Thus we get
\[ I^+_1 \leq c\|H^{±1}_t\|_{X^{0,1}_+} \|F\|_{X^{0,0.1}_+} \|G\|_{X^{0,0.1}_+}, \]

where we used the embedding \( X^{0,0.1}_+ \subset L^2_{x,t}. \)

**Estimate for \( I^+_2 \):** Using Parseval’s identity and Hölder’s inequality we get
\[ I^+_2 \leq c\left\| F^{-1} \left( F(\lambda, \eta) \right) \right\|_{L^2_x} \left\| F^{-1} \left( G_+ (\lambda - \tau, \eta - \xi) \right) \right\|_{L^2_x} \left\| H^{±1}_t \right\|_{L^2_x}. \]

The first factor is estimated using Sobolev’s embedding theorem by \( \|F\|_{L^2_x}. \)

Concerning the last factor we estimate by Sobolev and Minkowski’s inequality as follows:
\[ \|f\|_{L^2_x} = \|U(±t)f\|_{L^2_x} \leq \|U(±t)f\|_{L^2_x} \leq c\|U(±t)f\|_{L^2_x} \leq c\|f\|_{X^{0,0.1}_+}, \]

Thus the last factor can be estimated by \( c\|H^{±1}_t\|_{L^2_x} \leq c\|H^{±1}_t\|_{X^{0,1}_+}. \)

Concerning the second factor we start with Strichartz’ estimate
\[ \|U(t)u_0\|_{L^1_t(B^{−\frac{1}{2}}_{∞,2})} \leq c\|u_0\|_{L^2_x}, \]

which implies by Prop. 1.2
\[ \|f\|_{L^1_t(B^{−\frac{1}{2}}_{∞,2})} \leq c\|U(-t)f\|_{B^{−\frac{1}{2}}_{2,1} L^2_x} = c\|f\|_{X^{0,0.1}_+}. \]

Moreover we have
\[ \|f\|_{L^2_x(B^{0}_{∞,2})} = \|f\|_{X^{0,0}} = \|f\|_{X^{0,0.1}_+ L^2_x} \leq c\|U(-t)f\|_{B^{0}_{2,1}} = \|f\|_{X^{0,0.1}_+}. \]

We now use the complex interpolation method. By [BL], Thm. 6.4.5 we have
\[ (B^{−\frac{1}{2}}_{∞,2} B^{0}_{∞,2})_{\frac{3}{4}} = B^{−\frac{1}{2}}_{6,2} \quad \text{and also} \quad (X^{0,0.1}_+, X^{0,0.1}_+)_{\frac{3}{4}} = X^{0,\frac{1}{2},1}_+, \]

so that we get with \( B^{−\frac{1}{2}}_{6,2} \subset H^{−\frac{1}{2},0} \) ([BL], Thm. 6.4.4)
\[ \|f\|_{L^2_t(H^{−\frac{1}{2},0})} \leq c\|f\|_{L^2_t(B^{−\frac{1}{2}}_{6,2})} \leq c\|U(-t)f\|_{B^{−\frac{1}{2}}_{2,1} L^2_x} = \|f\|_{X^{0,\frac{1}{2},1}_+}, \]

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which implies
\[ \|f\|_{L^2_t(L^2_x)} \leq c\|f\|_{X^{\frac{1}{2},1}}. \] (24)

Thus the second factor is estimated by \( \|G+\|_{X^{0,0,0}}. \)

**Estimate for \( J_\cdot \):** If \( |\eta| \ll |\eta - \xi| \) we have \( |\xi| \sim |\eta - \xi| \), thus by Lemma 3.2:

\[ \Theta_+^2 \sim \frac{\rho_-}{\min(|\eta|,|\eta - \xi|)} \sim \frac{|\xi|\rho_-}{|\eta||\eta - \xi|}, \]

so that
\[ \Theta_- \leq c\frac{\langle \xi \rangle^{\frac{1}{2}}\rho_\frac{1}{2}}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}} \left( \langle A_{\pm} \rangle^{\frac{1}{2}} + \langle B \rangle^{\frac{1}{2}} + \langle C_- \rangle^{\frac{1}{2}} \right). \]

Because also \( \rho_- \leq 2 \min(|\eta|,|\eta - \xi|) \) the same estimates as for \( J_\pm \) can be given.

If \( |\eta| \gg |\eta - \xi| \), we have \( |\xi| \geq |\eta| - |\eta - \xi| \sim |\eta| \) and the same estimate for \( \Theta_- \) holds. This is also true if \( |\xi| \sim |\eta| \sim |\eta - \xi| \).

It remains to consider \( J_- \) in the case \( |\xi| \ll |\eta| \sim |\eta - \xi| \), which we assume from now on. We then have

\[ \Theta_- \leq c\frac{\rho_{\frac{1}{2}}}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}} \]

and thus
\[ J_- \leq c \int \int \frac{\bar{F}(\lambda,\eta) \bar{G}_-(\lambda - \tau, \eta - \xi) \bar{H}_{\pm\pm}(\tau, \xi)}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}(C_-)^{\frac{1}{2}}(\xi)^{\frac{1}{2}}(A_{\pm\pm})^{\frac{1}{2}}} \cdot d\lambda d\eta d\tau d\xi. \]

Using the estimates \( \rho_- \leq 2 \min(|\eta|,|\eta - \xi|) \) and \( \rho_- \leq |A_{\pm\pm}| + |B| + |C_-| \) (cf. Lemma 3.2) we get

\[ J_- \leq c(I_1^- + I_2^- + I_3^-), \]

where
\[
\begin{align*}
I_1^- &= \int \int \frac{\bar{F}(\lambda,\eta) \bar{G}_-(\lambda - \tau, \eta - \xi) \bar{H}_{\pm\pm}(\tau, \xi)}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}(C_-)^{\frac{1}{2}}(\xi)^{\frac{1}{2}}(A_{\pm\pm})^{\frac{1}{2}}} d\lambda d\eta d\tau d\xi, \\
I_2^- &= \int \int \frac{\bar{F}(\lambda,\eta) \bar{G}_-(\lambda - \tau, \eta - \xi) \bar{H}_{\pm\pm}(\tau, \xi)}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}(C_-)^{\frac{1}{2}}(\xi)^{\frac{1}{2}}(A_{\pm\pm})^{\frac{1}{2}}} d\lambda d\eta d\tau d\xi, \\
I_3^- &= \int \int \frac{\bar{F}(\lambda,\eta) \bar{G}_-(\lambda - \tau, \eta - \xi) \bar{H}_{\pm\pm}(\tau, \xi)}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}(C_-)^{\frac{1}{2}}(\xi)^{\frac{1}{2}}(A_{\pm\pm})^{\frac{1}{2}}} d\lambda d\eta d\tau d\xi.
\end{align*}
\]

The terms \( I_2^- \) and \( I_3^- \) are similar, so that we concentrate on \( I_1^- \) and \( I_2^- \).

**Estimate for \( I_1^- \):** We have
\[
\begin{align*}
I_1^- &\leq \|\bar{H}_{\pm\pm}\|_{L^2_t} \left\| \int (\xi)^{-\frac{1}{2}} \frac{\bar{F}(\lambda,\eta) \bar{G}_-(\lambda - \tau, \eta - \xi)}{(\eta)^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{2}}(C_-)^{\frac{1}{2}}} d\lambda d\eta \right\|_{L^2_t} \\
&= \|\bar{H}_{\pm\pm}\|_{L^2_t} \left\| \int (\xi)^{-\frac{1}{2}} \frac{\bar{G}'(\tau - \lambda - \xi - \eta)}{(\lambda + |\eta|)^{\frac{1}{2}}(\xi - \eta)^{\frac{1}{2}}(\tau - \lambda + |\eta - \xi|)^{\frac{1}{2}}} d\lambda d\eta \right\|_{L^2_t},
\end{align*}
\]
where $\tilde{G}'(\lambda, \eta) := \hat{G}_-(\lambda - \eta)$. This shows that we in fact are in the (+,+)-case. We also remark that we assumed $|\xi| << |\eta| \sim |\xi - \eta|$. Using Prop. 4.2 we arrive at

$$ I_2^- \leq c\|H_{\pm}||L_{\frac{3}{2}}\|F\|_{X_{\pm}^{0,0.1}}\|G'\|_{X_{\pm}^{0,0.1}} \leq c\|H_{\pm}\|_{X_{\pm}^{0,0.1}}\|F\|_{X_{\pm}^{0,0.1}}\|G_-\|_{X_{\pm}^{0,0.1}}. $$

**Estimate for $I_2^-$**: Parseval’s identity and H"older’s inequality imply

$$ I_2^- \leq c\|\mathcal{F}^{-1}\left(\frac{\tilde{F}}{\langle \eta \rangle^{3/2}}\right)\|_{L^2_\xi(L^2_x)}\|\mathcal{F}^{-1}\left(\frac{\tilde{G}_-}{\langle \eta - \xi \rangle^{3/2} \langle C_\pm \rangle^{1/4}}\right)\|_{L^2_\xi(L^2_x)}. $$

The first factor is controlled using Sobolev’s embedding $H^\frac{3}{2} \subset L^\frac{11}{5}$ by $\|F\|_{L^2_x}$, the last factor is handled as before using the estimate (24), and the second one similarly as before as follows. First, Sobolev’s embedding in $x$ gives

$$ \|\mathcal{F}^{-1}\left(\frac{\tilde{G}_-}{\langle \eta - \xi \rangle^{3/2} \langle C_\pm \rangle^{1/4}}\right)\|_{L^2_\xi(L^2_x)} \leq c\|\mathcal{F}^{-1}\left(\frac{\tilde{G}_-}{\langle C_\pm \rangle^{1/4}}\right)\|_{L^2_\xi(L^2_x)}. $$

Now we use (23) so that the second factor is estimated by $\|G_-\|_{L^2_x} \leq c\|G_-\|_{X_{\pm}^{0,0.1}}$.

This completes the proof of estimate (22).

The remaining cases $\pm_1 = -$ and $\pm_2 = \pm$ in (19) and (20) can be treated in the same way. Using $\Pi_\mp(\eta) = \Pi_{\pm}(\mp \eta)$ we in fact get by (21)

$$ \left| \int \int \phi(\beta \Pi_- (D) \psi, \Pi_{\pm}(D) \psi') dx dt \right| $$

$$ = \left| \int \int \phi(\beta \Pi_- (\eta) \tilde{\psi}(\lambda, \eta), \Pi_{\pm}(\eta - \xi) \tilde{\psi}'(\lambda - \tau, \eta - \xi)) d\lambda d\eta d\xi d\tau \right| $$

$$ = \left| \int \int \phi(\Pi_{\pm}(\eta - \xi) \beta \Pi_+(-\eta) \tilde{\psi}(\lambda, \eta), \tilde{\psi}'(\lambda - \tau, \eta - \xi)) d\lambda d\eta d\xi d\tau \right| $$

$$ \leq c \int \int \Theta_\mp |\tilde{\psi}(\lambda, \eta)||\tilde{\psi}'(\lambda - \tau, \eta - \xi)| d\lambda d\eta |\tilde{\psi}(\tau, \xi)| d\tau d\xi $$

$$ = I_\mp, $$

because by Lemma 3.1

$$ \Pi_{\pm}(\eta - \xi) \beta \Pi_+(-\eta) = \sigma_{\pm,+}(-\eta, \eta - \xi) $$

$$ = O(\mathcal{L}(-\eta, \pm(\eta - \xi)) = O(\mathcal{L}(\eta, \mp(\eta - \xi)) = O(\Theta_\mp), $$

which can be handled like $I_\mp$ above, namely as follows. Our aim is to show

$$ I_\pm \leq c\|\psi\|_{X_{\pm}^{0,1}}\|\psi'\|_{X_{\pm}^{0,1}}\|\phi\|_{X_{\pm}^{0,1}}. $$

This can be handled in the same way as before, provided the following Lemma holds.

**Lemma 3.3** Denoting

$$ A_{\pm} = \tau \pm_1 |\xi| \quad B_\pm = \lambda - |\eta| \quad C_\pm = \lambda - \tau \pm |\eta - \xi| $$

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we have
\[ \rho_\pm \leq |A_\pm| + |B_-| + |C_\mp| \]
where
\[ \rho_+ = |\xi| - |\eta| - |\eta - \xi|, \quad \rho_- = |\eta| + |\eta - \xi| - |\xi|. \]

Proof:
\[ \rho_+ \leq |\xi| + |\eta| + |\eta - \xi| = |\xi| \pm \lambda \mp \tau \mp |\eta - \xi| \]
\[ \leq |\xi| \pm \tau| + |\lambda - |\eta|| + |\lambda - |\eta - \xi|| = |A_\pm| + |B_-| + |C_\mp| \]
and
\[ \rho_- = |\eta| + |\eta - \xi| - |\xi| \]
\[ = -\lambda + |\eta| + |\lambda - \tau| + |\eta - \xi| + \tau - |\xi| \leq |B_-| + |C_+| + |A_-| \]
as well as for \( \tau \leq 0 \):
\[ \rho_- = |\eta| + |\eta - \xi| - |\xi| \leq |\eta| - \lambda + \lambda - \tau + |\eta - \xi| + \tau - |\xi| \]
\[ \leq |\eta| - \lambda + |\lambda - \tau + |\eta - \xi|| \]
\[ \leq |B_-| + |C_+| \]
and for \( \tau \geq 0 \):
\[ \rho_- = |\eta| + |\eta - \xi| - |\xi| \leq |\eta| - \lambda + \lambda - \tau + |\eta - \xi| + \tau - |\xi| \]
\[ \leq |\eta| - \lambda + |\lambda - \tau + |\eta - \xi|| + \tau + |\xi| \leq |B_-| + |C_+| + |A_-|. \]
This completes the proof of the Lemma and Prop. 2.3.

4 A bilinear Strichartz’ type estimate

The following bilinear refinement is crucial for the estimate of the term \( I_1^- \). It follows from the following proposition, which can be found in [AFS].

Defining
\[ [(f, g)_{HH \to L^2}]_s(\xi) := \int_{\mathbb{R}^2} \chi_{|\xi| < |\eta| + |\xi - \eta|} \hat{f}(\eta) \hat{g}(\xi - \eta) d\eta, \]
where \( \chi_A \) is the characteristic function of the set \( A \).

**Proposition 4.1** ([AFS], Theorem 6)

Let
\[ u_\pm(t) := e^{\mp it|D|} f \]
and
\[ v_\pm(t) := e^{\mp it|D|} g. \]

Then we have
\[ \| |D|^{-s_3} (u_\pm, v_\pm)_{HH \to L^2} \|_{L^2_t} \leq c \| f \|_{H^{s_4}} \| g \|_{H^{s_2}}, \]
where \( s_1 + s_2 + s_3 = \frac{1}{2}, \; s_1, s_2 < \frac{5}{8}, \; s_1 + s_2 > 0. \)
Using Prop. 1.3 we get

**Corollary 4.1** Under the assumptions of Prop. 4.1 the following estimate holds

\[
\|D\|^{-s_3}(u, v)_{H^\infty \to L^2_x} \leq c\|u\|_{X^{\frac{1}{2}, 1}_x} \cdot \|v\|_{X^{\frac{1}{2}, 1}_x}.
\]

where it is essential that the two signs on the right hand side are equal.

The following consequence is exactly what we need in order to control \(I_1^-\) in a suitable way.

**Proposition 4.2**

\[
\|\langle D \rangle^{-\frac{1}{2}}(u, v)_{H^\infty \to L^2_x} \|_{L^2_t} \leq c\|u\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x} \cdot \|v\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x}.
\]

**Proof:** The previous corollary is applied with \(s_1 = s_2 = s_3 = \frac{1}{6}\) leading to

\[
\|\langle D \rangle^{-\frac{1}{2}}(u, v)_{H^\infty \to L^2_x} \|_{L^2_t} \leq c\|u\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x} \cdot \|v\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x}.
\]

It is interpolated with the following estimate which follows from Sobolev and the estimate \(\|f\|_{L^4_t(L^2_x)} \leq c\|f\|_{X^{0, \frac{1}{4}}_x}\), which is proven like (23).

\[
\|\langle D \rangle^{-\frac{1}{2}}(u, v)_{H^\infty \to L^2_x} \|_{L^2_t} \leq c\|u\|_{L^1_t(L^\infty_x)} \|v\|_{L^1_t(L^\infty_x)} \leq c\|u\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x} \cdot \|v\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x} \leq c\|u\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x} \cdot \|v\|_{X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x}.
\]

Complex bilinear interpolation between (25) and (26) gives the result, using

\[
(X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x, X^{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}}_x)_{[\frac{1}{2}]} = X^{\frac{1}{3}, \frac{1}{2}, \frac{1}{3}}_x.
\]

**References**


