FIFTH INTERNATIONAL CONFERENCE ON ANALYTIC NUMBER THEORY AND SPATIAL TESSELLATIONS

September 16–20, 2013
National Pedagogical Dragomanov University
Kyiv, Ukraine

ABSTRACTS

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Fifth International Conference on Analytic Number Theory and Spatial Tessellations

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Abstracts

Institute of Mathematics, NAS of Ukraine
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Fifth International Conference on Analytic Number Theory and Spatial Tessellations : Abstracts. — Kyiv : Institute of Mathematics of National Academy of Sciences of Ukraine & Institute of Physics and Mathematics of the National Pedagogical Dragomanov University, 2013. — x + 126 p.

The proposed collection contains abstracts of the Fifth International Conference on Analytic Number Theory and Spatial Tessellations (September 16–20, 2013, Kyiv, Ukraine) devoted to the development of a scientific heritage of outstanding Ukrainian mathematician G. Voronoi (1868–1908). The conference covers the problems solved on the basis of the fundamental results by G. Voronoi: number theory, analysis, theoretical aspects and applications of Voronoi diagrams, and also includes some aspects of fractal analysis and fractal geometry.

The abstracts are intended for mathematicians and experts of many sciences and technology who use in their research the Voronoi methods, Voronoi results as well as for graduate students and senior students of the corresponding profiles.

П'ята міжнародна конференція з аналітичної теорії чисел і просторових мозаїк : Тези доповідей. — Київ : Інститут математики НАН України та Фізико-математичний інститут Національного педагогічного університету імені М. П. Драгоманова, 2013. — x + 126 с.

Збірник містить тези П'ятої міжнародної конференції з аналітичної теорії чисел і просторових мозаїк (16–20 вересня 2013 року, Київ, Україна), присвяченої розвитку наукового спадку видатного українського математика Георгія Вороного (1868–1908). Конференція охоплює проблеми, розв'язані на основі фундаментальних результатів Г. Вороного, з теорії чисел, аналізу, теоретичних аспектів і застосувань діаграм Вороного, а також включає деякі аспекти фрактального аналізу і фрактальної геометрії.

Тези призначені для математиків і фахівців у різноманітних сферах науки та техніки, які використовують у своїх дослідженнях методи й результати Вороного, а також для аспірантів і студентів відповідних спеціальностей.

As at the previous Voronoi conferences, the organizes has used for the emblem of the conference the following sources:

A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial tessellations. Concepts and applications of Voronoi diagrams*, John Wiley, 2000, Figure 3.7.22.

R. E. Miles and R. J. Maillardet, *The basic structures of Voronoi and generalized Voronoi polygons*, J. Appl. Probab. **19A** (1982), 97–111, Figure 5.

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Remarks

Some remarks concerning Georges Voronoï's family name

Georges Voronoï was Ukrainian-born, his name in Ukrainian is Heorhiy Voronyi. Here we reproduce the transliteration of the scientist's name as he himself used it in his scientific papers, in particular as it was used in the most remarkable Voronoï's papers published in French in the Journal für die reine und angewandte Mathematik, see

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The experts in the number theory usually use the spelling “Voronoï”, however, in the area of Voronoi diagrams it is accepted to use the spelling “Voronoi”.

G. Voronoï's father was registered as Theodosiy Voronyi at the list of the students of Kyiv St. Volodymyr University with symbol “Θ” in his first name Theodosiy. The symbol “Θ” was later on replaced by the letter “F” in the Russian alphabet and by the letter “T” in the Ukrainian alphabet. Therefore, there is some diversity in spelling of the full name of the scientist: Georges Feodosievich (Todosiyovych) Voronoï (Voronyi, Voronoi) in different publications.

We maintain a transliteration of names such as it is used by the author.

A few words about citations of archival documents

Different countries have their own national archival informational system. For citations from the Ukrainian archives, references are given as follows: archival abbreviation, fund number (*фонд* in Ukrainian), inventory or series (*опис* in Ukrainian) and file unit (*справа* in Ukrainian). *Опис* is a series within a fund, it helps to find the required file unit (*справа*).

In publications, abbreviated forms of references are usually used: Ф. (*Фонд*) No. ..., оп. (*опис*) No. ..., спр. (*справа*) No. ...

Contents

A NEW METHOD OF SUMMATION OF DIVERGENT SERIES AND EVALUATION OF SOME DIVERGENT SERIES OF ZETA AND RELATED FUNCTIONS

ARMEN BAGDASARYAN

The theory of infinite series is one of the main tools in analytic number theory, and the evaluation of the Riemann zeta function and infinite sums of related functions has been the subject of many works. This talk is centered around a new method of summation [1] which rests upon a new fundamental idea concerning the ordering of integers, and the definition of sum that generalizes and extends summations to the case when the upper limit of summation is less than the lower, or even negative. The method introduces a sufficiently large class of regular functions and allows one to assign limits to certain unbounded or oscillating functions.

Building upon the method, we establish several rules of operations with divergent series.

Theorem 1. *For any regular function* $f(x)$

$$\sum_{u=1}^{\infty} f(u) = \sum_{u=1}^{\infty} f(2u-1) + \sum_{u=1}^{\infty} f(2u) + \frac{1}{2}\lim_{n\to\infty}\big(f(n) - (-1)^n f(n)\big).$$

Theorem 2. *For any regular function* $f(x)$

$$\sum_{u=1}^{\infty} f(u) = \sum_{u=1}^{\infty}\big(f(2u-1) + f(2u)\big) + \frac{1}{2}\lim_{n\to\infty}\big(f(n) + (-1)^n f(n)\big).$$

Theorem 3. *Let* $f(x)$ *be a regular function and let* $\sum_{u=1}^{\infty} f(u) = A$ *and* $\sum_{u=1}^{\infty}(-1)^{u-1} f(u) = B$. *Then*

$$\sum_{u=1}^{\infty} f(2u) = \frac{1}{2}(A - B)$$

and

$$\sum_{u=1}^{\infty} f(2u-1) = \frac{1}{2}(A + B) - \frac{1}{2}\lim_{n\to\infty}\big(f(n) - (-1)^n f(n)\big).$$

Using the above theorems we obtain the summation formulas for some divergent series of Riemann's zeta and some zeta related functions. For example, we have

$$\sum_{u=1}^{\infty}(2u-1)^{k-1} = \frac{2\,(2^{k-1}-1)B_k + (-1)^k}{2k},$$

$$\sum_{u=1}^{\infty}(-1)^{u-1}(2u-1)^{k-1} = \frac{-1}{2k}\sum_{u=1}^{k}(-1)^u 2^u (2^u-1)\binom{k}{u}B_u,$$

$$\sum_{u=1}^{\infty}(4u-3)^{k-1} = \frac{1}{4k}\big((-3)^k + (2^k-2)B_k\big) - \sum_{u=1}^{k}(-1)^u 2^u (2^u-1)\binom{k}{u}B_u,$$

$$\sum_{u=1}^{\infty}(4u-3)^{2k-1} = \frac{1}{8k}\left(3^{2k} + (2^{2k}-2)B_{2k}\right).$$

We also give explicit formulas for the sums of infinitely ascending arithmetic and geometric progressions. The closed form evaluations of certain series involving Bernoulli numbers and polynomials are also presented.

We then evaluate at negative integer arguments the Riemann's zeta function and some other Dirichlet series in a really elementary fashion. The application of the method to some other related zeta functions and to Dirichlet L-series is also discussed.

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SOME CONGRUENCES AND IDENTITIES INVOLVING BERNOULLI AND EULER NUMBERS AND POLYNOMIALS

ARMEN BAGDASARYAN

We present several identities and congruences involving Bernoulli and Euler numbers, Bernoulli polynomials, and elementary symmetric functions. The results are obtained with use of series summation method [2], which also allows one to assign limits to certain unbounded or oscillating functions. For example, we have

Theorem 1. *Let* $f(x) = \sum_{u=1}^{2k} b_u x^u$ *be a polynomial of even degree that satisfies the condition* $f(-x) = f(x - \epsilon t)$, $\epsilon = \pm 1$, $t \in \mathbb{N}$ *fixed. Then*

$$\sum_{u=1}^{k} \frac{b_{2u-1}}{u} B_{2u} = -\epsilon \int_{-1}^{0} \left(\sum_{u=\delta}^{t-1+\delta} \left(f(x - \epsilon u) - f(-\epsilon u) \right) \right) dx,$$

where $\delta = (1 - \epsilon)/2$.

As corollaries several identities and congruences are derived, among which is the following

Proposition 1. *For any prime number* $p > 2$ *and odd* θ, $1 < \theta \leq p$, *the congruence*

$$\theta B_{\theta-1} \equiv - \sum_{u=0}^{\theta-2} S_u^p \binom{\theta}{u} \pmod{p}$$

holds, where $S_u^p = \frac{1}{p} \sum_{j=0}^{p-1} j^u$.

In particular, $pB_{p-1} \equiv -1 \pmod{p}$.

For the elementary symmetric functions $\sigma_u(t)$ of natural numbers of the interval $[1, t]$, $t = 2k - 1$, as well as the polynomials $f(x)$ involving $\sigma_u(t)$ as coefficients, where $\sigma_0(t) = 1$, $\sigma_u(t) = (1 \cdot 2 \cdots u) + \ldots + ((t+1-u) \cdots t)$, $u > 0$, we find a number of interesting formulas.

The following two are some of them. The one that involves $\sigma_u(t)$ and Bernoulli numbers

$$\sum_{u=1}^{[(k+1)/2]} \sigma_{k+1-2u}(k)\frac{2^{2u}-1}{u}B_{2u} = k!(1-2^{-k}) \quad \text{for any} \quad k \in \mathbb{N}$$

and the other one that relates together the functions $\sigma_u(t)$ and $f(x)$, and Bernoulli numbers

$$\sum_{u=0}^{t} \frac{\sigma_u(t)}{t+2-u}B_{t+2-u} = -\frac{1}{2}\int_{-1}^{0}\left(\sum_{u=0}^{t-1} f(x-u)\right)dx.$$

Some recurrence formulas for Bernoulli numbers, which in special cases reduce to some known identities on Bernoulli numbers, are also obtained, and possible relations to Euler and Fibonacci numbers are outlined.

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NUMBER SYSTEMS ON THE HEISENBERG GROUP

IEVGEN BONDARENKO

Positional number system on groups can be introduced as follows. Bases in these number systems are virtual endomorphisms of groups. A virtual endomorphism of a group G is a homomorphism $\varphi\colon H \to G$ from a subgroup H of finite index in G. A digit set for a ϕ-base number system is any set D of coset representatives for H in G. An element $g \in G$ is represented by a word $d_1 d_2 \dots d_n$ over D if $\phi(d_n^{-1}\phi(d_{n-1}^{-1} \dots \phi(d_1^{-1} g) \dots)) = e$. Multiplication in the group defines the action of G on words over D, which is closely related to self-similar group actions.

Let G be the discrete Heisenberg group, which consists of upper unitriangular matrices of dimension 3 with integer coefficients denoted (x, y, z), i.e., $G = \{(x, y, z)\colon x, y, z \in \mathbb{Z}\}$. Consider the number system on G associated to the homomorphism

$$\phi\colon H \to G, \quad \phi(x, y, z) = (x, y/2, z/2),$$

where $H = \{(x, 2y, 2z)\colon x, y, z \in \mathbb{Z}\}$ is a subgroup of index four in G, and digit set $D = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$. The corresponding action of G on finite and infinite words over D has the following properties (see [1, 2]): the action is transitive on finite words of length n for each $n \in \mathbb{N}$; the action on the infinite words is not free. Let Ω be the collection of all pre-periodic sequences from $D^{\mathbb{N}}$ together with all sequences obtained from those by arbitrary changes of letters $(0, 0, 0) \leftrightarrow (0, 0, 1)$ and $(0, 1, 0) \leftrightarrow (0, 1, 1)$. Then the growth of the action graph of G on the orbit of w for $w \in X^{\mathbb{N}} \setminus \Omega$ is polynomial of degree 4. The growth of the action graph of G on the orbit of w for $w \in \Omega$ is polynomial of degree 3.

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ON IRREDUCIBILITY OF SOME MATRICES OVER p-ADIC INTEGERS

VITALIJ BONDARENKO, RUSLANA DINIS,
AND ALEXANDER TYLYSHCHAK

Appelgate and Onishi [1] proved that matrices A, B over the p-adic integers are similar if and only if they are similar modulo p^s for all $s > 0$. The problem of describing up to similarity all irreducible matrices over the p-adic integers and irreducible matrices modulo p^s are still open.

Let k, l be positive integers, x, y be p-adic integers and

$$M(t,x,y,k,l) = \begin{array}{c} \overbrace{\qquad\qquad\quad}^{k}\overbrace{\qquad\qquad\quad}^{l} \\ \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & y \\ x & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & x & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & y & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & y & 0 \end{pmatrix} \end{array}.$$

We investigate the question: when the matrix $M(t,x,y,k,l)$ is reducible modulo p^s with $s > 0$?

Theorem 1. *If $(k,l) > 1$ then the matrix $M(t,x,y,k,l)$ is reducible.*

From this theorem it follows that

Theorem 2. *If $(k,l) > 1$ then the matrix $M(t,x,y,k,l)$ is reducible modulo p^s for any $s > 0$.*

Theorem 3. *Let $s \geq m \geq 2$, $\operatorname{ord}_p x = m-1$, $\operatorname{ord}_p x = m-2$, $n = k+l < 7$. The matrix $M(t,x,y,k,l)$ is reducible modulo p^s if and only if $(k,l) > 1$.*

For the proof of Theorem 3 we use the following lemmas.

Lemma 1. *Let $s \geq 2$, $\operatorname{ord}_p x = s-1$, $\operatorname{ord}_p x = s-2$. The matrix $M(t,x,y,3,4)$ is reducible modulo p^s (here $(k,l) = (3,4) = 1$).*

Lemma 2. *Let $s \geq 2$, $\operatorname{ord}_p x = s-1$, $\operatorname{ord}_p x = s-2$, and let $n > 6$ be odd. The matrix $M(t,x,y,n-4,4)$ is reducible modulo p^s (here $(k,l) = (n-4,4) = 1$).*

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THE ATKINSON TYPE FORMULA FOR THE PERIODIC ZETA-FUNCTION

SONDRA ČERNIGOVA AND ANTANAS LAURINČIKAS

Denote, as usual, by $\zeta(s)$, $s = \sigma + it$, the Riemann zeta-function, and define

$$E(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^2 \mathrm{d}t - T\log\tfrac{T}{2\pi} - (2\gamma_0 - 1)T,$$

where γ_0 is the Euler constant. In [1], F. V. Atkinson gave an explicit formula for $E(T)$.

Modified versions of the Atkinson formula were obtained by M. Jutila, T. Meurman and Y. Motohashi.

Kohji Matsumoto [2] and jointly with T. Meurman [3] considered a more complicated case. They proved the Atkinson type formula for the error term in the mean square formula for the Riemann zeta-function in the critical strip $\frac{1}{2} < \sigma < 1$.

Analogues of the Atkinson formula are also known for Dirichlet L-functions (T. Meurman, 1986), and for the periodic zeta-function $\zeta_\lambda(s)$ (J. Karaliūnaitė and A. Laurinčikas, 2007; J. Karaliūnaitė, 2008). We recall that the function $\zeta_\lambda(s)$, $\lambda \in \mathbb{R}$, is defined, for $\sigma > 1$, by the series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}}{m^s},$$

and by analytic continuation elsewhere. For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function, while, for $\lambda \notin \mathbb{Z}$, $\zeta_\lambda(s)$ is an entire function. In this case, we may suppose that $0 < \lambda < 1$.

The above mentioned authors studied the Atkinson type formula for the error term

$$E_\sigma(q,T) \stackrel{\mathrm{def}}{=} \sum_{a=1}^{q} \int_0^T \left|\zeta_{\frac{a}{q}}(\sigma + it)\right|^2 \mathrm{d}t \\ - q\zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}$$

with integers a and q, $1 \le a \le q$, and $\frac{1}{2} < \sigma < 1$. This report is devoted to the case $\frac{1}{2} < \sigma < \frac{3}{4}$, and gives a version of Atkinson formula with a new error term. For some positive constants $c_1 < c_2$, let $c_1 T < N < c_2 T$. Define

$$N_1 = N_1(q,N,T) = q\left(\frac{T}{2\pi} + \frac{qN}{2} - \left(\left(\frac{qN}{2}\right)^2 + \frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right),$$

denote by $\sigma_\alpha(m)$, $\alpha \in \mathbb{C}$, the generalized divisor function, and let

$$\begin{aligned}\Sigma_1(q,T) = {} & 2^{\sigma-1} q^{1-\sigma} \left(\frac{T}{\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m \le N} \frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \\ & \times \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi q m}{2\pi}}\right)\right)^{-1} \left(\frac{T}{2\pi q m} + \frac{1}{4}\right)^{-\frac{1}{4}} \\ & \times \cos\left(2T \operatorname{arsinh}\left(\sqrt{\frac{\pi q m}{2T}}\right) + \sqrt{2\pi q m T + T^2 q^2 m^2} - \frac{\pi}{4}\right), \\ \Sigma_2(q,T) = {} & -2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m \le N_1} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1} \\ & \times \cos\left(T\log\left(\frac{qT}{2\pi m}\right) - T + \frac{\pi}{4}\right),\end{aligned}$$

where $\operatorname{arsinh}(x) = \log\left(x + \sqrt{1+x^2}\right)$.

Theorem. *Suppose that* $\frac{1}{2} < \sigma < \frac{3}{4}$. *Then, for* $q \le T$,

$$E_\sigma(q,T) = \Sigma_1(q,T) + \Sigma_2(q,T) + O\left(q^{\frac{7}{4}-\sigma}\log T\right)$$

with the O*-constant depending only on* σ.

If $q = 1$, then we have the Atkinson formula for the Riemann zeta-function obtained in [3].

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APPROXIMATING FUNCTIONS BY THE RIEMANN ZETA-FUNCTION AND BY POLYNOMIALS WITH ZERO CONSTRAINTS

PAUL M. GAUTHIER

On certain compact sets K, we shall approximate functions having no zeros on the interior of K by translates of the Riemann zeta-function. As J. Andersson has shown recently, this is related to a natural problem in polynomial approximation.

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ALGEBRAIC AND ERGODIC PROPERTIES OF Ω-CONTINUED FRACTIONS

OLGA GORKUSHA

Every $x \in (0,1) \setminus \mathbf{Q}$ has a unique regular continued fraction (RCF) expansion of the form $x = [0; a_1, a_2, \ldots, a_n, \ldots]$, where for $n \geq 0$, a_n is a natural number. These so-called partial quotients a_n are defined by

$$a_n = a_n(x) = [T^{n-1}(x)], \quad \text{if} \quad T^{n-1}(x) \neq 0,$$

where T is the Gauss continued fraction transformation.

In 1981 Nakada [1] gave the natural extension of the Gauss transformation. Let $D = ([0,1] \setminus \mathbf{Q}) \times [0,1]$. The transformation $\mathfrak{T}\colon D \mapsto D$ is defined by

$$\mathfrak{T}(x,y) = \begin{cases} (T(x), 1/([1/x] + y)), & \text{if } (x,y) \in D \text{ and } x \neq 0 \\ (0,y), & \text{if } x = 0. \end{cases} \tag{1}$$

It is shown that the dynamical system $(D, \mathfrak{B}_D, \mu, \mathfrak{T})$ with the Borel algebra $\mathfrak{B}_D$ on D, Gauss measure $\mu(A)$ forms an ergodic system. This fact allows to study the approximation properties of RCF in more detail.

For $x \in [0,1) \setminus \mathbf{Q}$, the approximation coefficient of RCF $\Theta_n = \Theta_n(x)$ is defined by

$$\Theta_n = \Theta_n(x) = Q_n^2 |x - P_n/Q_n|, \quad \text{for} \quad n \geq 0, \tag{2}$$

where P_n/Q_n is the nth RCF-convergent of x. The limiting distribution of sequence $\{\Theta_n\}_{n\geq 1}$ was first investigated in the 1920s. In [2], Paul Lévy proved that

$$\lim_{n\to\infty} P\{[0; a_n, a_{n+1}, \ldots] \leq z\} = 1/((1+z)\log 2),$$

where $P\{A\}$ denotes the probability of A. In 1940 Wolfgang Doeblin published the work [3], in which in particular the limiting distribution of $1/\Theta_n(x)$ was investigated. The author proved that

$$\lim_{n\to\infty} P\{1/\Theta_n(x) \leq z\} = \begin{cases} 1 - 1/(x\log 2), & \text{if } z > 2 \\ (1 - z + z\log z)/(z\log 2), & \text{if } 1 < z \leq 2. \end{cases}$$

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In the main theorem of [4], Donald Knuth obtained the following result: for $z \in [0,1]$,

$$\mathrm{mes}\big\{x \in (0,1) \setminus \mathbf{Q} \mid \Theta_n(x) \le z\big\} = F(z) + O(g^n), \text{ with } g = (\sqrt{5}-1)/2,$$

$$F(z) = \begin{cases} z/\log 2, & z \in [0,1/2],\\ (1 - z + \log(2z))/\log 2, & z \in (1/2,1]. \end{cases}$$

In the same year Hendrik Lenstra conjectured that for RCF for almost all x and all $z \in [0,1]$, the limit

$$\lim_{n\to\infty} \#\big\{m \in [1,n] \mid \Theta_m(x) < z\big\}/n$$

exists and is equal to $F(z)$. In 1983 Wieb Bosma, Hendrik Jager, Freek Wiedijk [5] proved this proposition.

In 2008 the author [6] introduced Ω-continued fractions (ΩCF). In recently published paper [7] we prove the analogous result for this continued fraction expansion.

Our results. We show that ΩCF has an underlying dynamical system which is ergodic. We obtain the limiting distribution of the sequence of the approximation coefficients of ΩCF. Also we state and prove some results concerning the average behaviour of the approximation coefficients of ΩCF.

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ON π-SOLVABLE GROUP IN WHICH SOME MAXIMAL SUBGROUP OF π-HALL SUBGROUP IS MINIMAL NON-ABELIAN GROUP

DMITRY V. GRITSUK AND VICTOR S. MONAKHOV

All groups considered in this paper will be finite. All notation and definitions correspond to [1]. Let G be a π-solvable group. Then G has a subnormal series $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_{n-1} \supseteq G_n = 1$, whose factors G_{i-1}/G_i are π'-groups or abelian π-groups. The least number of abelian π-factors of all such subnormal series of a group G is called the derived π-length of a π-solvable group G and is denoted by $l^a_\pi(G)$. Clearly, if $\pi = \pi(G)$, then $l^a_\pi(G)$ coincides with the derived length $d(G)$ of G. The initial properties of the derived π-length established in [2].

Recall that a group is called a Miller–Moreno group if it is a non-abelian group and all of its proper subgroups are abelian. Nilpotent Miller–Moreno groups are the groups of prime-power order. Recall that a maximal subgroup of maximal subgroup of group is called a 2-maximal subgroup.

Theorem 1. *Let G be a π-solvable group. If some maximal subgroup M of π-Hall subgroup of G is Miller–Moreno group, then $l^a_\pi(G) \leq 4$. In particular, if M is a Hall subgroup, then $l^a_\pi(G) \leq 3$.*

Corollary 1. *Let G be a p-solvable group and let all 2-maximal subgroups of Sylow p-subgroup are abelian. Then $l^a_p(G) \leq 4$.*

Corollary 2. *If some maximal subgroup of a solvable group G is Miller–Moreno group, then $d(G) \leq 4$.*

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WEIGHTED DISCRETE UNIVERSALITY FOR THE MATSUMOTO ZETA-FUNCTION

ROMA KAČINSKAITĖ

For $m \in \mathbb{N}$, let $g(m) \in \mathbb{N}$, $f(j,m) \in \mathbb{N}$ and $a_m^{(j)} \in \mathbb{C}$, $1 \leq j \leq g(m)$. In [2], K. Matsumoto introduced and considered the zeta-function $\varphi(s)$ given by the polynomial Euler product

$$\varphi(s) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} p_m^{-sf(j,m)}\right)^{-1},$$

where $s = \sigma + it$ is a complex variable, and p_m denotes the mth prime number. It is assumed that the conditions

$$g(m) = c_1 p_m^{\alpha}, \quad \left|a_m^{(j)}\right| \leq p_m^{\beta} \tag{1}$$

hold. Here c_1 is a positive constant, and α and β are non-negative constants. Under condition (1), the product in the definition of the function $\varphi(s)$ converges absolutely for $\sigma > \alpha + \beta + 1$, and defines there a holomorphic function without zeros.

Suppose that the function $\varphi(s)$ is analytic in the strip

$$D = \{s \in \mathbb{C}\colon \varrho_0 < \sigma < \alpha + \beta + 1\}$$

with $\alpha + \beta + \frac{1}{2} < \varrho_0 < \alpha + \beta + 1$. For $\sigma \geq \varrho_0$, we assume that

$$\varphi(\sigma + it) = O(|t|^{c_2}), \tag{2}$$

and

$$\int_0^T |\varphi(\sigma + it)|^2 \, dt = O(T), \quad T \to \infty. \tag{3}$$

Here and latter c_2, c_3, ... are positive constants.

In [1], R. Kačinskaitė obtained discrete universality in the Voronin sense for the Matsumoto zeta-function.

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Let $N > 0$, and suppose that $h > 0$ is a fixed number such that $\exp\{\frac{2\pi k}{h}\}$ is an irrational number for all $k \in \mathbb{Z} \setminus \{0\}$. Let, for $m \geq 1$,

$$M(m) := \Bigg| \sum_{\substack{j=1 \\ f(j,m)=1}}^{g(m)} a_m^{(j)} \Bigg| p_m^{-\alpha-\beta}.$$

Theorem 1 ([1]). *Let the conditions* (1)–(3) *be satisfied. Moreover, we suppose that $M(m) \geq c_2 > 0$ for all $m \geq 1$. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be non-vanishing continuous function on K which is analytic in the interior of K. Then, for every $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\left\{ 0 \leq l \leq N\colon \sup_{s\in K} |\varphi(s+ilh) - f(s)| < \varepsilon \right\} > 0.$$

In the talk, we present weighted discrete universality theorem for the function $\varphi(s)$.

Let $w(x)$ be a non-negative function on $[0, \infty)$, and suppose that

$$U = U(N, w) = \sum_{l=0}^{N} w(l) \to \infty$$

as $N \to \infty$.

Theorem 2. *Suppose that $w(x)$ is a continuous non-vanishing function. Let the conditions as in Theorem* 1 *be satisfied. Then for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{U} \sum_{\substack{l=0 \\ \sup\limits_{s\in K}|\varphi(s+ilh)-f(s)|<\varepsilon}}^{N} w(l) > 0.$$

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GENERALIZED NUMBER SYSTEM

IMRE KÁTAI

Generalized number systems can be defined and some interesting theorems can be proved in the set of integers, integers of algebraic number fields, in the set of lattices in $\mathbb{R}_k$. Starting from these, we can define so called just touching covering systems, which is a special type of tessellation.

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A RELATIONSHIP BETWEEN SOME PROBLEMS IN PROBABILITY THEORY AND NUMBER THEORY

OLEG I. KLESOV

We discuss some applications of known asymptotics in the Dirichlet divisor problem for limit theorems in probability theory.

The divisor summatory function

$$D(x) = \sum_{n \le x} d(n) = \sum_{u,v:\, uv \le x} 1$$

counts the number of lattice points $(u; v)$ lying in the first quadrant under the hyperbola $uv = x$. The main asymptotic terms of D are obtained by Dirichlet, namely

$$D(x) = x \log x + (2\gamma - 1)x + O(\sqrt{x}), \qquad x \to \infty.$$

G. F. Voronoi [2] improved this result by showing that

$$D(x) = x \log x + (2\gamma - 1)x + O(x^{1/3} \log x).$$

Several applications of the asymptotic behavior of $D(x)$ as $x \to \infty$ are known in probability theory. Our aim is to survey some of them, namely

1. strong limit theorems for multiple sums,
2. renewal theory,
3. complete convergence.

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EXPONENTIAL DIVISOR FUNCTIONS

ANDREW V. LELECHENKO

Consider a set of arithmetic functions $\mathcal{A}$, a set of multiplicative prime-independent functions $\mathcal{M}_{PI}$ and operator $E\colon \mathcal{A} \to \mathcal{M}_{PI}$, which is defined as follows

$$(Ef)(p^a) = f(a).$$

The behaviour of Ef for various special cases of f has been widely studied by different authors, starting with the pioneering paper of Subbarao [1] on $E\tau$ and $E\mu$.

Our aim is to investigate asymptotic properties of multiple applications of E on arithmetic functions.

Let $\gamma(0) = 2$ and $\gamma(m) = 2^{\gamma(m-1)}$; let τ be the divisor function: $\tau(n) = \sum_{d|n} 1$. Denote also

$$\tau(a_1, \ldots, a_k; n) = \sum_{d_1^{a_1} \cdots d_k^{a_k} = n} 1,$$

and let $\Delta(\cdots; x)$ be an error term in the asymptotic estimate of the sum $\sum_{n \le x} \tau(\cdots; n)$. Further, $\theta(\cdots)$ denotes throughout the paper a real value such that $\Delta(\cdots; x) \ll x^{\theta(\cdots)+\varepsilon}$.

Theorem 1. *For a fixed integer $m > 1$ we have*

$$\sum_{n \le x} E^m \tau(n) = A_m x + B_m x^{1/\gamma(m)} + R_m(x),$$

where A_m and B_m are computable constants and

$$R_m(x) \ll x^{\alpha_m + \varepsilon}, \qquad R_m(x) \gg x^{1/2(\gamma(m)+1)}.$$

Also under the Riemann hypothesis

$$\alpha_m = \frac{1 - \theta(1, \gamma(m))}{\gamma(m) + 2 - 2(\gamma(m)+1)\theta(1, \gamma(m))},$$

We get precise unconditional estimates of α_m as a result of the following lemma.

Lemma 1. *For a fixed integer $r \ge 5$*

$$\theta(1, 2^r) = \frac{2^r - 2r}{2^{2r} - r \cdot 2^r - 2r^2 + 2r - 4} < \frac{1}{2^r + r}.$$

Further, abbreviate $a \times k$ for a sequence of arguments $\underbrace{a, \ldots, a}_{k \text{ times}}$. We obtain an analog of Theorem 1 on asymptotic properties of $E^m\tau(1 \times k; \cdot)$ and improve Tóth's theorem [2] showing that

$$\theta(1, l \times (k-1)) = \frac{1}{l + 1 - \theta(1 \times (k-1))}.$$

In the case of $E^m\tau(1 \times 3; \cdot)$ sharper estimates can be given. To achieve this goal we proved following lemmas.

Lemma 2. *For a fixed integer $r \geq 10$ we have*

$$\theta(1, 2^r, 2^r) = \frac{26 \cdot 2^{2r} - (29r + 41)2^r + 16r^2 + 12r + 32}{26 \cdot 2^{3r} - (16r + 41)2^{2r} + (24r - 3)2^r + 16r + 12} < \frac{1}{2^r + 1}.$$

Lemma 3. *Consider a multiplicative function f such that*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(as)\zeta^r(bs)}{\zeta^k(cs)} := F(s),$$

where $2a \leq b < c < 2(a + b)$. Let $\Delta(x)$ be defined implicitly by the equation

$$S(x) := \sum_{n \leq x} f(n) = \left(\operatorname*{res}_{s=1/a} + \operatorname*{res}_{s=1/b} \right) F(s) x^s s^{-1} + \Delta(x).$$

Then under RH for any $1 \leq y \leq x^{1/c}$

$$\Delta(x) = \sum_{l \leq y} \mu_k(l) \Delta(a, b \times r; x/l^c) + O(x^{1/2a+\varepsilon} y^{1/2-c/2a} + x^{\varepsilon}),$$

where μ_k is a multiplicative function such that $\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \zeta^{-k}(s)$.

Finally, we investigate properties of $E^m f$ for arbitrary arithmetic function f and large enough m. We show that they can be estimated in the quite similar manner as $E^m\tau(1 \times k; \cdot)$ was.

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ZETA-FUNCTIONS OF WEIGHT LATTICES OF COMPACT CONNECTED SEMISIMPLE LIE GROUPS

KOHJI MATSUMOTO

This is a joint work with Y. Komori and H. Tsumura.

As a multi-variable generalization of Witten's zeta-functions, I will introduce the notion of zeta-functions of weight lattices of compact connected semisimple Lie groups, and will discuss their properties.

Main results are their explicit forms, functional relations, and parity results.

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DERIVED AND NILPOTENT LENGTH OF FINITE GROUPS

VICTOR S. MONAKHOV

All groups considered in this paper are finite. All notation and definitions correspond to [1,2].

Let $\mathbb{P}$ be the set of all prime numbers, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. By π also denote a function defined on the set of natural numbers $\mathbb{N}$ as follows:

$\pi(a)$ is the set of primes dividing a positive integer a.

For a group G and a subgroup H of G we believe that $\pi(G) = \pi(|G|)$ and $\pi(G : H) = \pi(|G : H|)$.

Fix a set of prime numbers π. If $\pi(m) \subseteq \pi$, then a positive integer m is called a π-number. The group G is called a π-group if $\pi(G) \subseteq \pi$, and a π'-group if $\pi(G) \subseteq \pi'$. The chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{n-1} \supseteq G_n = 1 \tag{1}$$

is called subnormal series of a group G, if subgroup G_{i+1} is normal subgroup of G_i for every i. The quotient groups G_i/G_{i+1} are called factors of the series (1).

The group is called a π-solvable group if it has a subnormal series (1) whose factors are solvable π-groups or π'-groups.

The least number of π-factors of all such subnormal series of a group G is called the π-length of a π-solvable group G and is denoted by $l_\pi(G)$.

Since π-factors of subnormal series (1) of a π-solvable group G are solvable, then every π-solvable group has a subnormal series in which all π-factors are nilpotent. The least number of nilpotent π-factors of all such subnormal series of G is called the nilpotent π-length of G and denoted by $l_\pi^n(G)$. Clearly, in the case $\pi = \pi(G)$, $l_\pi^n(G)$ coincides with the nilpotent length of G.

In case when π consists of a single prime $\{p\}$, i. e., $\pi = \{p\}$, we will obtain $l_\pi(G) = l_\pi^n(G) = l_p(G)$ for every π-solvable group. The equality $l_\pi(G) = l_\pi^n(G)$ is cleared to be a true for a π-solvable group with nilpotent π-Hall subgroup.

Let G be a π-solvable group. Then G has a subnormal series (1) whose factors are π'-groups or abelian π-groups. The least number of abelian π-factors of all such subnormal series of G is called the derived

π-length of group G and denoted by $l_\pi^a(G)$. Clearly, in the case $\pi = \pi(G)$, $l_\pi^a(G)$ coincides with the derived length of G. The initial properties of the derived π-length and some of the results related to this notion established in [4–6].

Theorem 1. *Let G be a π-solvable group. If the derived subgroup of π-Hall subgroup of G is nilpotent, then*

$$l_\pi^n(G) \leq 1 + \max_{r\in\pi} l_r(G) \quad \textit{and} \quad l_\pi^a(G) \leq 1 + \max_{r\in\pi} l_r^a(G).$$

Since the derived subgroup of a supersolvable group is nilpotent, Theorem 1 implies

Corollary 1. *Let G be a π-solvable group. If a π-Hall subgroup of G is supersolvable, then*

$$l_\pi^n(G) \leq 1 + \max_{r\in\pi} l_r(G) \quad \textit{and} \quad l_\pi^a(G) \leq 1 + \max_{r\in\pi} l_r^a(G).$$

Corollary 2. *Let G be a π-solvable group. If a Sylow p-subgroup of G is cyclic for every $p \in \pi$, then $l_\pi^n(G) \leq l_\pi^a(G) \leq 2$.*

Corollary 3. *Let G be a π-solvable group, and let a Sylow p-subgroup of G be bicyclic for every $p \in \pi$. If $2 \notin \pi$, then $l_\pi^n(G) \leq l_\pi^a(G) \leq 3$.*

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APPROXIMATION QUALITY OF COMPLEX CONTINUED FRACTIONS

NICOLA OSWALD

We consider the convergents to a given complex number which arise from its Hurwitz continued fraction expansion. It appears that with respect to approximation quality these convergents behave rather different from the real case.

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CONTINUED FRACTIONS OF INHOMOGENEOUS LINEAR FORMS

VLADIMIR PARUSNIKOV

Let α, β be real numbers, $0 \leq \alpha < 1$, $0 \leq \beta < 1$. They define at the plane $(y,z) \in \mathbb{R}^2$ the inhomogeneous linear form $L_{\alpha,\beta}(y,z) = -\beta+\alpha y+z$. We propose the algorithm of an expansion of this linear form into the 'inhomogeneous continued fraction'

$$L_{\alpha,\beta} \sim [0; b_1, b_2, \dots] \bmod [0; a_1, a_2, \dots].$$

Inhomogeneous continued fraction is a generalization of the classic regular continued fraction: for $\beta = 0$ every $b_n = 0$, we get the continued fraction expansion of the number α: $L_{\alpha,0} \sim [0] \bmod [0; a_1, a_2, \dots]$. Some base properties of inhomogeneous continued fractions are proved.

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A NEW CHARACTERISTIC OF THE IDENTITY FUNCTION

BUI MINH PHONG

In 1997, Koninck, Kátai, Phong proved that if f is a multiplicative function with $f(1)=1$ satisfies

$$f(p+m^2)=f(p)+f(m^2)$$

for all primes p and all positive integers m, then $f(n)=n$ for all positive integers n.

Recently the following result is proved:

Assume that f, g are multiplicative functions with $f(1)=1$. If

$$f(p+m^2)=g(p)+g(m^2) \quad \text{and} \quad g(p^2)=g(p)^2$$

for all primes p and positive integers m, then either

$$f(p+m^2)=0, \quad g(p)=-1 \quad \text{and} \quad g(m^2)=1$$

for all primes p and positive integers m, or

$$f(n)=n \quad \text{and} \quad g(p)=p, \quad g(m^2)=m^2$$

for all primes p and positive integers n, m.

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APPLICATION OF VORONOI THEOREM TO DIAGONAL CONTRACTIONS OF LIE ALGEBRAS

DMYTRO R. POPOVYCH

Let V be an n-dimensional vector space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, $n < \infty$, and $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on V. We identify $\mu \in \mathcal{L}_n$ with the corresponding Lie algebra $\mathfrak{g} = (V, \mu)$.

Definition 1. Consider a parameterized family of the Lie algebras $\mathfrak{g}_\varepsilon = (V, \mu_\varepsilon)$ isomorphic to $\mathfrak{g} = (V, \mu)$. The family of the new Lie brackets μ_ε, $\varepsilon \in (0, 1]$, is defined via the Lie bracket μ using a continuous function $U\colon (0, 1] \to \mathrm{GL}(V)$ by the rule $\mu_\varepsilon(x, y) = U_\varepsilon^{-1}\mu(U_\varepsilon x, U_\varepsilon y)$ for any $x, y \in V$. If for any $x, y \in V$ there exists the limit

$$\lim_{\varepsilon\to+0} \mu_\varepsilon(x, y) = \lim_{\varepsilon\to+0} U_\varepsilon^{-1}\mu(U_\varepsilon x, U_\varepsilon y) =: \mu_0(x, y)$$

then μ_0 is a well-defined Lie bracket. The Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *one-parametric continuous contraction* (or simply a *contraction*) of the Lie algebra $\mathfrak{g}$. The procedure $\mathfrak{g} \to \mathfrak{g}_0$ providing $\mathfrak{g}_0$ from $\mathfrak{g}$ is also called a *contraction*.

Definition 2. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ (over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \mathrm{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i\colon (0, 1] \to \mathbb{F} \setminus \{0\}$. In the specific case $W_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n})$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, the contraction is called *a generalized Inönü–Wigner contraction* (*generalized IW-contraction*). Then the tuple of exponents $(\alpha_1, \dots, \alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathfrak{g} \to \mathfrak{g}_0$.

All continuous contractions appearing in the physical literature are generalized IW ones. The question whether every contraction can be realized by a generalized IW-contraction was answered in the negative in [1, 3]. Considering different subclasses of Lie algebras closed with respect to contractions or imposing restrictions on contraction matrices, we can pose the problem on partial universality of generalized IW-contractions in specific subsets of contractions. Diagonal contractions arose in [5] as an intermediate step in realizing general contractions via generalized IW-contractions.

Theorem 1. *Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.*

Given a diagonal contraction, the existence of an equivalent generalized IW-contraction is tied to the consistency of a system of homogeneous strict linear inequalities with integer coefficients [2], which is proved by using a modification of the well-known Voronoi theorem [4].

Corollary 1. *Any diagonal contraction whose matrix possesses a finite limit at* $\varepsilon \to +0$ *is equivalent to a generalized IW-contraction with non-negative integer exponents.*

Theorem 1 is obviously extended to the class of contractions wider than the class of diagonal contractions. In particular, it implies that any contraction $\mathfrak{g} \to \mathfrak{g}_0$ whose matrix can be represented in the form $U_\varepsilon = \hat{U}_\varepsilon A W_\varepsilon \check{U}_\varepsilon$ is equivalent to a generalized IW-contraction with an integer signature. Here $\hat{U}_\varepsilon$ is an automorphism matrix of the algebra $\mathfrak{g}$, $\check{U}_\varepsilon$ is a nonsingular matrix with the well-defined componentwise limit $\lim_{\varepsilon\to+0} \check{U}_\varepsilon =: P$, both the matrices $\hat{U}_\varepsilon$ and $\check{U}_\varepsilon$ are continuously parameterized by $\varepsilon \in (0,1]$, A and P are constant nonsingular matrices and $W_\varepsilon = \operatorname{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i\colon (0,1] \to \mathbb{F} \setminus \{0\}$. The problem on the widest class of parameterized matrices which are associated with contractions equivalent to generalized IW-contractions is still open.

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MEAN VALUES OF MULTIPLE DIRICHLET SERIES AT NON-POSITIVE INTEGERS

BOUALEM SADAOUI

In this talk, we relate the special value at a non-positive integer $\underline{\mathbf{s}} = (s_1, \ldots, s_r) = -\underline{\mathbf{N}} = (-N_1, \ldots, -N_r)$ obtained by meromorphic continuation of the multiple Dirichlet series

$$Z(\underline{\mathbf{P}}, \underline{\mathbf{s}}) = \sum_{\underline{m} \in \mathbb{N}^{*n}} \frac{1}{\prod_{i=1}^{r} P_i^{s_i}(\underline{m})}$$

to special values of the function

$$Y(\underline{\mathbf{P}}, \underline{\mathbf{s}}) = \int_{[1,+\infty[^n} \prod_{i=1}^{r} P_i^{-s_i}(\underline{\mathbf{x}})\, d\underline{\mathbf{x}}$$

where $\underline{\mathbf{P}} = (P_1, \ldots, P_r)$ are polynomials in several variables, verified a certain conditions.

We prove a simple relation between $Z(\underline{\mathbf{P}}_{\underline{\mathbf{a}}}, -\underline{\mathbf{N}})$ and $Y(\underline{\mathbf{P}}_{\underline{\mathbf{a}}}, -\underline{\mathbf{N}})$, such that for all $\underline{\mathbf{a}} \in \mathbb{C}^n$, we denote $\underline{\mathbf{P}}_{\underline{\mathbf{a}}} := (P_{1\underline{\mathbf{a}}}, \ldots, P_{r\underline{\mathbf{a}}})$, where $P_{\underline{\mathbf{a}}}(\underline{\mathbf{x}}) := P(\underline{\mathbf{x}} + \underline{\mathbf{a}})$ is the shifted polynomial.

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ON A BIQUADRATIC DIOPHANTINE EQUATION

ANDRZEJ SCHINZEL AND MARIUSZ SKAŁBA

The following theorems are proved.

Theorem 1. *Let* $p \equiv 3 \pmod 4$ *be a prime,* α odd*,* $D = p^\alpha$*. If the equation* $(x^2+y^2)^2 - D(z^2+t^2)^2 = 1$ *has at least one solution in integers with* $z^2+t^2 > 0$*, then it has infinitely many.*

Theorem 2. *Let* $p \equiv 3 \pmod 8$ *be a prime,* α odd*,* $D = p^\alpha$*. If the equation* $(x^2+y^2)^2 - D(z^2+t^2)^2 = -2$ *has at least one solution in integers, then it has infinitely many.*

Theorem 3. *Let* $p \equiv 7 \pmod 8$ *be a prime,* α odd*,* $D = p^\alpha$*. If the equation* $(x^2+y^2)^2 - D(z^2+t^2)^2 = 2$ *has at least one solution in integers, then it has infinitely many.*

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ALGEBRAIC INDEPENDENCE OF CERTAIN POWER SERIES OF EXPONENTIAL TYPE

IEKATA SHIOKAWA

We exhibit here main results in [1].

Theorem 1 ([1, Theorem 1]). *Let $q \geq 3$ be an integer. If α is a nonzero algebraic number, then among q numbers*

$$e_r(\alpha) = \sum_{\substack{n=0 \\ n \equiv r \pmod q}}^{\infty} \frac{\alpha^n}{n!} \qquad (r = 0, 1, \dots, q-1) \tag{1}$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q)+1$ of the q functions

$$e_0(z),\ e_1(z),\ \dots,\ e_{q-1}(z)$$

are algebraically dependent over $\mathbb{Q}$.

Corollary 1. *Let $q \geq 3$ be an integer and let α be a nonzero algebraic number. Then any $\varphi(q)$ of the numbers*

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{(qn+r)!} \qquad (r = 0, 1, \dots, q-1)$$

are algebraically independent over $\mathbb{Q}$.

For any real number α we denote by $\{\alpha\}$ the fractional part of α.

Theorem 2 ([1, Theorem 4]). *Let q and a are coprime integers with $q \geq 3$ and $0 < a < q$. Let*

$$f_b(z) = \sum_{n=0}^{\infty} \left\{ \frac{an+b}{q} \right\} \frac{z^n}{n!} \qquad (b = 0, 1, \dots, q-1).$$

If α is a nonzero algebraic number, then among q numbers

$$f_0(\alpha),\ f_1(\alpha),\ \dots,\ f_{q-1}(\alpha)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q)+1$ of the functions

$$f_0(z),\ f_1(z),\ \dots,\ f_{q-1}(z)$$

are algebraically dependent over $\mathbb{Q}$.

Let F_n and L_n be the *Fibonacci numbers* and the *Lucas numbers*, respectively.

Theorem 3 ([1, Theorem 5]). *Let $f_s(\alpha)$ and $g_s(\alpha)$ be power series defined by*

$$f_s(z) = \sum_{n=0}^{\infty} F_n^s \frac{z^n}{n!}, \qquad g_s(z) = \sum_{n=0}^{\infty} L_n^s \frac{z^n}{n!}.$$

If α is a nonzero algebraic number, then all the numbers in the set

$$\{f_s(\alpha) \mid s \in \mathbb{N}\} \cup \{g_s(\alpha) \mid s \in \mathbb{N}\}$$

are distinct and any two are algebraically independent over $\mathbb{Q}$. Moreover, any three functions in the set

$$\{f_s(z) \mid s \in \mathbb{N}\} \cup \{g_s(z) \mid s \in \mathbb{N}\}$$

are algebraically dependent over $\mathbb{Q}$.

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ON THE NUMBER OF ZEROS OF SOME ANALYTIC FUNCTIONS

DARIUS ŠIAUČIŪNAS

Let $V > 0$ be an arbitrary positive number, $D_V = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1,\ |t| < V\}$, and let $H(D_V)$ denote the space of analytic functions endowed with the topology of uniform convergence on compacta. Define $S_V = \{g \in H(D_V)\colon g^{-1}(s) \in H(D_V)$ or $g(s) \equiv 0\}$. We say that the continuous operator $F\colon H^n(D_V) \to H(D_V)$ belongs to the class U_V if, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V^n$ is non-empty. In the report we consider the number of zeros of the function $F(L(s,\chi_1),\dots,L(s,\chi_n))$, where, as usual, $L(s,\chi)$, $s = \sigma + it$, denotes a Dirichlet L-function. More precisely, we have the following result.

Theorem. *Suppose that $\chi_1, \dots, \chi_n$ be pairwise non-equivalent Dirichlet characters, and $F \in U_V$ with sufficiently large V. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\chi_1,\dots,\chi_n,F,\sigma_1,\sigma_2) > 0$ such that, for sufficiently large $T < V$, the function $F(L(s,\chi_1),\dots, L(s,\chi_n))$ has more than cT zeros lying in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.*

We give an example of $F \in U_V$. For $g_1, g_2 \in H(D_V)$, let $F(g_1,g_2) = g_1^2 + g_2^2$, and let $p(s)$ be an arbitrary polynomial. Then there exists a constant $C > 0$ such that $|p(s)| \le C$ for $s \in D_V$. We take $K > C$, and

$$p_1(s) = \frac{p(s)+K}{2\sqrt{K}}, \qquad p_2(s) = \frac{p(s)-K}{2i\sqrt{K}}.$$

Then we have that $p_j(s) \neq 0$ on D_V, $j = 1,2$, and $p_1^2(s) + p_2^2(s) = p(s)$. Thus, $(p_1,p_2) \in (F^{-1}\{p\}) \cap S_V^2$.

A proof of the theorem is based on a universality theorem obtained in [1].

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ON THE LIMIT DISTRIBUTIONS FOR SOME SETS OF ADDITIVE ARITHMETIC FUNCTIONS

GEDIMINAS STEPANAUSKAS AND JONAS ŠIAULYS

The limit behaviour of distributions $(1/[x])\sum_{\substack{n\le x\\ f_x(n)-\alpha(x)<u}} 1$ was considered in the probabilistic number theory very often. There were considered various classes of additive functions $f_x(n)=\sum_{p^r||n} f_x(p^r)$ with different centering functions $\alpha(x)$. But the class of examined additive functions f_x was of a special expression:

$$f_x(n)=h(n)/\beta(x), \tag{1}$$

where f is some additive function and $\beta(x)$ is some unboundedly increasing function. In the books [1] and [2], and works cited there, one can find almost all classical results and their historical context.

An object of our talk is strongly additive functions taking the values 0 or 1 on the set of primes and maybe depending on x (therefore we call usually f_x by a set of additive functions), and in general case it is impossible to express f_x by relation (1). We will discuss about the weak convergence of distributions

$$(1/[x])\sum_{\substack{n\le x\\ f_x(n)<u}} 1$$

for the set of strongly additive functions f_x as $x\to\infty$ and about the class of possible limit laws.

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DIOPHANTINE APPROXIMATION OF COMPLEX NUMBERS

JÖRN STEUDING

Whereas the convergents to the continued fraction expansion of a given real number provide the best possible rational approximations the situation with complex numbers is rather different for various reasons; e.g., for certain imaginary fields there is no Euclidean algorithm at hand.

In the talk we survey previous work on this topic and provide a new approach to circumvent the complex obstacles by geometric methods.

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ON STATISTICAL PROPERTIES OF 3-DIMENSIONAL VORONOI–MINKOWSKI CONTINUED FRACTIONS

ALEXEY USTINOV

There exist two geometric interpretations of classical continued fractions admitting a natural generalization to the multidimensional case. In one of these interpretations, which is due to Klein, a continued fraction is identified with the convex hull (the Klein polygon) of the set of integer lattice points belonging to two adjacent angles (1895–1896). The second interpretation, which was independently proposed by Voronoi and Minkowski, is based on local minima of lattices, minimal systems, and extremal parallelepipeds (1896). The vertices of Klein polygons in plane lattices can be identified with local minima; however, beginning with the dimension 3, the Klein and Voronoi–Minkowski geometric constructions become different.

The constructions of Voronoi and Minkowski is simpler from the computational point of view. In particular, they make it possible to design efficient algorithms for determining fundamental units in cubic fields. In both Voronoi's and Minkowski's approaches, the three-dimensional theory of continued fractions is based on interesting theorems of the geometry of numbers.

Analytical approach based on the method of trigonometric sums and estimates of Kloosterman sums allows to solve different problems concerned with classical continued fractions. The talk will be devoted to analogous 3-D tool. It is also based on the estimates of Kloosterman sums and uses Linnik–Skubenko ideas from their work "Asymptotic distribution of integral matrices of third order" (1964). This tool, in particular, allows to study statistical properties of Minkowski–Voronoi 3-D continued fractions.

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TWISTED EXPONENTIAL SUMS OVER THE RING OF GAUSSIAN INTEGERS

PAVEL VARBANETS AND SERGEY VARBANETS

Let $G := \{a + bi \mid a, b \in \mathbb{Z},\ i^2 = -1\}$ be the ring of Gaussian integer numbers. Denote by G_γ (respectively, G_γ^*) a residue (reduced residue) system modulo γ, $\gamma \in G$. Let χ be an arbitrary multiplicative character modulo γ. We will estimate the twisted exponential sums

$$S(f, \chi, \gamma) := \sum_{x \in G_\gamma^*} \chi(x) e^{\pi i \operatorname{Sp}\left(\frac{f(x)}{\gamma}\right)},$$

where f is a rational function over G, and $\operatorname{Sp}(u)$ denotes a trace of $u \in \mathbb{Q}(i)$ into $\mathbb{Z}$.

Moreover, for $q > 1$, $q \in \mathbb{N}$ and $\chi_q \in \widehat{\mathbb{Z}}_q$ we consider the n-dimensional norm Kloosterman sum

$$K_{\chi_q}(\alpha_0, \ldots, \alpha_n) := \sum_{\substack{x_0, x_1, \ldots, x_n \in G_q^* \\ N(x_0, \ldots, x_n) \equiv 1 \pmod{q}}} e^{\pi i \operatorname{Sp}\left(\frac{\alpha_0 x_0 + \cdots + \alpha_n x_n}{q}\right)}$$

and apply estimations of that sum to the problem of distribution of values of the divisor function $\tau_n(\alpha)$, $\alpha \in G$, on the arithmetical progression $N(\alpha) \equiv a_0 \pmod{q}$.

Also we will estimate the character sum on the special subgroup of norm residues modulo p^n, p is a prime rational integer.

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A TAUBERIAN THEOREM FOR INGHAM SUMMATION METHOD

VYTAS ZACHAROVAS

In many problems of number theory sums of type

$$\frac{1}{n}\sum_{k=1}^{n} a_k \left[\frac{n}{k}\right], \tag{1}$$

where $a_k \in \mathbb{C}$ is some sequence of complex numbers, naturally appear. We say that a formal series

$$\sum_{k=1}^{\infty} \frac{a_k}{k} \tag{2}$$

is convergent in sense of Ingham if the sum (1) converges to some finite number C as $n \to \infty$. In such case we say that the series (2) is summable in sense of Ingham and that its Ingham sum is equal to C. Several papers were devoted to finding conditions that are sufficient for Ingham summability. We investigate the case when the sequence a_k has a convergent Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^\sigma}$$

for all $\sigma > 1$.

Theorem 1 ([2]). *Suppose a_m is a fixed sequence of complex numbers such that $g(\sigma) = \sum_{m=1}^{\infty} \frac{a_m}{m^\sigma}$ converges for all $\sigma > 1$. Then*

$$\frac{1}{n}\sum_{k \leqslant n} a_k \left[\frac{n}{k}\right] = C + o(1) \quad \textit{as} \quad n \to \infty$$

if and only if the following two conditions are satisfied:

$$\sum_{k \leqslant n} a_k \left[\frac{n}{k}\right] \log k = o(n \log n) \quad \textit{as} \quad n \to \infty \tag{3}$$

and

$$\lim_{\sigma \downarrow 1} g(\sigma) = \lim_{\sigma \downarrow 1} \sum_{m=1}^{\infty} \frac{a_m}{m^\sigma} = C.$$

Condition (3) *alone implies that*

$$\frac{1}{n}\sum_{k\leqslant n} a_k \left[\frac{n}{k}\right] = g\left(1+\frac{1}{\log n}\right) + o(1).$$

Although at a first glance the condition (3) might seem artificial and difficult to check we show that it can easily checked for a wide class of sequences a_k that are multiplicative functions of k. This as a consequence allows us to derive from our theorem several classical results on the mean values of multiplicative functions.

Theorem 1 is a direct analogue of the very first Tauber's result that gave rise to the whole class of so called Tauberian theorems and is of the same nature as the earlier result we obtained for a certain class of Voronoi summation methods [1].

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BIRTH AND DEATH EVOLUTIONS ON SPACE OF FINITE CONFIGURATIONS

VIKTOR BEZBORODOV AND YURI KONDRATIEV

We would like to construct Markov process on the space of finite configurations corresponding to birth and death generator

$$LF(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)\big[F(\eta\cup x) - F(\eta)\big]dx \\ - \sum_{x\in\eta} d(x,\eta)\big(F(\eta\setminus x) - F(\eta)\big). \quad (1)$$

Such processes were first considered in [1]. In order to do this, we consider stochastic equation

$$\eta_t(B) = \int\limits_{B\times[0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) \\ - \int\limits_{[0;t]\times[0;\infty]} \sum_{x\in B\cap\eta_{r-}} I_{[0;d(x,\eta_{r-})]}(v)dN_2(x,r,v) + \eta_0(B), \quad (2)$$

where $B \in \mathscr{B}(\mathbb{R}^d)$ is a Borel set, N_1, N_2 are Poisson point processes on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$, the mean measure of N_1 is $dx \times ds \times du$, of N_2 is $|dx| \times ds \times du$, $|A| = \#A$, η_0 is a (random) finite configuration, b, d are measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\Gamma_0(\mathbb{R}))$. We require processes N_1, N_2, η_0 to be independent of each other. We are looking for the solution of the equation (1) in the configuration space over $\mathbb{R}^d$,

$$\Gamma(\mathbb{R}^d) = \left\{\gamma \in \mathbb{R}^d \colon |\gamma\cap K| < +\infty \text{ for all compact } K \subset \mathbb{R}^d\right\}.$$

We assume that the birth rate b satisfies the following conditions: linear growth on the second variable in the sense that

$$\int\limits_{\mathbb{R}^d} \bar{b}(x,\eta)ds \le c_1|\eta| + c_2,$$

where $\bar{b}(x,\eta) := \sup\limits_{\xi\subset\eta} b(x,\xi)$.

Theorem. *For each initial condition* η_0 *with* $\mathsf{E}|\eta_0| < \infty$ *equation* (2) *has the unique solution* $\eta = \{\eta_t, t \geq 0\}$*. Particularly,* $\eta_t \in \Gamma_0(\mathbb{R}^d)$ *almost surely for all* $t \geq 0$*. The solution is a Markov process with the generator given in* (1)*.*

We also consider special case of (2), with

$$d(x, \eta) = \exp\Big\{\sum_{y \in \eta} a(x - y)\Big\}.$$

For this model we prove under some additional assumption that the expectation of number of points of the point process with generator (1) grows exponentially fast, and, moreover, the number of points of the solution has almost surely only two possible types of asymptotic behavior: the process either extincts in finite time or the number of points grows exponentially fast almost surely.

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ABSOLUTE VORONOI SUMMABILITY OF FOURIER INTEGRALS OF FUNCTIONS OF BOUNDED VARIATION

LILIYA BOITSUN AND TAMARA RYBNIKOVA

There is a lot of works in mathematical literature on the theory of summability of series. Application of summability theory to Fourier series was also developed to a considerable extent.

Voronoi summability of Fourier series was considered in many papers. G. F. Voronoi introduced this method in 1901 in his work "Extension of the notion of limit to the sum of terms of an infinite series" [1].

However, in the foreign mathematical literature this method of summability is known as the Nörlund method, although N. E. Nörlund considered it 18 years later [2]. So in the literature this method is denoted as (N, p_n), and such literature is growing constantly.

Functional summability of integrals was also considered by G. F. Voronoi in his work [1].

Let $f(t)$ be a function integrable over any finite interval $[0, A]$, $A > 0$, and $S(t) = \int_0^t f(u)du$. Let $p(t)$ be a function integrable over any finite interval, and $P(y) = \int_0^y p(u)du \neq 0$.

The integral $\int_0^\infty f(u)du$ is said to be summable by Voronoi method, i.e., $(W, p(y))$-summable to I if

$$\lim_{y\to\infty} \tau(y) = \lim_{y\to\infty} \frac{1}{P(y)} \int_0^y P(y-u)f(u)du$$
$$= \lim_{y\to\infty} \frac{1}{P(y)} \int_0^y p(y-u)S(u)du = I.$$

If $\int_0^\infty |\tau'(y)|dy < \infty$, the integral is said to be absolutely $(W, p(y))$-summable, or $|W, p(y)|$-summable.

In the special case, if $p(t) = \alpha t^{\alpha-1}$, $\alpha > 0$, the method is reduced to the well-known Cesàro method (C, α), $\alpha > 0$.

In the special case, if $p(t) = \frac{1}{1+t}$, and, therefore, $P(y) \sim \ln y$ as $y \to \infty$, it is reduced to the harmonic method of summability.

The functional Voronoi summability is studied systematically at the Department of Mathematical Analysis and Theory of Functions of Dnipropetrovsk National University (L. G. Boitsun, T. I. Rybnikova).

We proved the following theorem.

Theorem. *Let*

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

be a function of bounded variation over $(0, \infty)$*, and let* $p(y)$ *be a positive, non-increasing, continuously differentiable function such that* $P(y) \to \infty$ *as* $y \to \infty$ *and*

$$\int_y^\infty \frac{du}{uP(u)} \le \frac{K}{P(y)} \quad \text{for all} \quad y > 0.$$

Then the Fourier integral of the function $f(t) \in L(-\infty, \infty)$

$$\frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(t) \cos u(x-t) dt$$

is $|W, p(y)|$*-summable.*

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LIMIT BEHAVIOR OF EXPECTED ESSCHER TRANSFORM

VITALIY DROZDENKO

Esscher transform of random variable X with parameter $\alpha \in \mathbb{R}$ is a random variable defined in the following way

$$ET[X;\alpha] := \frac{Xe^{\alpha X}}{\mathsf{E}[e^{\alpha X}]}.$$

Esscher transform was introduced by Swedish mathematician Fredrik Esscher in the paper [2]. Results of the just mentioned paper were later extended in the paper [3].

Expected Esscher transform of a random variable X with parameter $\alpha \in \mathbb{R}$ is defined as expectation of corresponding Esscher transform, namely

$$EET[X;\alpha] := \mathsf{E}[ET[X;\alpha]] = \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]}.$$

Expected Esscher transform with parameter $\alpha \geq 0$ is often used as a method of pricing of insurance contracts (in this case variable X indicates size of insurance compensation related to some insurance pact) and is called in actuarial literature Esscher premium. Application of Esscher transform to pricing of insurance contracts probably was initiated by Swiss mathematician Hans Bühlmann in the paper [1].

Review of known results associated with Esscher transform can be found, for example, in the work [4].

We will also need the following two definitions:
essential supremum of random variable X with distribution function $F(x)$

$$\operatorname{ess\,sup}[X] := \sup\{\delta\colon F(\delta) < 1\};$$

essential infimum of random variable X with distribution function $F(x)$

$$\operatorname{ess\,inf}[X] := \inf\{\delta\colon F(\delta) > 0\}.$$

Limit behavior of expected Esscher transform is following.

Theorem 1. *For any random variable X holds limit relation*

$$\lim_{\alpha \to +\infty} EET[X;\alpha] = \operatorname{ess\,sup}[X].$$

Theorem 2. *For any random variable X holds limit relation*

$$\lim_{\alpha \to -\infty} EET[X;\alpha] = \operatorname{ess\,inf}[X].$$

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INFINITE ARITHMETIC SUM OF FINITE SUBSETS OF THE COMPLEX PLANE

VALERIY KOVALENKO

Recall that the arithmetic sum of number sets A and B (symbolically $A \oplus B$) is the set consisting of elements in the form $c = a + b$, where $a \in A$, $b \in B$, i.e., $A \oplus B = \{a + b \colon a \in A, b \in B\}$.

Definition 1. Infinite arithmetic sum of an infinite sequence of sets A_1, A_2, …, A_k, … (symbolically $\bigoplus\limits_{k=1}^{\infty} A_k$) of complex plane $\mathbb{C}$ is a limit of a sequence of arithmetic sums $S_k = \bigoplus\limits_{m=1}^{k} A_m$ as $k \to \infty$ (if it exists), namely $\bigoplus\limits_{k=1}^{\infty} A_k := \lim\limits_{k\to\infty} \bigoplus\limits_{m=1}^{k} A_m$.

Definition 2. A set S is called a Σ-set if $S = \bigoplus\limits_{k=1}^{\infty} Z_k$, where Z_k ($k \in \mathbb{N}$) are finite subsets of $\mathbb{C}$ satisfying the following conditions:

(1) $\sum\limits_{k=1}^{\infty} \max\{|z| \colon z \in Z_k\} = M < \infty$;
(2) any set Z_k ($k = 1, 2, \dots$) contains at least two elements.

Uniform self-similar fractals and their arithmetic sums, spectra of distributions of some random variables of Jessen–Wintner type (including infinite Bernoulli convolutions), sets of incomplete sums of absolutely convergent series with real or complex terms are examples of such sets.

In order to create a unified approach to study the properties of the specified sets we have introduced the notion of Σ-set.

In the talk we consider topological and metric properties of Σ-sets. In particular, the following propositions are proved.

Lemma 1. *Any Σ-set is continuum, compact and perfect set.*

Let $d(Z)$ be a diameter of the set Z.

Theorem 1. *If for any positive integer k the following inequality holds:*

$$l(Z_k) := \min_{\substack{u,v\in Z_k \\ u\neq v}} |u - v| > \frac{2}{\sqrt{3}} \sum_{m=k+1}^{\infty} d(Z_m),$$

then Σ-set $S = \bigoplus_{k=1}^{\infty} Z_k$ is totally disconnected.

Theorem 2. *If for Σ-set $S = \bigoplus_{k=1}^{\infty} Z_k$ the following equality holds:*

$$\lim_{k\to\infty} \prod_{j=1}^{k} m_j \Bigl(\sum_{n=k+1}^{\infty} d(Z_n) \Bigr)^2 = 0,$$

where m_j is a number of elements of the set Z_j $(j = 1, 2, \dots)$, then the set S is of zero two-dimensional Lebesgue measure.

Consider the partition of the set of positive integers $\mathbb{N} = A_1 \cup A_2$, where $A_1 = \{a_{11}, a_{12}, \dots, a_{1k}, \dots\}$, $A_2 = \{a_{21}, a_{22}, \dots, a_{2k}, \dots\}$.

Theorem 3. *If the sets $Z_{a_{ik}}$ satisfy conditions:*

(1) *$Z_{a_{ik}} \subset l_i$, where l_i $(i = 1, 2)$ is a line in the complex plane;*
(2) *for any positive integer k,*

$$d(Z_{a_{ik}}) \leq \sum_{p=k+1}^{\infty} d(Z_{a_{ip}}) \qquad (i = 1, 2);$$

(3) *straight lines l_1 and l_2 are not parallel and not coinciding,*

then set S has interior points and therefore is of positive two-dimensional Lebesgue measure.

We also give examples of application of these results to the study of topological and metric properties of arithmetic sums of certain classes of self-similar sets in the complex plane.

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BOUNDARY VERSIONS OF THE WORPITZKY THEOREM AND OF PARABOLA THEOREMS

KHRYSTYNA KUCHMINSKA

What happens to the limit regions in the Worpitzky theorem and parabola theorems when the element regions are replaced by their boundaries? Haakon Waadeland answered on these questions by his theorems [1].

Theorem 1. *Let ρ be a fixed positive number, $0 < \rho \leq 1/2$, and let F_ρ be the family of continued fractions*

$$\mathop{\mathrm{D}}_{n=1}^{\infty} \frac{a_n}{1}, \tag{1}$$

defined by the condition

$$|a_n| = \rho(1-\rho) \tag{2}$$

for all n. Then the set of all possible values f of continued fractions in F_ρ is the annulus A_ρ, given by

$$\rho\frac{1-\rho}{1+\rho} \leq |f| \leq \rho.$$

Theorem 2. *Let α be a fixed real number, $-\pi/2 < \alpha < \pi/2$, and let P_α be the family of continued fractions*

$$\mathop{\mathrm{D}}_{n=1}^{\infty} \frac{a_n}{1}, \tag{3}$$

defined by the condition

$$|a_n| - \mathrm{Re}\left(a_n e^{-i\alpha}\right) = \frac{1}{2}\cos\alpha \tag{4}$$

for all n. Then the set of all possible values f of all continued fractions in P_α is the halfplane V_α, given by

$$\mathrm{Re}\left(f e^{-i\alpha}\right) \geq -\frac{1}{2}\cos\alpha,$$

minus the origin.

For a two-dimensional continued fraction

$$\underset{i=0}{\overset{\infty}{\mathrm{D}}} \frac{a_{i,i}}{\Phi_i}, \quad \Phi_i = 1 + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i+j,i}}{1} + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i,i+j}}{1},$$

a Worpitzky-like theorem and parabola theorem are also proved [2], and the same questions for two-dimensional continued fractions are answered by obtained results.

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RATIONAL Q_2-REPRESENTATION AND ITS APPLICATIONS

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Researchers who study and modeling objects with complicated local structure having everywhere dense set of singularities (fractal sets, continuous nowhere differentiable functions, singular functions and probability measures, probability distributions on the fractal sets, complex systems with chaotic dynamics etc.) are often need for new systems of representation of real numbers.

Let $0 < q_0 < 1$, $q_1 = 1 - q_0$, $\beta_0 = 0$, $\beta_1 = q_0$. Then for any $x \in [0,1)$ there exists a sequence (a_n), $a_n \in A = \{0,1\}$, such that

$$x = \beta_{a_1} + \sum_{k=2}^{\infty}\left(\beta_{a_k}\prod_{j=1}^{k-1} q_{a_j}\right) = \Delta^{Q_2}_{a_1a_2\ldots a_n\ldots}. \tag{1}$$

Expression (1) is called a Q_2-*expansion* of the number x and its formal notation $x = \Delta^{Q_2}_{a_1a_2\ldots a_n\ldots}$ is called a Q_2-*representation* of the number x.

Note that Q_2-expansion is a generalization of the classic binary expansion

$$x = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_k}{2^k} + \ldots \equiv \Delta^2_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$$

and coincides with it when $q_0 = 1/2$.

A Q_2-representation is an encoding system with a two-letter alphabet $A = \{0,1\}$ having zero redundancy (this means that almost all numbers have a unique representation, and only a countable subset of the set of rational numbers has two representations).

The number is called Q_2-*rational* if there exists its Q_2-representation including period (0). The rest of numbers are called Q_2-*irrational.*

If q_0 is a rational number, then we say that a Q_2-representation is a *rational* Q_2-*representation* of real numbers.

Lemma 1. *If rational Q_2-representation of the number x is periodic, then the number x is rational.*

From Lemma 1 it follows that the rational Q_2-representation of the irrational number is non-periodic.

Theorem 1. *The number* $\dfrac{1}{s-p}$ *has a non-periodic* Q_2*-representation for* $q_0 = \dfrac{p}{s}$.

Theorem 2. *The set of rational numbers having non-periodic* Q_2*-representations is everywhere dense set in* $[0, 1]$.

Theorem 3. *Function*

$$F_i(x) = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left(\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)} \right), \tag{2}$$

where $\alpha_k(x)$ *is a kth binary digit of the number* x*,* $\beta_0 = 0$*,* $\beta_1 = p_0$*,* $0 < p_0 < 1$*,* $p_1 = 1 - p_0 \neq p_0$*, is strictly increasing singular function mapping every rational point into rational and only a "small" number of irrational points into rational.*

This function transforms a binary representation of argument into Q_2-representation of the value of function.

Function (2) is the Salem function if p_0 is the rational number.

Let $(c_1, c_2, \ldots, c_m)$ is a fixed ordered set of numbers from $\{0, 1\}$. *Cylinder of rank* m with a base $(c_1, c_2, \ldots, c_m)$ is the set $\Delta^{Q_2}_{c_1 c_2 \ldots c_m}$ of all numbers $x \in [0, 1]$ having the following Q_2-representation:

$$\Delta^{Q_2}_{c_1 c_2 \ldots c_m a_{m+1} a_{m+2} \ldots a_{m+k} \ldots}$$

For calculating the Hausdorff–Besicovitch dimension of the subsets of the interval $[0, 1]$ one can use their covering by cylinders of Q_2-representation, which leads to an equivalent definition of dimension.

Theorem 4. *The value of nondegenerate distribution function of the random variable*

$$\xi = \Delta^2_{\eta_1 \eta_2 \ldots \eta_k \ldots}$$

with independent identically distributed binary digits η_k $(\mathsf{P}\{\eta_k = 0\} = p_0$, $\mathsf{P}\{\eta_k = 1\} = 1 - p_0 = p_1)$ *is a* Q_2*-representation with* $q_0 = p_0$*, namely:*

$$F_\xi(x) = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left(\beta_{\alpha_k}(x) \prod_{j=1}^{k-1} q_{\alpha_j}(x) \right),$$

where $x = \Delta^2_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}$.

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ON TWO FUNCTIONS WITH COMPLICATED LOCAL STRUCTURE

SYMON SERBENYUK

Let $s>1$ be a positive integer, and let $A \equiv \{0,1,2,\dots,s-1\}$ be an alphabet of s-adic numeral system. Let us consider a function

$$x=\Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f} \Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}=f(x)=y. \tag{1}$$

Remark 1. The function f is not well defined on the set of s-adic rational numbers from $[0;1]$.

Let we have the functions $\Phi_1=\Phi_1(x)$ and $\Phi_2=\Phi_2(x)$:

$$x=\Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{\Phi_1} \Delta^{s}_{\alpha_1 0\alpha_3 0\ldots 0\alpha_{2n-1}0\alpha_{2n+1}0\ldots}=\Phi_1(x),$$
$$x=\Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{\Phi_2} \Delta^{s}_{0\alpha_2 0\alpha_4\ldots 0\alpha_{2n}0\alpha_{2n+2}0\ldots}=\Phi_2(x).$$

Lemma 1. *The function f has the following properties:*

(1)
$$[0;1] \xrightarrow{f} \left[-\frac{s}{s+1};\frac{1}{s+1}\right]\setminus T,$$
where $T=\left\{y\colon y=\Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n-1)(s-1)(s-1)(s-1)\ldots}\right\}$, $\alpha_n\neq 0$, $y\neq \Delta^{-s}_{(s-1)(s-1)\ldots}$;

(2) *for each* $x\in[0;1]$,
$$f(x)=\Phi_2(x)-\Phi_1(x)=x-2\Phi_1(x)=2\Phi_2(x)-x,$$
and $x=\Phi_1(x)+\Phi_2(x)$;

(3) *for each* $x\in[0;1]$,
$$f(x)+f(1-x)=-\frac{s-1}{s+1};$$

(4) *the function f is not monotonic on* $[0;1]$;

(5) *the function f is not a bijective mapping on* $[0;1]$;

(6) *the set of invariant points of the function f is a self-similar fractal and its Hausdorff–Besicovich dimension is equal to* $\frac{1}{2}$;

(7) *the function f is continuous in s-adic irrational points and s-adic rational points are points of discontinuity of the function;*

(8) *the functions f and g are nowhere differentiable;*

(9) *the Hausdorff–Besicovich dimension of graphs of* $f(x)$ *and* $g(x)$ *is equal to* 1*;*

(10)
$$\int\limits_{[0;1]} f(x)dx = \int\limits_{\left[-\frac{s}{s+1};\frac{1}{s+1}\right]} g(x)dx = \frac{1}{2},$$

where
$$g\colon x = \Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{g} \Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = g(x).$$

Remark 2. The function g is not well defined on the set of nega-s-adic rational numbers from $[0;1]$.

Let we have the functions f^s_k ($k, s > 1$ are fixed positive integers):
$$x = \Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f^s_k} \Delta^{s}_{\beta_1\beta_2\ldots\beta_n\ldots} = f^s_k(x) = y^s_k,$$
where $\left(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}\right) = \theta\left(\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{(m+1)k}\right)$, k is a fixed positive integer, $m = 0, 1, 2, \ldots$, and $\theta(\gamma_1, \gamma_2, \ldots, \gamma_k)$ is a certain function of k variables (it is biunique correspondence), that the set
$$A^k = \underbrace{A \times A \times \ldots \times A}_{k}.$$
is domain and range of θ.

It is obvious that
$$f_k = f \circ f^s_k,$$
where
$$x = \Delta^{s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f_k} \Delta^{-s}_{\beta_1\beta_2\ldots\beta_n\ldots} = f_k(x),$$
$$\left(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}\right) = \theta\left(\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{(m+1)k}\right).$$

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TERNARY-QUINARY RATIONAL NUMBERS WITH A NEGATIVE BASE

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Let $A_s = \{0, 1, 2, \dots, s-1\}$ be the alphabet of s-adic number system.

Lemma 1. *For any number $x \in (0;2)$ there exists a sequence (α_k), $\alpha_k \in A_5$, such that*

$$x = \frac{3}{2} + \sum_{k=1}^{\infty} \frac{\alpha_k}{(-3)^k} = \frac{3}{2} - \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} - \frac{\alpha_3}{3^3} + \frac{\alpha_4}{3^4} - \dots \qquad (1)$$
$$\equiv \Delta_{\alpha_1\alpha_2\dots\alpha_k\dots}.$$

For example, $1 = \Delta_{1(04)}$, $\frac{1}{3} = \Delta_{3(04)}$, $2 = \Delta_{(04)}$.

A representation of the number x in the form (1) is called a *negative-ternary expansion* with two extra digits 3 and 4 or *negative-ternary-quinary expansion*, or *ternary-quinary expansion with a negative base*.

Brief (symbolic) notation $x \equiv \Delta_{\alpha_1\alpha_2\dots\alpha_k\dots}$ of equality (1) is called a *negative-ternary-quinary representation* or *ternary-quinary representation with a negative base*, or just 5_{-3}*-representation* of the number x.

Definition 1. The number $x \in (0;2)$ is called a 5_{-3}*-rational* if there exists 5_{-3}-representation containing period (40), all other numbers are called 5_{-3}*-irrational*.

Theorem 1. *The number $x \in (0,2)$ is rational if and only if its 5_{-3}-representation is periodic.*

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NOWHERE DIFFERENTIABLE TAKAGI FUNCTION IN PROBLEMS AND APPLICATIONS

NATALYA VASYLENKO

One of the first and most famous example of a continuous nowhere differentiable function was constructed by T. Takagi [1]:

$$T(x) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi_0(2^n x), \tag{1}$$

where $\varphi_0(x)$ is the distance between the point x and the nearest integer point, i.e.,

$$\varphi_0(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \tag{2}$$

moreover, $\varphi_0(x+1) = \varphi_0(x)$ for all real numbers x, i.e., $\varphi_0(x)$ is a periodic function with period 1.

Let $A_2 = \{0, 1\}$ be an alphabet and

$$[0,1] \ni x = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_n}{2^n} + \ldots \equiv \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \quad \alpha_n \in A_2. \tag{3}$$

Lemma 1. *For functions $\varphi_n(x)$ $(n \in \mathbb{N}_0)$ the following equalities hold:*

$$\varphi_0(x) = \begin{cases} \Delta^2_{0\alpha_2\alpha_3\ldots\alpha_n\ldots} & \textit{if } \alpha_1 = 0, \\ \Delta^2_{0[1-\alpha_2][1-\alpha_3]\ldots[1-\alpha_n]\ldots} & \textit{if } \alpha_1 \neq 0, \end{cases}$$

$$\varphi_n(x) = \begin{cases} 2\varphi_{n-1}(x) & \textit{if } \alpha_n = \alpha_{n-1}, \\ 1-2\varphi_{n-1}(x) & \textit{if } \alpha_n \neq \alpha_{n-1}. \end{cases}$$

Lemma 2. *The Takagi function $T(x)$ can be rewritten in the following equivalent form*

$$T(x) = \frac{\beta_1}{2} + \frac{\beta_2}{2^2} + \ldots + \frac{\beta_k}{2^k} + \ldots, \quad \beta_k \in \mathbb{N}_0,$$

where

$$\beta_1 = 0, \quad \beta_n = \begin{cases} N_1(x,k) & \textit{if } \alpha_n = 0, \\ N_0(x,k) & \textit{if } \alpha_n = 1, \end{cases} \tag{4}$$

or in other words

$$\beta_1 = 0, \quad \beta_n = \begin{cases} \beta_{n-1} & \text{if } \alpha_n = \alpha_{n-1}, \\ n-1-\beta_{n-1} & \text{if } \alpha_n \neq \alpha_{n-1}. \end{cases} \tag{5}$$

Lemma 3. *Function T has the following properties:*

(1) *it is continuous on $[0,1]$ and takes all values from $[0, \frac{2}{3}]$;*
(2) *it is non-monotonic;*
(3) *it has neither finite nor infinite derivative at binary rational points;*
(4) *it does not have a finite derivative at irrational points;*
(5) *it is the solution of the system of two functional equations*

$$\begin{cases} T(\Delta^2_{0\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^2_{0\alpha_1\alpha_2\ldots\alpha_k\ldots} + \dfrac{1}{2}T(\Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) - \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \\ T(\Delta^2_{1\alpha_1\alpha_2\ldots\alpha_n\ldots}) = (1 - \Delta^2_{\alpha_1\alpha_2\ldots\alpha_k\ldots}) + \dfrac{1}{2}T(\Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) \end{cases}$$

in the class of continuous on $[0,1]$ functions.

In the talk we study arithmetic, differential, integral and fractal properties of the Takagi function and solve the following problems:

(1) we determine asymptotic value $T(\epsilon)$ for small values of ϵ;
(2) we apply combinatorial analysis and graph theory to find the set of solutions of the equations $T(x) = \frac{1}{2}$ and $T(x) = \max\limits_{0 \le x \le 1} T(x)$.

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INVARIANTS OF THE DIRICHLET/VORONOI TILINGS OF HYPERSPHERES IN $\mathbb{R}^n$ AND THEIR DUAL DELONE GRAPHS

FRANÇOIS ANTON

In this paper, we are addressing the geometric and topological invariants that arise in the exact computation of the Delone graph and the Dirichlet/Voronoi tiling of n-dimensional hyperspheres using Ritt–Wu's algorithm. Our main contribution is first a methodology for automated derivation of geometric and topological invariants of the Dirichlet tiling of $n+1$-dimensional hyperspheres and its dual Delone graph from the invariants of the Dirichlet tiling of n-dimensional hyperspheres and its dual Delone graph (starting from $n = 3$). The 3D case was treated in Anton et al. [1]. The reader is invited to refer to that paper for an explanation of the previous work and the 3D case.

Starting from the system of equations defining the zero-dimensional algebraic set of the problem, we are applying Wu's algorithm to transform the initial system into an equivalent Wu characteristic (triangular) set. In the corresponding system of algebraic equations, in each polynomial (except the first one), the variable with higher order from the preceding polynomial has been eliminated (by pseudo-remainder computations) and the last polynomial we obtain is a polynomial of a single variable. By regrouping all the formal coefficients for each monomial in each polynomial, we get polynomials that are invariants for the given problem. We rewrite the original system by replacing the invariant polynomials by new formal coefficients. We repeat the process until all the algebraic relationships (syzygies) between the invariants have been found by applying Wu's algorithm on the invariants.

Starting from $n = 3$ (3D), we need to change the following polynomial set [1]:

$$\begin{cases} x^2 + y^2 + z^2 - (v)^2\,, \\ -2a'x - 2b'y - 2c'z - 2s'v + (a'^2 + b'^2 + c'^2 - s'^2), \\ -2d'x - 2e'y - 2f'z - 2t'v + (d'^2 + e'^2 + f'^2 - t'^2), \\ -2g'x - 2h'y - 2i'z - 2u'v + (g'^2 + h'^2 + i'^2 - u'^2). \end{cases}$$

This work was done while the author was Visiting Full Professor at the 3D GIS Research Group, Faculty of Geoinformation and Real Estate, Universiti Teknologi Malaysia, for which he is very thankful.

However, after addition of one variable for the $n+1$ dimension and rewriting of each polynomial with this new variable and addition of the corresponding polynomial for the radical plane with the first hypersphere into the system, the triangular system obtained using Wu's algorithm is still formed by a quadratic polynomial in the offset parameter v and linear polynomials in all the variables of the system:

$$\begin{cases} C_1 = Jv^2 + Kv + L, \\ C_2 = Ax + Hv + I, \\ C_3 = -Ay + Ev + F, \\ C_4 = Az + Bv + C, \\ C_5 = Aw + Nv + O, \end{cases}$$

where w is the variable for the fourth dimension.

The difficult invariants present in C_1 are expressed as algebraic polynomials in the simple invariants in C_2, C_3, C_4 and C_5:

$$\begin{cases} J = B^2 + E^2 + H^2 - A^2 + N^2, \\ K = 2BC + 2EF + 2HI + 2NO, \\ L = C^2 + F^2 + I^2 + O^2. \end{cases}$$

We find again the same invariants of isometries that correspond to scalar products of some vectors formed by invariants and powers of points formed by invariants with respect to one of the spheres.

Finally, we can conclude that the same methodology that allowed us to get the invariants of the Voronoi diagram of hyperspheres could be applied on ellipses starting from 2D in order to get the invariants of the Voronoi diagram of ellipsoids.

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THREE-DIMENSIONAL STRAIGHT SKELETONS FROM BISECTOR GRAPHS

FRANZ AURENHAMMER AND GERNOT WALZL

The *straight skeleton* of a polygon P in the plane is a versatile skeletal structure, composed of angular bisectors of P and thus of piecewise linear components [1]. A well-known procedural definition exists, by a parallel offsetting (shrinking) process for P which yields the *mitered offset* of P, in contradistinction to the Minkowski sum offset specified by the medial axis of P. Applications arise in diverse areas, including computer graphics, robotics, architecture, and geographical information systems.

To construct a straight skeleton for a given polytope $\mathcal{Q}$ in 3-space, one tries to proceed in a way analogous to the planar case. During the offsetting process, the edges and vertices of $\mathcal{Q}$ trace out the facets and edges, respectively, of the spatial skeleton. Unlike polygon offsets, however, parallel offsets of polytopes in $\mathbb{R}^3$ are in general not unique.

Barequet et al. [3] and Martinez et al. [4] studied straight skeletons in 3-space, mainly for the special case of orthogonal (i.e., axes-aligned) polytopes, where the skeleton is the medial axis of the polytope for the L_∞-metric (and thus can be defined via distances). Also, a nice offsetting possibility by means of the planar weighted straight skeleton [2] is observed, but the offset treatment of general vertices remained unclear. In fact, it remains open whether a boundary-continuous and non-selfintersecting offset polytope always has to exist.

In this paper, we tackle the offsetting process by means of generalized *vertex figures* of a given polytope $\mathcal{Q}$ in $\mathbb{R}^3$ and their spherical *bisector graphs*. These graphs result from certain arrangements of great circles on the sphere, and they project to different local offset surfaces at the polytope vertices.

The main problem is to resolve vertices of degree higher than three, which either are already part of the input $\mathcal{Q}$ or arise during shrinking $\mathcal{Q}$. In the generic case, such a vertex splits into several vertices of degree three in the offset surface, which are connected by an unrooted binary tree. Our straight skeleton construction algorithm produces all such trees that form a valid offset surface.

Supported by ESF Programme EuroGIGA–Voronoi, Austrian Science Foundation.

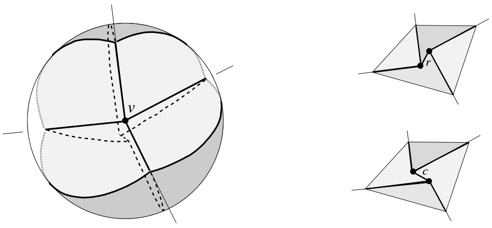

FIGURE 1. Two different offsets for saddle point v. The spherical bisector graph corresponds to the lower solution.

Each valid choice leads to a different way of handling the corresponding offset event, and eventually to a different straight skeleton for the polytope $\mathcal{Q}$. Certain optimality criteria – for example, concerning edge convexity or polytope volume – can be applied to guide the choice of offsetting. As a byproduct, our algorithm provides a means for converting boundary-triangulated (i.e., *simplicial*) nonconvex polytopes in $\mathbb{R}^3$ into 3-regular (i.e., *simple*) ones via ε-thinning.

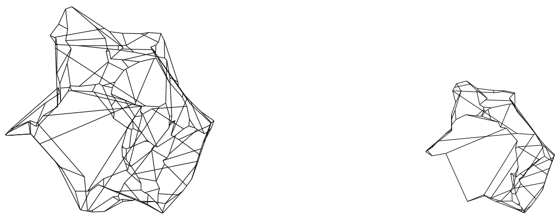

FIGURE 2. Shrinking a polytope with our offsetting algorithm.

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LOCAL RULES AS THE MECHANISM OF GLOBAL ORDER FORMATION IN CRYSTALS

MIKHAIL BOUNIAEV AND NIKOLAI DOLBILIN

One of the most fundamental problems of crystallography is the problem of crystal formation. The Russian crystallographer N. Belov named this problem the "501st element problem".

Since the crystallization is a process which results from mutual interaction of just nearby atoms, it was believed (L. Pauling, R. Feynman et al.) that the long-range order of atomic structures of crystals (and quasi-crystals too) comes out local rules restricting the arrangement of nearby atoms. However, before 1970's there were no whatever rigorous statements and results until B. Delone (Delaunay) and R. Galiulin initiated a problem and his students N. Dolbilin and M. Shtogrin developed a nice and mathematically rigorous *local theory of crystals*.

The motivation and a sketch of the local theory are as follows. An appropriate concept for describing an arbitrary discrete atomic structure is the *Delone set* (or an (r, R)*-system*). Structures with long-range order such as crystals involves a concept of the *space group* as well. A mathematical model of an *ideal monocrystalline matter* is defined now as a *Delone set which is invariant with respect to some space group.*

One should emphasize that under this definition the commonly-known periodicity of crystal in all 3 dimensions is not an additional requirement. By the celebrated Schoenflies–Bieberbach theorem, *any space group contains a translational subgroup with a finite index.* Thus, a mathematical model of an ideal crystal uses two concepts: a Delone set (which is *of local character*) and a space group (which is *of global character*).

The main aim of this theory was *rigorous derivation* of space group symmetry of a crystalline structure from the pair-wise identity of local arrangements around each atoms. The core of the local theory is rigorously proved statement (Dolbilin, Shtogrin) that local identity within certain radius R implies the global regularity of the structure. In frames of the

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theory, in some cases, the magnitude of R has been established precisely. However, in many cases, regardless a serious progress, a rigorously so far obtained upper bound for R remains overestimated.

At present, the goal of the local theory is seen twofold. On one, "mathematical", hand, one should improve upper bounds for R and, thus, try to draw a line between the local conditions of the crystals and quasicrystals.

Another goal is to fill the gap between mathematical local theory and empirical concepts of self-assembly that occurs during formation of the crystal/quasicrystal at the nanoscale. This goal obviously will require the close cooperation of mathematicians and specialists in crystallography and structural chemistry.

In the talk it is supposed to discuss some points of the local theory and challenges ahead.

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APPLICATION OF VORONOI DIAGRAMS FOR CONSTRUCTING GRAPHS OF EXHIBITIONS OVERLOOK COMFORT

VALERIY DOOBKO AND ELENA TSOMKO

Exhibition is a technical object formed with the purpose to transmit certain information and conception within the limited time. However, the comfort of receiving this information is also very important.

Let us consider an approach which allows us to draw analytical methods for constructing graphs of improved exhibition overlook comfort.

It is known that the most stressful condition occurs while solving the problem of choice (problem about Buridan's donkey). If we consider the exhibits' placement, then, the problem of choice decreases for the edges of the movement graphs, where equidistance to the surfaces of the overlooked objects is disordered. Equidistance lines should be considered as pathogenic ones. Therefore, it is necessary to shift lines of movements towards dominant object which is determined and coordinated with plan and graph of the overlook. In this interpretation, the problem is reduced to constructing the lines, in general, the bands of the most probable discomfort. For this purpose we suggest to use algorithms for constructing a graph based on Voronoi polygons.

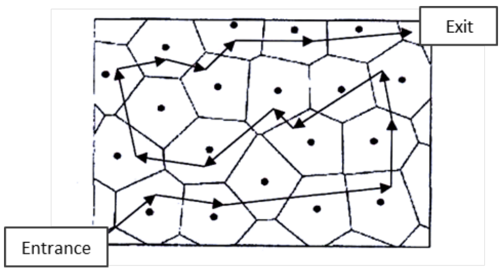

FIGURE 1

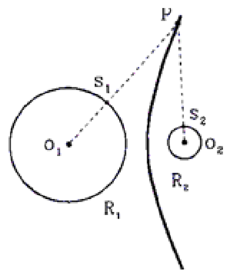

FIGURE 2

Consider given finite set of scattered points on a limited surface. These points are further interpreted as centers of Voronoi cells.

Edges of Voronoi cells bound a set of the points, any of which is closer to the selected center. Example of such constructed cells is shown in Figure 1.

This is the basic network for further constructing the comfortable movement graph not coinciding with the network. Any deviation from the edges of Voronoi polygon brings the observer to one of the centers. Therefore, we can avoid the problem of equally likely choice between the centers.

There are many algorithms for constructing Voronoi cells and polygons. In our problem we should take into account the fact that the stands are not point-like objects.

As it was mentioned before, the network of two-dimensional cells' edges is used as a "navigation map" of undesirable movement in the inter-element space. Note that using this mapping we can also explore other problems: symmetric and asymmetric movement during observation, detecting available space for a particular visitor (test particle) in the "thick" stand systems, modeling visitor's movement past a set of stands.

We present some general geometric approaches for the structural description of the empty inter-element space in the exhibition hall, namely, generalized two-dimensional Voronoi cells.

Consider the basic geometric construction which helps to understand the properties of the generalized two-dimensional Voronoi cells.

Generalized Voronoi edge for two given spheres is the geometric hodograph of points equidistant to the surfaces of a given pair of spheres. It divides the space into two parts. All points of one half-space are closer to the surface of the first sphere, and all points of another half-space are closer to the surface of the second sphere. Figure 2 illustrates this. Generalized Voronoi edge is the surface of the second order, namely, a hyperboloid. Let us call it as Voronoi hyperboloid. In the particular case of similar spheres, Voronoi hyperboloid becomes a line and coincide with the Voronoi edge for the centers of the given pair of spheres.

In general case, Voronoi edge is an arched line. It can be closed or open arched line. It may also be a straight line, but only for equivalent spheres. Important thing is that the generalized Voronoi edges are defined unambiguously and without breaks. As in the previous case, the observation paths are suggested to be formed shifting from these lines.

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VORONOI DIAGRAM AS MODEL OF QUASICRYSTALS OF LOBACHEVSKIAN GEOMETRY

O. A. DYSHLIS, N. V. VAREKH, O. I. GERASIMOVA, AND M. V. TSIBANIOV

In this work we use the concept of crystalline set of Euclidean space. This concept can be applied to an arbitrary geometry [1], in particular, to Lobachevskian geometry L_2. In geometry L_2, it is possible to introduce the concept of Voronoi diagram. We establish the following results.

1. We can consider an ideal two-dimensional quasicrystal as a crystalline set of plane L_2.

2. Let K be a crystalline set of geometry L_2 which is a model of an ideal two-dimensional quasicrystal. Then the fundamental domain of its symmetry group $\mathrm{Sym}\,K$ is a triangle with two angles being equal to 0. The third angle is equal to $2\pi/n$, $n = 5$, 8, 10, 12.

3. Group $\mathrm{Sym}\,K$ is generated by the reflection space of L_2 in sides of triangle from item 2. Decomposition of this space is obtained by action of this group on some of such triangles.

4. Using Voronoi method of lifting of decomposition on a paraboloid [2] we obtain a Voronoi diagram of the upper sheet of a hyperboloid in the pseudo-Euclidean metric.

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GENERALIZATION OF THE NOTION OF VENKOV BELT FOR PARALLELOHEDRA

ROBERT ERDAHL

If a d-dimensional polytope P has the property that a selection of translates fit together facet-to-facet to tile space, then P is called a parallelohedron. Minkowski knew that a parallelohedron must be centrally symmetric, and have centrally symmetric facets, two properties that combine so that facets appear as parallel pairs related by a translation. Moreover, if G^{d-2} is a $(d-2)$-dimensional face of the facet F^{d-1}, then there is a parallel copy of G^{d-2} in F^{d-1} related to G^{d-2} by an inversion through the center of F^{d-1}. Therefore, the facets that contain two parallel copies of G^{d-2} related by inversion link together to form a belt around P that is labeled by G^{d-2}. In 1898 Minkowski knew that such belts must contain either four or six parallel copies of G^{d-2} and correspondingly four or six facets. In 1953 Venkov established that Minkowski's three conditions, namely, the central symmetry of both P and its facets, and all belt lengths being either four or six, are sufficient to characterize parallelohedra.

In this talk I will show how the notion of Venkov belt admits a broad generalization that gives a new way to read information off the parallelohedron. These new belts are labeled by commensurate lattice triangles, namely lattice triangles that have translates that can be inscribed vertex-to-vertex in P. As an example of how this information might be used I will show how the 3-cells of the dual tiling can be determined.

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AN ORBIFOLD OF VORONOI PARALLELOTOPES

ANDREY GAVRILYUK

Consider an arbitrary parallelotope P. Its *affine class* $\mathcal{A}(P)$ is a set of all parallelotopes affinely equivalent to P. A hundred year old problem is whether $\mathcal{A}(P)$ contains any Voronoi parallelotopes i.e. Dirichlet cells for some lattice. The question is still open and is well-known as Voronoi conjecture [1]. We show that a subset of Voronoi parallelotopes in $\mathcal{A}(P)$ is an orbifold with dimension (as of a real manifold with singularities) depending just on a combinatorial type of P (see also [2, 3]). We prove the dimension equals to the number of connected components of so-called *Venkov subgraph* of the parallelotope [4]. Nevertheless structure of the orbifold depends on affine properties of the parallelotope, beside combinatorial ones. We provide exhaustive description of the structure [5].

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CRITICAL LATTICES IN METRICS OF FOUR-DIMENSIONAL REAL SPACE AND APPLICATIONS

NIKOLAJ GLAZUNOV

Voronoï in paper [1] have investigated extreme lattices and showed that a lattice is extreme if and only if it is both *perfect* and *eutactic*. The notion of critical lattice is a special case of extreme lattice. Given any set $\mathcal{D} \subset \mathbb{R}^n$, a lattice Λ is *admissible* for $\mathcal{D}$ (or is $\mathcal{D}$-*admissible*) if $\mathcal{D} \bigcap \Lambda = \varnothing$ or $\{0\}$. The infimum $\Delta(\mathcal{D})$ of the determinants (the determinant of a lattice Λ is written $d(\Lambda)$) of all lattices admissible for $\mathcal{D}$ is called the *critical determinant* of $\mathcal{D}$. A lattice Λ is *critical* for $\mathcal{D}$ if $d(\Lambda) = \Delta(\mathcal{D})$ (see [2]).

For $p > 1$, let $|x|^p + |y|^p < 1$ be the set of metrics in real plane and D_p be the corresponding domains in the metrics:

$$D_p = \{(x, y) \in \mathbb{R}^2 \mid |x|^p + |y|^p < 1\}.$$

Minkowski [3] raised a question about critical determinants and critical lattices of regions D_p for varying $p > 1$. In papers (see [4] and references their in) the Minkowski conjecture about the critical determinant of the region $|x|^p + |y|^p < 1$, $p > 1$, has investigated and proved. Important ingredient of these considerations is the *Minkowski–Cohn* (*MC*) *moduli space* [5].

In the communication we investigate admissible and critical lattices in metrics

$$|x|^p + |y|^p + |z|^p + |u|^p < 1, \quad p > 1 \tag{1}$$

of four-dimensional real space and construct the generalized Minkowski–Cohn moduli space for the set of mentioned metrics.

Next we consider the function $G(x, y, z.u) = (|xyzu|)^{1/4}$.

Lemma. *Functions of the forms* $F(x, y, z.u) = |x|^p + |y|^p + |z|^p + |u|^p$, $p > 1$ *and* $G(x, y, z.u) = (|xyzu|)^{1/4}$ *are beam functions.*

Follow to [2] we construct from beam functions corresponding star bodies. Let

$$(|xyzu|)^{1/4} < 1 \tag{2}$$

be the star body. We investigate critical lattices of the star body.

Corollary. *Star bodies* (1) *and* (2) *define metric functions on tangent bundles of real four-dimensional smooth differentiable manifolds.*

Lattices, partially admissible and critical lattices are discrete anisotropic spaces. We will discuss connection of star bodies with their critical lattices.

Numerical examples are included.

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A COMPUTATION OF A TYPE DOMAIN OF A PARALLELOTOPE

VIACHESLAV GRISHUKHIN

A *type* of a polytope P, in particular, of a parallelotope, is an isomorphism class of the partial ordered set of all faces of P.

There corresponds a Voronoi polytope $P_V(f)$ to each positive semidefinite quadratic form f. Voronoi defined in his famous paper [1] an L-type domain of a Voronoi polytope $P_V(f_0)$ as a set of quadratic forms f such that polytopes $P_V(f)$ have the same type as $P_V(f_0)$. Voronoi conjectured in [1] that each parallelotope is affinely equivalent to a Voronoi polytope. Since the Voronoi conjecture is not yet proved, it is useful to define a type domain of a parallelotope not using quadratic forms.

Call a face of codimension 1 by a *facet*. Each n-dimensional polytope P can be described by the following system of inequalities

$$P = P(\alpha) = \{x \in \mathbb{R}^n \colon \langle p, x\rangle \le \alpha(p),\, p \in \mathcal{P}\}. \tag{1}$$

Here $\langle p, x\rangle$ is scalar product of vectors $p, x \in \mathbb{R}^n$, $\mathcal{P} \subset \mathbb{R}^n$ is a set of vectors including all facet vectors such that if $p \in \mathcal{P}$, then $-p \in \mathcal{P}$. The function $\alpha \in \mathbb{R}^{\mathcal{P}}_{+}$ is symmetric and non-negative, i.e. $\alpha(-p) = \alpha(p) \ge 0$ for all $p \in \mathcal{P}$. Call the function α by *parameter*.

A *type domain* $\mathcal{D}(P)$ of a parallelotope P is a set of all parameters α such that parallelotopes $P(\alpha)$ have the same type as P for all $\alpha \in \mathcal{D}(P)$. The domain $\mathcal{D}(P)$ is determined by equalities and inequalities between values $\alpha(p)$ for distinct p.

Let $P = P(\alpha)$ be a parallelotope described by Eq. (1). Let $\mathcal{P}(P) \subseteq \mathcal{P}$ be a set of all facet vectors of P. Let $\mathcal{P}(G) \subseteq \mathcal{P}(P)$ be a set of facet vectors of all facet containing G. The following assertion describes some equalities between parameters $\alpha(p)$.

Proposition 1. *Let G be a k-dimensional face of a polytope P. Then*

(i) *if $t \in \mathcal{P} - \mathcal{P}(G)$ is such that the hyperplane*

$$H(\alpha, t) = \{x \in \mathbb{R}^n \colon \langle t, x\rangle = \alpha(t)\}$$

supports P at the face G, then

$$\alpha(t) = \sum_{p \in \mathcal{P}(G)} \mu_t(p)\alpha(p),$$

where $\mu_t(p) \geq 0$ *are coefficients of the decomposition* $t = \sum_{p \in \mathcal{P}(G)} \mu_t(p)p$ *of the vector* t *by the facet vectors* $p \in \mathcal{P}(G)$*;*

(ii) *if* $|\mathcal{P}(G)| > n - k$*, then*

$$\sum_{p \in \mathcal{P}(G)} \mu(p)\alpha(p) = 0,$$

where $\mu(p)$ *are coefficients of a linear dependence* $\sum_{p \in \mathcal{P}(G)} \mu(p)p = 0$ *between vectors* $p \in \mathcal{P}(G)$*.*

The following assertion describes linear inequalities between $\alpha(p)$ related to 6-belts of a parallelotope.

Proposition 2. *Let* (p_1, p_2, p_3) *be facet vectors of a* 6*-belt of a parallelotope* P *such that* $p_3 = \mu_1 p_1 + \mu_2 p_2$*, where* $\mu_1, \mu_2 > 0$*. Then this* 6*-belt determines the following inequalities between the three parameters* $\alpha(p_i)$*,* $i = 1, 2, 3$*,*

$$\mu_1 \alpha(p_1) + \mu_2 \alpha(p_2) \geq \alpha(p_3). \tag{2}$$

Proposition 1 gives nothing for a primitive parallelotope. If $P(\alpha)$ is a primitive parallelotope, then $\alpha(p) = \langle p, Dp \rangle$ for all $p \in \mathcal{P}$, where D is a positive definite matrix, and the type domain $\mathcal{D}(P)$ is determined by matrices D.

If $P = P(\alpha)$ is not primitive, then P has k-faces G such that $|\mathcal{P}(G)| > n - k$. In this case, the type domain $\mathcal{D}(P)$ is a face of the type domain of a primitive parallelotope.

Call a parallelotope P *rigid* if its type domain $\mathcal{D}(P)$ is one-dimensional. For a rigid parallelotope P, Proposition 1 allows to prove its rigidity, since a rigid parallelotope has sufficiently many k-faces G with $|\mathcal{P}(G)| > n - k$. In particular, one can show that the Voronoi polytopes $PV(D_4)$, $P_V(E_n)$, $P_V(E_n^*)$ for $n = 6, 7$, are rigid.

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EFFECTIVENESS-BASED SPATIAL SHARE-TITIONING: A NEW TOOL FOR COVERAGE OPTIMIZATION

K. R. GURUPRASAD

There are several problems of practical importance where a space is divided into cells and each cell is allotted to a facility such as in multi-robot coverage problems [1, 2] and a facility location problem [3]. In these problems, it is assumed that only one facility can serve a given region, leading to requirement of interiors cells being mutually disjoint. However in many applications it is possible that some of the regions in space are served by multiple facilities. In this article, we propose such a spatial division scheme to formulate and solve a spatial coverage problems. The ideas presented in here are based on and extension of generalized Voronoi partition presented in earlier work [4, 5].

Effectiveness-based share-titioning (EBS)

Let $Q \subset \mathbb{R}^d$ be a polytope; $\mathcal{P} = \{p_1, p_2, \dots, p_N\}$ be a finite set of nodes, with $p_i \in Q$; and $I_N = \{1, \dots, N\}$ be an index set. Consider a collection functions $\mathcal{F} = \{f_i \colon \mathcal{P} \times Q \mapsto \mathbb{R}, i \in I_N\}$, where f_i is called a *node function* for the i-th node. Define a collection $\mathcal{V}^E = \{V_i^E\}$, $i \in I_N$, such that $Q = \cup_{i \in I_N} V_i^E$, where V_i^E is defined as

$$V_i^E = \{q \in Q \mid f_i(p_i, q) \geq f_j(p_j, q) \; \forall j \neq i, \, j \in I_N\}. \tag{1}$$

For every $i, j \in I_N$, $i \neq j$, such that $V_i^E \cap V_j^E \neq \emptyset$, define the boundary between cells V_i^E and V_j^E as

$$\partial V_{ij}^E = \{q \mid f_i(p_i, q) = f_j(p_j, q)\} \tag{2}$$

If for each (i, j), $i \neq j$, $i, j \in I_N$, ∂V_{ij}^E is a set of measure zero, then the collection $\mathcal{V}^E$ reduces to a partition of Q. Observe that the standard Voronoi partition and its generalizations can be extracted from (1).

Coverage optimization

Consider N facilities required to cover a region Q in the sense that each $q \in Q$ should be served by at least one facility (not necessarily by a single facility). Let $p_i \in Q$ be the location of the i-th facility; $f_i(p_i, q)$ be its effectiveness at $q \in Q$; $\phi \colon Q \mapsto [0, 1]$ be a continuous density distribution function, indicating the importance of serving a point $q \in Q$; $\mathcal{W} = \{W_i \subset Q\}$, $i \in I_N$, be such that $Q = \cup_{i \in I_N} W_i$. Now consider an objective function which when maximized maximizes the weighted coverage of Q,

$$\mathcal{H}(\mathcal{P}, \mathcal{W}) = \sum_{i \in I_N} \mathcal{H}(\mathcal{P}, W_i) = \sum_i \int\limits_{W_i} f_i(q, p_i)\phi(q) dQ. \tag{3}$$

Theorem. *A necessary condition for maximizing $\mathcal{H}(\mathcal{P}, \mathcal{W})$ is that $\mathcal{W} = \mathcal{V}^E(\mathcal{P})$, with $f_i(p_i, q)$ as node functions.*

Proof. Let $\mathcal{W}^* \neq \mathcal{V}^E(\mathcal{P})$ be such that for a given $\mathcal{P}$, $\mathcal{W}^*$ maximizes the objective function (3). Consider any point $q \in V_i^E \cap W_j^*$. Now

$$f_i(p_i, q) \geq f_j(p_j, q) \tag{4}$$

by definition of $\mathcal{V}^E(\mathcal{P})$ in Eqn. (1). Since $\mathcal{W}^* \neq \mathcal{V}(\mathcal{P})$, and the equality is valid only when $q \in \partial V_{ij} \cap W_j^*$ which is a proper subset of W_j^*,

$$\mathcal{H}(\mathcal{P}, \mathcal{V}^E) > \mathcal{H}(\mathcal{P}, \mathcal{W}^*). \tag{5}$$

This contradicts our assumption that $\mathcal{W}^*$ maximizes the objective function (3) for a given $\mathcal{P}$. $\square$

Thus, the proposed effectiveness-based share-titioning scheme can be used for coverage optimization even when some of the regions are served or shared by more than one facility.

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BIOLOGICAL CELL MODELS BASED ON VORONOI TESSELLATION ARE ANTECEDENTS OF MODERN VERTEX CELL MODELS

HISAO HONDA

Genes is believed to determine shape of living organisms. To understand a determination route from genes to morphogenesis of multicellular organisms, geometrical cell models play an important role. Geometrical cell models describe processes of self-construction of a cell aggregate. I will review that the first useful cell model was based on Voronoi tessellation, and it has developed into vertex cell models. Then, I will show a new concept, successive self-construction of cell aggregates to understand morphogenesis of multicellular organisms.

Cells show a spherical shape in free space. When they are packed in restricted space, they become polyhedral (or polygonal in two dimensional space). These cell shapes can be described by Voronoi tessellations, where cells correspond to Voronoi centers one to one (Honda, 1978, 1983). Cell divisions are described by addition of Voronoi centers (Honda et al., 1984). Cell disappearance is by deletion of centers (Honda, 1978), cell locomotion (Honda et al., 1982) and regular arrangement (Honda et al., 1996) are by migration of centers.

Cells in an epithelium often show a polygonal pattern on epithelial surface. Along the polygonal edges, contractile actomyosin bundles are running. Then, boundaries of cells on the epithelial surface are contracting while neighboring cells adhere with each other. We have observed a process of cell pattern change (from Voronoi pattern to edge contracting pattern) during embryogenesis of the starfish and recorded quantitatively (Honda et al., 1983).

Recently we introduced differential equations to the vertex cell models and extended to three-dimensional system. 3D vertex dynamics have been applied to (a) formation of spherical cell aggregates (Honda et al., 2004), (b) formation of mammalian blastocysts (Honda et al., 2008a), (c) tissue elongation by cell intercalation (Honda et al., 2008b) and (d) epithelial invagination for tube formation.

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Actual morphogenesis during embryogenesis is complicated and seems to be too far to understand by self-construction. However, I would like to introduce a concept of successive self-construction, which is useful to understand complex morphogenesis by self-construction. Initially cell properties are determined by genes and cells with given properties make self-construction of cell aggregates. On the next stage, cell properties is modified by expression of other genes and cells with modified properties make self-construction further. These self-constructions continue successively and complicated morphogenesis takes place. For example of our studies, initially a cell aggregate becomes spherical (a), the aggregate has a cavity (b), the sphere including the cavity elongates (c), finally several invaginations appear at specific sites on epithelial surface (d). The cell aggregate becomes complicated structure during embryogenesis.

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QUASI-SIMPLICIAL COMPLEX: THE CORNERSTONE TO MOLECULAR GEOMETRY

DEOK-SOO KIM

Voronoi diagrams are everywhere in universe and are useful computational tools for many problems related with the spatial reasoning among particles which are otherwise difficult to solve. A particular class of Voronoi diagram, the Voronoi diagram of spheres $\mathcal{VD}$ [1], usually referred to the additively-weighted Voronoi diagram, has been proven useful for solving geometry problems defined on molecular structure which consists of spherical atoms of various radii in $\mathbb{R}^3$ [2]. It has been well-known that the Delaunay triangulation is dual to the ordinary Voronoi diagram of points and so is the regular triangulation to the power diagram. However, the dual of $\mathcal{VD}$ was only recently defined as the *quasi*-triangulation $\mathcal{QT}$ in $\mathbb{R}^3$ which was named because $\mathcal{QT}$ was *almost* simplicial with a very few cases, called anomalies, violating the conditions to be simplicial [3, 4, 6].

To understand the quasi-triangulation in a more general setting, we define a special class of non-simplicial complexes to which $\mathcal{QT}$ belongs [4, 6]. $\mathcal{C}$ is a quasi-simplicial complex if it satisfies the following two conditions: i) Any face of an element in $\mathcal{C}$ is also in $\mathcal{C}$ and ii) two elements in $\mathcal{C}$ intersect at *one or more* lower dimensional face, if it does. Quasi-simplicial complex possesses the following properties: Simplexes are connected; two simplexes can share more than one facet; the underlying spaces of two simplexes can intersect; a quasi-simplicial complex is decomposable to a set of simplicial complexes. Other properties are to be found.

This presentation will cover the definitions, properties, and algorithms for $\mathcal{VD}$, $\mathcal{QT}$, and the beta-complex $\mathcal{BC}$ which is a subset of $\mathcal{QT}$ satisfying a certain condition related with the size of a spherical probe [5, 9]. The issue of query efficiency on $\mathcal{QT}$, for example, is discussed in [7]. Then, this presentation will show how these three computational constructs can be used to solve potentially *all* molecular structure problems correctly, efficiently, and conveniently. Important examples include the computation of surfaces defined on molecules, the volume and the surface area

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of molecules [8], docking problem, side-chain prediction and the design of proteins, protein structure determination, the measure of molecular shape, etc. The presented algorithms are entirely implemented into the `BetaMol` program and is freely available at the Voronoi Diagram Research Center (VDRC, `http://voronoi.hanyang.ac.kr/`). For better understanding of $\mathcal{VD}$, $\mathcal{QT}$, and $\mathcal{BC}$, readers are encouraged to download and test the `BetaConcept` program from VDRC for those on the plane. The proposed theory shall be the cornerstone of the new discipline *Molecular Geometry*.

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PARALLELOHEDRA AND THE CONJECTURE BY G. VORONOI

ALEXANDER MAGAZINOV

The theory of parallelohedra began with E. S. Fedorov [4], who introduced the notion, in 1885. A *parallelohedron* is a convex polytope P that admits a face-to-face tiling of a Euclidean space by its translates. We will denote such tiling by $T(P)$.

In 1897 H. Minkowski [6] proved that every d-parallelohedron P is centrally symmetric, all facets of P are centrally symmetric, and projection of P along any its face of dimension $d-2$ is a parallelogram or a centrally symmetric hexagon. Later Venkov [8] proved that these three conditions are sufficient for a convex polytope to be a parallelohedron.

The projection property implies that every $(d-2)$-face of $T(P)$ is incident to 3 or 4 parallelohedra of $T(P)$. Such faces are sometimes called *triple* and *quadruple*. We will also apply these terms accordingly to the $(d-2)$-faces of P.

Conjecture 1 below has been stated by G. Voronoi [9]. Despite its long history, it has not been solved so far in full generality.

Conjecture 1 (Voronoi's conjecture)**.** *Every d-dimensional parallelohedron P is a Dirichlet–Voronoi domain for $\Lambda(P)$ with respect to some Euclidean metric in $\mathbb{E}^d$ (given by some positive quadratic form).*

The method of *generatrissa* allowed Voronoi to prove a special case of his conjecture.

Theorem 1. *Voronoi's conjecture holds for every* primitive *parallelohedron, i.e., a parallelohedron P* ($\dim P = d$) *such that every vertex of $T(P)$ is incident exactly to $d+1$ parallelohedra of $T(P)$.*

Later on, several improvements have been made. Among the contributors to the method of generatrissa we will mention O. Zhitomirskii [10] and A. Ordine [7]. We also mention the paper [2] by M. Deza and V. Grishukhin containing one of the modern viewpoints on this method.

Another known proofs of Voronoi's conjecture for certain classes of parallelohedra include the case $d \leq 4$ by B. Delaunay and the case of

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zonotopal parallelohedra by R. Erdahl [3]. P. McMullen [5] proved the zonotopal case with a unimodular system of zone vectors, however, appealing to Voronoi's generatrissa extends McMullen's case to all space tiling zonotopes.

In the talk we will discuss the method of generatrissa and, especially, the notion of *canonical scaling.* In this we will mainly follow [7]. Several new applications will be presented, including the two main results reported in the talk.

Theorem 2. *Delete all closed standard $(d-2)$-faces of P from ∂P. The resulting surface is a $(d-1)$-dimensional manifold δP, or the δ-surface of P. Suppose δP is simply connected. Then P is affinely equivalent to some Voronoi parallelohedron.*

Theorem 3. *Let I be a segment. Suppose that P and $P+I$ are parallelohedra and P is Voronoi in some Euclidean metric of $\mathbb{E}^d$. Then $P+I$ is Voronoi in some other Euclidean metric.*

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A DETAILED ANALYSIS OF THE SHORT RANGE ORDER IN LIQUID BINARY AND TERNARY ALLOYS USING VORONOI POLYHEDRA

OLEKSII MURATOV, OLEKSII YAKOVENKO, OLEKSANDR ROIK, VOLODYMYR KAZIMIROV, AND VOLODYMYR SOKOLSKII

During the past few decades a great effort has been devoted to the research of rapidly solidified alloys [1]. Nanocrystalline, quasicrystall and amorphous Al-based materials are interesting for their excellent mechanical, physical and magnetic properties. It should be noted that the structure and properties of the rapidly solidified materials depend on the short-range order (the SRO) of the liquid state from which they have been obtained. X-ray diffraction is presently the most widely used and powerful method to study the structure of metallic melts. However the structure factor (SF) or the radial distribution function (RDF), which obtained from X-ray scattering data, gives limited information about the SRO. Hence the study of the atomic structure of the liquid metallic alloys includes the following steps: (1) the diffraction experiment, (2) reconstruction of the 3D-models from the SF curves using the Reverse Monte Carlo (RMC) method [2], (3) analysis of the SRO in the models by means Voronoi and Delaunay tessellations.

The liquid binary and ternary Al-based alloys have been studied by X-ray diffraction technique and by RMC simulations at near liquidus temperatures. The study of the SRO in the models was carried out using Voronoi polyhedra [3]. The most informative characteristic appeared the sphericity coefficient ($K_{\mathrm{sph}} = 36\pi V^2/S^3$, where V is the volume, and S is the surface area of the VP). Since each VP is the geometrical image of the local environment of the given atom, the value of K_{sph} characterises the packing density and homogeneity of the local atomic environment. The high value of K_{sph} may be attributed to dense non-crystalline packing of atoms. Therefore, comparative analysis the most probable values $[K_{\mathrm{sph}}]$ and the standard deviation σ of the K_{sph}-distribution was carried out.

The interaction between the atoms of different types and atomic packing density are two factors that affect on the formation of the SRO of metallic melts. A difference in the electronegativity of such atoms causes strong interatomic interactions and the chemical short-range order (CSRO) can be realized. Simultaneously, the high atomic packing density of the metallic melts can be realized by the formation of dense non-crystalline atomic clusters. The change of the local atomic order in the liquid Al-based alloys, which is realized via the partial substitution of the one kind of TM atom by another TM atom, was analysed in detail by means the partial pair correlation function and the K_{sph}-distributions [4]. It was found that the presence of a covalent contribution to the Al–TM metallic bonding leads to the short R_1(Al–TM) distance and have a considerable effect on the formation of the partial local atomic order of the liquid alloys. For example, the VP parameters of the liquid $Al_{81.6}Ni_{14.9}Fe_{3.5}$, and $Al_{80}Co_{10}Ni_{10}$ alloys as well as the values R_1(Al–TM) are close to the ones in the corresponding liquid binary alloys, that point to the slight changes of the local atomic structure (in particular the density of atomic packing) at the transition from binary to corresponding ternary alloys. On the other hand, the partial substitution of Mn by Ni ($Al_{80}Mn_{20} \to Al_{80}Mn_{14.7}Ni_{5.3}$) leads to the significant changes in the SRO that correlates with the reduction of the R_1(Al–TM).

Also the same analysis of the SRO in the liquid binary Al–Fe, Fe–Si alloys and liquid ternary Al–Fe–Si alloys was carried out. The $[K_{\mathrm{sph}}]$ and σ values calculated for VPs, that were built around different types of atoms, are more similar than in case of the liquid Al–Fe–Si and Fe–Si alloys. The largest values of $[K_{\mathrm{sph}}]$ and smallest values of σ inherent polyhedra containing Fe. On the other hand, the VPs containing Si demonstrate the smallest values of $[K_{\mathrm{sph}}]$ and largest values of σ in comparison with other kind of VPs in structure model of liquid alloys. Addition of iron increases the atomic packing density of liquid alloys. On the contrary, the addition of silicon reduces the one.

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KALEIDOSCOPICAL CONFIGURATIONS

IGOR PROTASOV AND KSENIA PROTASOVA

Let X be a set, $\mathfrak{F}$ be a family of subsets of X. The pair $(X, \mathfrak{F})$ is called a *hypergraph*. Following [2], we say that a coloring $\chi\colon X \to \kappa$ (i.e. a mapping of X onto a cardinal κ) is *kaleidoscopical* if $\chi|_F$ is bijective for all $F \in \mathfrak{F}$. A hypergraph $(X, \mathfrak{F})$ is called *kaleidoscopical* if there exists a kaleidoscopical coloring $\chi\colon X \to \kappa$. The adjective "kaleidoscopical" appeared in definition [5] of an s-regular graph $\Gamma(V, E)$ (each vertex $v \in V$ has degree s) admitting a vertex $(s+1)$-coloring such that each unit ball $B(v, 1) = \{u \in V\colon d(u, v) = 1\}$ has the vertices of all colors (d is the path metric on V). These graphs define the kaleidoscopical hypergraphs $(V, \{B(v, 1)\colon v \in V\})$ and can be considered as the graph counterparts of the Hamming codes [3].

We survey some recent results and open problems on kaleidoscopical configurations in G-spaces.

Let G be a group. A *G-space* is a set X endowed with an action $G \times X \to X$, $(g, x) \mapsto gx$. All G-spaces are suppose to be *transitive*: for any $x, y \in X$, there exists $g \in G$ such that $gx = y$. For a subset $A \subseteq X$, we denote $G[A] = \{gA\colon g \in G\}$ where $gA = \{ga\colon a \in A\}$.

A subset $A \subseteq X$ is called a *kaleidoscopical configuration* if the hypergraph $(X, G[A])$ is kaleidoscopical, in words, if there exists a coloring $\chi\colon X \to |A|$ such that $\chi|_{gA}$ is bijective for every $g \in G$.

We discus a relationship between the kaleidoscopical configurations in a G-space X and transversals of the family $\{gA\colon g \in G\}$, $A \subseteq G$. We present also an effective method (namely, the splitting), of construction of kaleidoscopical configurations in a G-space X from the finite chains of G-invariant equivalence relations on X.

The main results are about kaleidoscopical configurations in $\mathbb{R}^n$ considered as a G-space with respect to the group $G = \mathrm{Iso}(\mathbb{R}^n)$ of all Euclidean isometries. For $n = 1$, it is easy to find a kaleidoscopical configuration in $\mathbb{R}$ of any size $\leq$ the cardinality of the continuum. The problem is much more difficult for $n \geq 2$. Surprisingly, the subsets $\mathbb{Z} \times \{0\}$, $\mathbb{Q} \times \{0\}$, $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Z} \times \mathbb{Z}$ are kaleidoscopical in $\mathbb{R}^2$. The most intriguing open problem: for $n \geq 2$, does there exist a finite kaleidoscopical configuration K, $|K| \geq 2$ in $\mathbb{R}^n$? We show that if such a K exists in $\mathbb{R}^2$ then $|K| \geq 5$.

Each group G can be considered as a (left) regular G-space $X = G$, where $(g, x) \longmapsto gx$ is the group product. We show that kaleidoscopical configurations in G are tightly connected with factorizations of $G = AB$ by subsets A, B. The factorizations were introduced by Hajós [1] to solve the famous Minkowski's problem on tiling of $\mathbb{R}^n$ by the copies of a cube. For the modern state of factorizations see [6, 7]. Also we establish a connection between kaleidoscopical configurations and T-sequences from [4].

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VORONOI TESSELATION AND MIGRATION WAY OF IONS IN CRYSTAL

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In this paper the program package TOPOS 4.0 Standard [1] of the automatic stereo-atomic crystal structure analysis was applied. The algorithm of detecting voids in the crystal structure is implemented. Possible migration paths of ions Ca, Mo and oxygen in the scheelite structure (space group C_{4h}^6 – $I4_1/a$) at room temperature (rt) and 1273 K were constructed.

To solve the problem was used the Voronoi tessellation. The atomic Voronoi polyhedra with unambiguous physical meaning [1] were formed. The map of voids and channels is consistent with experimental data. The basic concepts [1, 2] for description of the voids and channels in terms of Voronoi tessellation are the following: Voronoi polyhedron (VP), elementary void (channel), and closely related terms of form and radius of void, significant elementary void (channel). The VP of atom (geometric image atom), as suggested by G. Voronoi for ordered system of centers [3] is convex polyhedron, all points in space which are closer to this center than the other center of system.

Elementary crystal void is an area of crystal unit cell, the center of which is one of the vertex of VP. The major (ZA) and minority (ZC) elementary voids with sequence numbers N (ZAN and ZCN) are considered. They form characterized by the second moment of inertia (G) of VP. An atom can pass through the elementary channel if the sum of its radius (r_i) and the average radius of the atoms forming the channel (r_a), is less than the radius of the channel cross section (r_c). In consideration of a possible polarization (deformation) of ions when they passes through the channel was used the coefficient of deformation $\gamma_{ia} \leq 1$. An ion passes through the channel if $\gamma_{ia}(r_i + r_a) \leq r_c$.

The migration path is determined as the set of elementary voids and lines of elementary channels. It can be infinite along the 1D-channel, or 2D-, or 3D-channel-net. For superionics the endless ways of migration are interested. The conductivity map formed by the ways of migration.

Calculation algorithm includes four steps:

(i) Construction of VP for all atoms of the structure.

(ii) Determination of atomic coordinates of vertices of VP and positioning elementary voids.
(iii) Identification of all the independent edges of the VP and all basic channels.
(iv) The calculation of basic characteristics of voids and channels.

Presence of structural data for the crystal $CaMoO_4$ at rt and 1273 K [4] causes the choice of this crystal as model object for mapping the conductivity. For the calculation we used the radii of the ions Mo^{6+} (0,55 Å) at the coordination number (cn) 4, Ca^{2+} (1,26 Å) at cn 8 and O^{2-} (1,36 Å) [5] and the values of the second moment of inertia G – 0.089(5) and 0.0830(1) for Mo and Ca ions, respectively [6].

Calculation at rt for the Mo atoms shown that they can pass through the channels if $r_i > 0.9 \times (0{,}55 + 1{,}36) = 1{,}72$ (Å). It is shown that the migration way consists of two parts: ZA5–ZA6 (0,82 Å) and ZA6–ZA8 (0,2 Å). At 1273 K nearly continuous chains of conductivity are formed.

Thus, within the used approach at rt the ion conductivity in perfect structure of $CaMoO_4$ is low probable. At 1273 K and further higher temperatures can form Mo-ion conductivity channels. Note that the defects that always exist in real crystals provide significant opportunities for increasing the mobility of ions.

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ON POLYHEDRA WITH ISOLATED SYMMETRIC FACES

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The face of a closed convex polyhedron in three-dimensional Euclidean space is called symmetric if axis of symmetry of the polyhedron pass through it and asymmetric otherwise. Symmetric face F of the polyhedron is called isolated if all faces belonging to the star of F are asymmetric. We say that a closed convex polyhedron in three-dimensional Euclidean space is a polyhedron with isolated symmetric faces if each its symmetric face is isolated.

Belt of asymmetric faces is a finite sequence of asymmetric faces having only one common edge (connection edge) with previous and next face and the last face of the belt also has only one common edge with the first face. Moreover, each face of the belt have not common elements with non-adjacent face of the belt.

The following theorem is proved.

Theorem. *Let each asymmetric face of polyhedron with isolated symmetric faces belongs to only one belt of asymmetric faces and has only one common edge with some symmetric face. Then the maximum number of faces of the polyhedron is* 302 (*not counting the infinite series of truncated prisms, pyramids, truncated pyramids, bipyramids and antiprisms*).

In [1], accurate estimates of the number of faces of polyhedra with isolated asymmetric faces are given.

Note that the polyhedra with isolated symmetric faces are not metrically dual to polyhedron with isolated asymmetric faces.

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FAST CALCULATION OF THE EMPTY VOLUME IN MOLECULAR SYSTEMS BY THE VORONOI–DELAUNAY SUBSIMPLEXES

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The calculation of occupied and empty volume in an ensemble of overlapping spheres is not a simple task in general. There are analytical and numerical methods specialized in specific problems, in particular to calculate the van der Waals volume of a molecule, or the volume of internal voids in atomic systems. Voronoi diagrams are the helpful instrument assisting in solution of this problem [1, 2, 3]. An interesting approach was proposed in paper [4] for calculation the accessible volume of cavities in simple liquids. It uses triangle pyramids (subsimplexes) defined on the intersection of Voronoi polyhedron and Delaunay simplex. Later the subsimplexes were successfully applied for the volume calculation of union of strongly overlapped spheres [5].

In this work we discuss more wide applications of the subsimplexes for calculation of the occupied and empty volumes of different constructions selected on a molecular system. In particular, it can be the Voronoi and Delaunay shells defined around a solute molecule in solution as well as their intersection [6].

There are analytical formulas to calculate the occupied (or empty) volume inside the subsimplex. Summing the subsimplexes (using *the rule of signs*) the occupied (empty) volume can be calculated for whole constructions. A convenient data structure for efficient calculation of this volume is proposed. It is also discussed that the calculated volume might have an inaccuracy because of some peculiarities of the Voronoi–Delaunay tessellation, but it is not significant for the models of the molecular systems.

We calculated the components of the *partial molar volume* of a polypeptide molecule (hIAPP) in water for different temperatures on the molecular dynamics models of the solution, which is important for interpretation of the experimental volumetric data for protein solutions. The obtained results helps to explain the nature of the thermal expansion coefficient

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of hIAPP molecule: it is related to the surrounding water, but not to conformational and density changes of the solute molecule itself.

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TESSELLATION-BASED VELOCITY FIELD RECONSTRUCTIONS: MIGRATION FLOWS IN THE LOCAL UNIVERSE

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We use Voronoi and Delaunay tessellations to reconstruct the density and velocity field in our Local Universe. The Delaunay Tessellation Field Estimator, DTFE, translates the spatial distribution of galaxies into a piecewise linear representation of the volume-weighted density and velocity field [4, 6, 2]. On scales of millions of lightyears the velocities of the galaxies, with respect to the Hubble flow, provide a sensitive probe of the large-scale gravitational force field.

Within each Delaunay tetrahedron, the velocity flow is determined from the value of the measured velocity field at its 4 vertices. The vertex velocities are processed into a piecewise linear velocity field representation. The value of the density and velocity field gradient in each Delaunay tetrahedron is directly and uniquely determined from the location $\mathbf{r} = (x, y, z)$ of the four measurement points forming the Delaunay tetrahedra's vertices, $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, and the value of the estimated density and sampled velocities at each of these locations. The four vertices of the Delaunay tetrahedron are both necessary and sufficient for

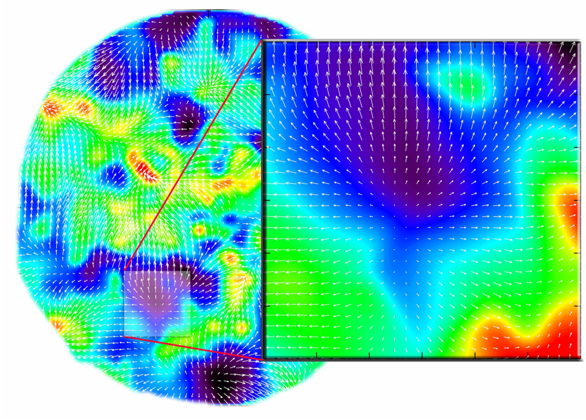

FIGURE 1. Density and velocity field map of the local Universe determined by DTFE on the basis of the PSCz galaxy redshift survey. Romano-Díaz & van de Weygaert 2007.

computing the entire 3×3 velocity gradient tensor $\partial v_i / \partial x_j$,

$$\begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_z}{\partial x} \\[2ex] \dfrac{\partial v_x}{\partial y} & \dfrac{\partial v_y}{\partial y} & \dfrac{\partial v_z}{\partial y} \\[2ex] \dfrac{\partial v_x}{\partial z} & \dfrac{\partial v_y}{\partial z} & \dfrac{\partial v_z}{\partial z} \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \Delta v_{1x} & \Delta v_{1y} & \Delta v_{1z} \\[2ex] \Delta v_{2x} & \Delta v_{2y} & \Delta v_{2z} \\[2ex] \Delta v_{3x} & \Delta v_{3y} & \Delta v_{3z} \end{pmatrix}, \qquad (1)$$

where $\mathbf{A}$ is the matrix whose elements specify the coordinate differences between the vertices and where $\Delta \mathbf{v}_n \equiv \mathbf{v}_n - \mathbf{v}_0$ $(n = 1, 2, 3)$ is the velocity difference between the n^{th} and 0^{th} vertex. From this, we may directly infer the shear, vorticity and divergence [1], which specify the deformation, rotation and expansion/compresssion of mass elements in the mass distribution.

We will discuss the principle behind the use of tessellations for velocity field reconstructions, and demonstrate the remarkable accuracy achieved for studying the components and scale-dependence of the field. The analysis will subsequently proceed to the cosmography of the Local Universe: the presentation of a 3-D map of the density and velocity fields in the Local Universe, along with the identification of the structures involved [3]. In addition, we will discuss the use of the DTFE velocity field reconstruction for our understanding of cosmic structure formation [5].

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ON SINGULARITY OF PROBABILITY DISTRIBUTIONS CONNECTED WITH CONTINUED FRACTIONS

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Let ξ be a random variable with independent symbols of continued fractions expansion:

$$\xi = \cfrac{1}{\xi_1 + \cfrac{1}{\xi_2 + \cfrac{1}{\xi_3 + \ldots + \cfrac{1}{\xi_k + \ldots}}}},$$

where $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are independent random variables taking values $1, 2, \ldots, n, \ldots$ with probabilities $p_{1k}, p_{2k}, \ldots, p_{nk}, \ldots$ respectively,

$$p_{ik} \in [0,1], \quad \sum_n p_{nk} = 1 \quad \text{for any } k \in \mathbb{N}.$$

In the talk we discuss different approaches to the proof of singular continuity of the random variable ξ with independent identically distributed symbols of continued fractions expansion: ergodic approach [5], frequency approach [2], dimensional approach [4]. We shall also discuss some problems related to the using of usual derivative for the proof of the singularity of ξ.

For the general independent case (i.e., if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are independent and not necessarily identically distributed random variables) we also develop the frequency approach to the proof of singularity of ξ.

We also analyze spectral structure of the distribution of ξ in the sense of [1] and prove that μ_ξ is of pure spectral type.

Finally, we shall pay a special attention to the Lebesgue structure of the probability measure μ_ξ for the case where ξ_k are markovian.

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MULTIFRACTAL FORMALISM FOR PROBABILITY MEASURES WITH INDEPENDENT TERNARY DIGITS

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Let μ be the probability measure corresponding to the random variable $\xi = \sum\limits_{k=1}^{\infty} \frac{\alpha_k}{3^k} = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$, where α_k are independent and identically distributed random variables: $\mathsf{P}(\alpha_k(x) = i) = p_i$, $i = \overline{0,2}$.

Let $B(x, \varepsilon)$ be a ball of radius ε centered at x.

Definition. If limit $\lim\limits_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon}$ exists, then its value $\dim_{\mathrm{loc}}\mu(x)$ is said to be the local dimension of measure μ at point $x \in [0,1]$.

For many purposes it is convenient to redefine local dimension of measure in terms of ternary intervals:

$$\lim_{k\to\infty} \frac{\log\mu(\Delta^3_{\alpha_1(x)\alpha_2\ldots\alpha_k(x)})}{\log|\Delta^3_{\alpha_1(x)\alpha_2\ldots\alpha_k(x)}|}.$$

It is natural to investigate connection of different definitions of local dimension of measure. There are some results in particular cases (like simple Cantor set), but we don't know any general results.

We formulate hypothesis for probability measures with independent ternary digits:

$$\underline{\lim}_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon} = \underline{\lim}_{k\to\infty} \frac{\log\mu(\Delta^3_{\alpha_1(x)\alpha_2\ldots\alpha_k(x)})}{\log|\Delta^3_{\alpha_1(x)\alpha_2\ldots\alpha_k(x)}|},$$

for μ-almost all x.

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ON RELATIONS BETWEEN SYSTEMS OF NUMERATIONS AND FRACTAL PROPERTIES OF SETS OF NON-NORMAL AND ESSENTIALLY NON-NORMAL NUMBERS

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The report is devoted to analysis of the dependence of fractal and topological properties of sets of non-normal and essentially non-normal numbers on systems of numerations.

Till 1994 the set of numbers, which are non-normal w.r.t. s-adic expansion (i.e., those numbers for which the asymptotic frequencies of some digits from the alphabet do not exist), was considered as a "rather small" one in the sense of Lebesgue measure as well as in the sense of the Hausdorff–Besicovitch dimension. After the proof of the superfractality of sets of non-normal and essentially non-normal numbers for s-adic and some other expansions [1] and construction of systems of representation such that the set of essentially non-normal numbers is of full Lebesgue measure, the conjecture about the superfractality and topological massivity of the set of essentially non-normal numbers (independently of the choice of a numeration system) became dominating.

Probabilistic approach is shown to be very useful to prove the superfractality of the set of essentially non-normal numbers for Q-expansions [2], Q_∞-expansions (provided that the stochastic vector Q_∞ satisfies the condition $\sum\limits_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < +\infty$) [3]. Very recently the superfractality for the set of Q^*-essentially non-normal numbers has been proven by M. Ibragim and G. Torbin under additional assumptions $\inf\limits_{i,k} q_{ik} > 0$ on the matrix Q^*.

We shall try to answer the following open problems, which are well motivated by the above arguments:

1. Is the condition $\inf\limits_{i,k} q_{ik} > 0$ necessary for the superfractality of the set of Q^*-essentially non-normal numbers?

2. Are there systems of numeration, for which the corresponding set of essentially non-normal numbers is not a superfractal?

We present a counterexample to the above mentioned conjecture. We show, in particular, that there are Q^*-expansions of real numbers, for which the corresponding set of essentially non-normal numbers has zero Hausdorff–Besicovitch dimension: $\dim_H(L_{Q^*}) = 0$.

Theorem 1. *Let*

$$Q^* = \|q_{ik}\|, \quad q_{2k} = \frac{1}{(k+1)^{k+1}}, \quad q_{0k} = q_{1k} = \frac{1 - q_{2k}}{2}, \quad i \in \{0, 1, 2\}.$$

Then

$$\dim_H(L_{Q^*}) = 0.$$

Moreover, we prove the existence of such Q^*-expansions of real numbers, for which even the whole set D_{Q^*} of all non-normal numbers has zero Hausdorff–Besicovitch dimension: $\dim_H(D_{Q^*}) = 0$.

Theorem 2. *Let* $Q^* = \|q_{ik}\|$, $i \in \{0, 1, 2, \ldots, s-1\}$. *Let*

$$A = \{n \colon n = 10^k, k \in \mathbb{N}\}, \quad B = \{n \colon n \neq 10^k, k \in \mathbb{N}, n \in \mathbb{N}\},$$

and let

$$q_{1k} = q_{2k} = \ldots = q_{s-1,k} = \begin{cases} \frac{1}{(s-1)(k+1)^{k+1}}, & \text{if } k \in B, \\ \frac{1}{s}, & \text{if } k \in A, \end{cases}$$

$$q_{0k} = \begin{cases} 1 - \frac{1}{(k+1)^{k+1}}, & \text{if } k \in B, \\ \frac{1}{s}, & \text{if } k \in A. \end{cases}$$

Then

$$\dim_H(D_{Q^*}) = 0.$$

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ON THE HAUSDORFF–BESICOVITCH DIMENSION FAITHFULNESS FOR THE FAMILY OF Q^*-CYLINDERS

MUSLEM IBRAGIM AND GRYGORIY TORBIN

The notion of the Hausdorff–Besicovitch dimension is well-known now and is of great importance in mathematics as well as in different applied problems. In many situations the determination (or even estimations) of this dimension for sets from a given family or even for a given set is a rather complicated problem. To simplify the calculation of the Hausdorff–Besicovitch dimension of a given set it is extremely useful to have an appropriate and a relatively narrow family of admissible coverings which lead to the same value of the dimension.

A fine covering family Φ is said to be *faithful family of coverings* (*non-faithful family of coverings*) for the Hausdorff–Besicovitch dimension calculation on $[0,1]$ if

$$\dim_H(E,\Phi) = \dim_H(E) \text{ for any } E \subseteq [0,1]$$
$$(\text{resp. there exists } E \subseteq [0,1] \text{ such that } \dim_H(E,\Phi) \neq \dim_H(E)).$$

The talk will be devoted to the open problem for the Hausdorff–Besicovitch dimension faithfulness of the family $\Phi(Q^*)$-cylinders generated by Q^*-expansion for real numbers (see [2,9] for definitions and properties of Q^*-expansion).

Conditions for a fine covering family to be faithful were studied by many authors (see, e.g., [1, 5–8] and references therein). First steps in this direction have been done by A. Besicovitch [4], who proved the faithfulness for the family of cylinders of binary expansion. His result was extended by P. Billingsley [5] to the family of s-adic cylinders, by M. Pratsiovytyi [10] to the family of Q-S-cylinders, and by S. Albeverio and G. Torbin [1] to the family of Q^*-cylinders for those matrices Q^* whose elements p_{0k}, $p_{(s-1)k}$ are bounded from zero.

The following theorem is a generalization of the above mentioned results and to prove it we use new technics for the verification of the Hausdorff–Besicovitch dimension faithfulness.

Theorem. *Let* $q^*_k = \max\{q_{0k}, q_{1k}, \dots, q_{(s-1)k}\}$.

If for any $\delta > 0$ the following conditions hold

$$\begin{cases} \lim\limits_{k\to\infty} \dfrac{1}{q_{0k}}(q_1^* q_2^* \dots q_k^*)^\delta = 0, \\ \lim\limits_{k\to\infty} \dfrac{1}{q_{(s-1)k}}(q_1^* q_2^* \dots q_k^*)^\delta = 0, \end{cases} \tag{1}$$

then the family $\Phi(Q^*)$ of Q^*-cylinders is faithful for the Hausdorff–Besicovitch dimension calculation on the unit interval, i.e.,

$$\dim_H(E) = \dim_H(E, \Phi(Q^*)) \quad \textit{for any} \quad E \subset [0, 1].$$

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ON A NEW FAMILY OF INFINITE BERNOULLI CONVOLUTIONS WITH ESSENTIAL OVERLAPS

GANNA IVANENKO AND GRYGORIY TORBIN

Let ξ_k be a sequence of independent random variables taking values 0 and 1 with probabilities p_{0k} and p_{1k} respectively, and let $\sum_{k=1}^{\infty} a_k$ be a positive convergent series. The distribution of the random variable

$$\xi = \sum_{k=1}^{\infty} \xi_k a_k \tag{1}$$

is known to be the infinite Bernoulli convolution. Properties of ξ were studied by many authors during last 80 years (see, e.g., [2, 6] and references therein). Since the purity (in the sense of Lebesgue decomposition) of the distribution μ_ξ follows from the Jessen–Wintner theorem, and Lévy theorem [5] gives necessary and sufficient conditions for μ_ξ to be atomic, main problems for μ_ξ are the following:

(1) to find criteria for absolute resp. singular continuity;
(2) to study fractal properties of the spectrum S_ξ and fine fractal properties of the measure μ_ξ.

For the case where $a_k \geq r_k := \sum_{j=k+1}^{\infty} a_j$ for all sufficiently large k, properties of μ_ξ (including fine fractal ones) are studied in detail (see [2] for the last results in this direction). The situation where $a_k < r_k$ for an infinite number of k is essentially more complicated, especially if a generic part of points from the spectrum have continuum many different variants to be represented in the form $\sum_{k=1}^{\infty} \varepsilon_k a_k$, where $\varepsilon_k \in \{0, 1\}$. Such a convolution is said to be an infinite Bernoulli convolution with essential overlaps. Special cases of this problem were studied by P. Erdös, B. Solomyak, Yu. Peres and many other mathematicians for the case where $a_k = \lambda^k$, $\lambda \in (0, \frac{1}{2})$ (see, e.g., [5, 6, 8] and references therein). A series of papers by S. Albeverio, G. Ivanenko, M. Lebid, M. Pratsiovytyi, G. Torbin (see, e.g., [2–4]) was devoted to Bernoulli convolutions, which can be represented as measures with independent $\tilde{Q}$-symbols [1]. Necessary and sufficient conditions for μ_ξ to be measures with independent $\tilde{Q}$-symbols have been recently found by G. Ivanenko and G. Torbin.

In the talk we introduce and describe properties of probability measures from a new family of Bernoulli convolutions with essential overlaps. A special attention will be paid to a subfamily, which is generated by sequences $\{a_k\}$ of the following form:

$$a_{7k-i} = \Delta^2_{\underbrace{00\ldots0\beta_2(i)\beta_1(i)\beta_0(i)}_{n_k}(0)}, \quad i \in \{0,1,2,3,4,5,6\}, \tag{2}$$

where $\beta_j(i)$ are uniquely defined digits from $\{0,1\}$ such that

$$i+1 = \sum_{j=0}^{2} \beta_j(i) 2^j,$$

and $\{n_k\}$ is an increasing sequence of positive integers.

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TOPOLOGICAL, METRIC AND FRACTAL PROPERTIES OF THE SET OF INCOMPLETE SUMS OF SERIES OF GENERALIZED FIBONACCI NUMBERS

DMYTRO KARVATSKY

Let us consider the series

$$\sum_{n=1}^{\infty} u_n, \tag{1}$$

whose terms satisfy the following conditions: $u_{n+2} = pu_{n+1} + su_n$, $n \in \mathbb{N}$, where $u_1, u_2, p, s \in \mathbb{R}$.

Theorem 1. *If series* (1) *satisfies the condition*

$$\begin{cases} -2 < p < 2, \\ -\dfrac{p^2}{4} < s < \min\{1-p; 1+p\}, \end{cases}$$

than series is convergent.

Theorem 2. *If series* (1) *satisfies the condition*

$$\begin{cases} -1 < p < 1, \\ -\dfrac{p^2}{4} < s < \min\left\{\dfrac{1-2p}{4}; \dfrac{1+2p}{4}\right\}, \end{cases}$$

than series is convergent and its set of incomplete sums has the following properties:

(1) *it is a perfect set* (*closed set without isolated points*)*;*
(2) *it is a nowhere dense set;*
(3) *it is of zero Lebesgue measure.*

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THE RANDOM INCOMPLETE SUMS OF ALTERNATING LÜROTH SERIES WITH ELEMENTS FORMING A HOMOGENEOUS MARKOV CHAIN

YURIY KHVOROSTINA

We consider the random variable (see [1])

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\tau_k}{a_1(a_1+1)a_2(a_2+1)\dots a_{k-1}(a_{k-1}+1)a_k},$$

where (a_k) is a fixed sequence of positive integers, (τ_k) are random variables forming a homogeneous Markov chain and taking two values 0 and 1. The report contains the results of research of Lebesgue structure of the random variable ξ. We specify necessary and sufficient conditions for a singular distribution of ξ to be of Cantor or Salem type.

Theorem 1. *Random variable ξ has a discrete distribution if the transition probability matrix has two zeros or exactly one zero $p_{ij} = 0$ $(i \neq j)$.*

Theorem 2. *Random variable ξ has the singular distribution of Cantor type if the transition probability matrix has exactly one zero $p_{ii} = 0$.*

Theorem 3. *If the transition probability matrix $\|p_{ik}\|$ has no zeros, then the distribution of the random variable ξ is:*

(1) *the singular distribution of Cantor type if infinitely many $a_k \neq 1$;*
(2) *the absolutely continuous if*

$$\|p_{ik}\| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \equiv B;$$

(3) *the singular distribution of Salem type if $\|p_{ik}\| \neq B$ and infinitely many $a_k \neq 1$.*

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CONDITIONS FOR EXISTENCE OF ASYMPTOTIC MEAN OF DIGITS OF REAL NUMBER

SVITLANA O. KLYMCHUK

It is well known that any real number $x \in [0;1]$ can be represented in the form

$$x = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \dots + \frac{\alpha_n}{s^n} + \dots \equiv \Delta^s_{\alpha_1\alpha_2\dots\alpha_n\dots},$$

where $\alpha_n \in \mathcal{A} = \{0, 1, \dots, s-1\}$. The last expression is called an *s-adic representation*, and $\alpha_k = \alpha_k(x)$ is called a *kth s-adic digit* of number x.

Some numbers can be represented in two ways:

$$\Delta^s_{c_1\dots c_{k-1}c_k(0)} = \Delta^s_{c_1\dots c_{k-1}[c_k-1](s-1)}.$$

These numbers are called *s-adic rational.* Other numbers have only one representation and are called *s-adic irrational.* A kth s-adic digit of number is well defined if we agree to use s-adic representation with period (0).

Let $N_i(x,k)$ be a number of $i \in \mathcal{A}$ in first k digits of the s-adic representation of $x \in [0;1]$.

Definition 1. *Frequency* (*asymptotic frequency*) of digit i in s-adic representation of real number $x \in [0;1]$ is the limit

$$\nu_i(x) = \lim_{k\to\infty} \frac{N_i(x,k)}{k},$$

if it exists.

Function of frequency $\nu_i(x)$ of digit i in s-adic representation of number $x \in [0;1]$ is well defined for s-adic irrational numbers and s-adic rational numbers (with agreement to use representation with period (0)).

Definition 2. Let α_i be the digits of s-adic representation of number $x \in [0;1]$. Number $r_n(x) \equiv \frac{1}{n}\sum\limits_{i=1}^{n} \alpha_i(x)$ is called a *relative mean* of digits of x. If $\lim\limits_{n\to\infty} r_n(x) = r(x)$ exists, then its value (number $r(x)$) is called an *asymptotic mean* (or just *mean*) of digits of x.

We study topological, metric and fractal properties of number sets represented in 4-adic numeral system with given asymptotic mean of digits.

Let (s_k) be a sequence of positive numbers such that $\lim\limits_{k\to\infty} s_k = \infty$, $\lim\limits_{k\to\infty} \dfrac{s_{k+1}}{\sum\limits_{i=1}^{k} s_i} = 0$, $\lim\limits_{k\to\infty} \dfrac{k}{\sum\limits_{i=1}^{k} s_i} = 0$. Let $\|\tau_{in}\|$ be a $4\times\infty$ matrix. Let us consider the following representation of real number $x \in [0;1]$:

$$\hat{x} = \Delta^4_{\underbrace{\underbrace{0\ldots0}_{[\tau_{01}s_1]}\underbrace{1\ldots1}_{[\tau_{11}s_1]}\underbrace{2\ldots2}_{[\tau_{21}s_1]}\underbrace{3\ldots3}_{[\tau_{31}s_1]}}_{\text{1st set}}\ldots\underbrace{\underbrace{0\ldots0}_{[\tau_{0k}s_k]}\underbrace{1\ldots1}_{[\tau_{1k}s_k]}\underbrace{2\ldots2}_{[\tau_{2k}s_k]}\underbrace{3\ldots3}_{[\tau_{3k}s_k]}}_{k\text{th set}}\ldots}.$$

Theorem 1. *If $\|\tau_{in}\|$ is a $4\times\infty$ matrix and for all $n\in\mathbb{N}$ the conditions*

$$\tau_{0n}+\tau_{1n}+\tau_{2n}+\tau_{3n}=1 \quad \textit{and} \quad \tau_{1n}+2\tau_{2n}+3\tau_{3n}=\theta$$

are satisfied, then

$$\lim_{n\to\infty} r_n(\hat{x}) = \theta.$$

Theorem 2. *If $\|\tau_{in}\|$ is a stochastic $4\times\infty$ matrix and $\lim\limits_{n\to\infty}\tau_{jn}=\lambda$ for fixed $j\in\{0,1,2,3\}$, then*

$$\nu_j(\hat{x}) = \lambda.$$

Theorem 3. *Suppose $(s_k^{(1)})$, $(s_k^{(2)})$ are the sequences of positive numbers such that $\lim\limits_{k\to\infty} s_k^{(r)} = \infty$, $r\in\{1,2\}$, $\|p^{(1)}\| = \|p_{in}^{(1)}\|$, $\|p^{(2)}\| = \|p_{in}^{(2)}\|$ are the stochastic $4\times\infty$ matrices, and*

$$x(\|p^{(r)}\|;\|s_k^{(j)}\|) = \Delta^4_{\underbrace{\underbrace{00...0}_{[p_{01}^{(r)}s_1^{(j)}]}\underbrace{11...1}_{[p_{11}^{(r)}s_1^{(j)}]}\underbrace{22...2}_{[p_{21}^{(r)}s_1^{(j)}]}\underbrace{33...3}_{[p_{31}^{(r)}s_1^{(j)}]}}_{1st\ set}\ldots\underbrace{\underbrace{00...0}_{[p_{0k}^{(r)}s_k^{(j)}]}\underbrace{11...1}_{[p_{1k}^{(r)}s_k^{(j)}]}\underbrace{22...2}_{[p_{2k}^{(r)}s_k^{(j)}]}\underbrace{33...3}_{[p_{3k}^{(r)}s_k^{(j)}]}}_{kth\ set}\ldots}$$

If $\lim\limits_{k\to\infty}|s_k^{(1)}-s_k^{(2)}|=\infty$, then $x(\|p^{(1)}\|;\|s_k^{(1)}\|)\neq x(\|p^{(2)}\|;\|s_k^{(2)}\|)$.

If $\overline{\lim\limits_{n\to\infty}}\sum\limits_{i=0}^{3}|p_{in}^{(1)}-p_{in}^{(2)}|>0$, then $x(\|p^{(1)}\|;\|s_k^{(1)}\|)\neq x(\|p^{(2)}\|;\|s_k^{(2)}\|)$.

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ON DP-TRANSFORMATIONS GENERATED BY RANDOM VARIABLES WITH INDEPENDENT SYMBOLS OVER DYNAMIC ALPHABETS

MYKOLA LEBID

The talk will be devoted to the study of connections between fractal properties of one-dimensional singularly continuous probability measures and the preservation of the Hausdorff dimension of any subset of the unit interval under the corresponding distribution functions. A special attention will be paid to the case of random variables with independent $\tilde{Q}$-symbols and for a special subcase of random Cantor series expansion. Based on these results we give a precise characterization of DP-properties of the distribution functions generated by the above probability measures. We show, in particular, that under some natural assumptions the above distribution functions preserve the Hausdorff dimension on the unit interval if and only if the corresponding measures μ_ξ are of full Hausdorff dimension.

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ON SUPERPOSITION OF THE ABSOLUTELY CONTINUOUS AND SINGULARLY CONTINUOUS DISTRIBUTION FUNCTIONS

MARINA LUPAIN[1] AND GRYGORIY TORBIN[1,2]

It is well known that the function which is inverse to the continuous strictly increasing function is also continuous and strictly increasing. It is not hard to prove (see, e.g., [1]) that the same implication for singularly continuous functions is also true. So, it is naturally to ask whether such a relation holds for absolutely continuous functions.

But it is not always for the absolutely continuous function.

Proposition 1. *There exist strictly increasing absolutely continuous functions such that the corresponding inverse functions are not absolutely continuous.*

To prove the above proposition let us construct a strictly increasing absolutely continuous function F such that the $F'(x) = 0$ on the set of positive Lebesgue measure. To this end we consider the following Q^*-expansion of real numbers:

$$Q^* = \begin{pmatrix} \frac{1}{2}-\frac{1}{4} & \frac{1}{2}-\frac{1}{8} & \dots & \frac{1}{2}-\frac{1}{2^{n+1}} & \dots \\ \frac{1}{2} & \frac{1}{4} & \dots & \frac{1}{2^n} & \dots \\ \frac{1}{2}-\frac{1}{4} & \frac{1}{2}-\frac{1}{8} & \dots & \frac{1}{2}-\frac{1}{2^{n+1}} & \dots \end{pmatrix}.$$

That is $q_{0n} = q_{2n} = \frac{1}{2} - \frac{1}{2^{n+1}}$, $q_{1n} = \frac{1}{2^n}$, $\forall n \in \mathbb{N}$. Then the set of Cantor type $C_0 = C_0(Q^*, \{0,2\}) = \{x\colon x = \Delta^{Q^*}_{\alpha_1(x)\dots\alpha_n(x)\dots}, \alpha_j(x) \in \{0,2\}\}$ is of positive Lebesgue measure: $\lambda(C_0) = \prod\limits_{k=1}^{\infty}(1-q_{1k}) = \prod\limits_{n=1}^{\infty}(1-\frac{1}{2^n}) > 0$.

The function F is defined to be linearly increasing on the following cylinders of the Q^*-expansion

$$\Delta^{Q^*}_{1}, \quad \Delta^{Q^*}_{01}, \quad \Delta^{Q^*}_{21}, \quad \Delta^{Q^*}_{001}, \quad \Delta^{Q^*}_{021}, \quad \Delta^{Q^*}_{201}, \quad \Delta^{Q^*}_{221}, \quad \dots$$

and

$$F(\Delta^{Q^*}_{1}) = \Delta^3_1, \quad F(\Delta^{Q^*}_{01}) = \Delta^3_{01}, \quad F(\Delta^{Q^*}_{21}) = \Delta^3_{21}, \quad \dots,$$

where $\Delta^3_{\alpha_1\alpha_2\dots\alpha_n}$ is the cylinder of ternary expansion.

Let $C_1 := [0,1] \setminus C_0$, $A_0 := F(C_0)$, $A_1 := F(C_1)$. Then

$$\lambda(A_1) = \lambda\big(F(\Delta^{Q^*}_{1} \cup \Delta^{Q^*}_{01} \cup \Delta^{Q^*}_{21} \cup \Delta^{Q^*}_{001} \cup \dots)\big) = 1.$$

Therefore, $\lambda(A_0) = 0$, and the function F is absolutely continuous.

The function F is strictly increasing. So, there exists the inverse function $\varphi = F^{-1}$, and $\varphi(A_0) = F^{-1}((F(C_0)) = C_0$. From $\lambda(A_0) = 0$ and $\lambda(C_0) > 0$ it follows that φ is not absolutely continuous. Let us remark that $F'(x) = 0$ for almost all $x \in C_0$ with respect to Lebesgue measure. Therefore, $F'(x) = 0$ on a set of positive Lebesgue measure.

Theorem 1. *Let $F_1(x)$ be a strictly increasing distribution function such that F^{-1} is also absolutely continuous, and let F_2 be a strictly increasing singularly continuous distribution function. Then the superposition $F = F_2(F_1)$ is singularly continuous.*

Remark 1. The condition of the absolute continuity of the inverse function F^{-1} is essential. To stress the importance of the absolute continuity of F^{-1}, let us construct strictly increasing functions F_2 and F_1 such that F_1 is absolutely continuous, F_2 is singularly continuous and their superposition $F = F_2(F_1)$ is not singularly continuous.

Let F_{ac} be a strictly increasing continuous distribution function on $[0, 1]$, which has been described above. We define a strictly increasing singularly continuous distribution function F_{sc} in the following way.

Let the Q^*-expansion of real numbers be such that $q_{0n} = q_{2n} = \frac{1}{2} - \frac{1}{2^{n+1}}$, $q_{1n} = \frac{1}{2^n}$, and define

1) $F_{sc}(\Delta^3_{\alpha_1(x)\ldots\alpha_k 1(x)}) = \Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k 1(x)}$, $\alpha_j \in \{0, 2\}$;

2) the graph of F_{sc} is an affine copy of the classical Salem function on the segment $\Delta^3_{\alpha_1(x)\ldots\alpha_k(x)1}$. For other $x \in C_0$ function F_{sc} is defined by continuity.

It is easy to see that F_{sc} is singular on $\Delta_{\alpha_1(x)\ldots\alpha_k(x)1}$. So, F_{sc} is strictly increasing and singularly continuous on the whole segment $[0, 1]$. On the other hand the superposition $F = F_{sc}(F_{ac})$ is a mixture of the absolutely continuous and singularly continuous distribution functions.

Theorem 2. *Let $F_1(x)$ be an absolutely continuous distribution function and let F_2 be a singularly continuous distribution function. The superposition $F = F_1(F_2)$ is singularly continuous.*

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REPRESENTATION OF REAL NUMBERS BY SYLVESTER SERIES AND SECOND OSTROGRADSKY SERIES AND ITS FRACTAL ANALYSIS

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In 1883 J. Sylvester introduced a representation of real numbers by series of positive terms: *for any real* $x \in (0,1]$ *there exists a unique sequence of positive integers* (q_k) *such that* $q_1 \geq 2$, $q_{n+1} \geq q_n(q_n - 1) + 1$ *and*

$$x = \frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{q_n}. \tag{1}$$

In 1911 W. Sierpiński proved the following proposition.

Theorem 1. *For any real* $x \in (0,1]$ *there exists a unique sequence* (d_k) *of positive integers such that* $d_{n+1} \geq d_n(d_n + 1)$ *and*

$$x = \frac{1}{d_1} - \frac{1}{d_2} + \ldots + \frac{(-1)^{n+1}}{d_n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_n}. \tag{2}$$

Representation of real number in the form (1) or (2) is called the *Sylvester expansion* or the *second Ostrogradsky expansion* respectively. The right-hand side of Eq. (1) is denoted by $S(q_1, q_2, \ldots, q_n, \ldots)$ and is called the *S-representation.* Similarly, the right-hand side of Eq. (2) is denoted by $O_2(d_1, d_2, \ldots, d_n, \ldots)$ and is called the *O_2-representation.*

This method permits to construct the theory of real numbers without previous construction of the theory of rational numbers and to build metric and probabilistic theory of real numbers.

So, there are “similar” representations of real numbers by alternating series and series of positive terms such that their terms are reciprocals of the positive integers. They have common properties and some differences. This permits to construct the theory of real number encoding with an infinite alphabet – set of positive integers.

Representations of real numbers by Sylvester series and second Ostrogradsky series are “fast” covergent and play an important role in the theory of Diophantine approximation. Also they can be effectively used in fractal analysis where the main notion is a fractal Hausdorff–Besicovitch dimension. Non-self-similar geometry generated by these representations leads to differences from the similar problems for s-adic representation.

The set $\Delta_{c_1c_2\ldots c_m}$ of all $x \in (0,1]$ such that in the Sylvester expansion $q_i(x) = c_i$ (or in the second Ostrogradsky series $d_i(x) = c_i$ respectively), $i = 1, 2, \ldots, m$, is called a *cylinder* of rank m with base $c_1c_2\ldots c_m$.

For $\Delta^S_{c_1c_2\ldots c_m}$ the following relation holds:

$$\frac{|\Delta^S_{c_1\ldots c_m c_{m+1}}|}{|\Delta^S_{c_1\ldots c_m}|} = \frac{c_m(c_m-1)}{c_{m+1}(c_{m+1}-1)} \le \frac{1}{c_m(c_m-1)+1} \to 0 \quad (m \to \infty).$$

For $\Delta^{O_2}_{c_1c_2\ldots c_m}$ the following relation holds:

$$\begin{aligned}\frac{|\Delta^{O_2}_{c_1\ldots c_m c_{m+1}}|}{|\Delta^{O_2}_{c_1\ldots c_m}|} &= \frac{c_m(c_m+1)}{c_{m+1}(c_{m+1}+1)} \le \frac{c_m(c_m+1)}{c_m(c_m+1)(c_m(c_m+1)+1)} \\ &= \frac{1}{c_m(c_m+1)} \to 0 \quad (m \to \infty).\end{aligned}$$

Let $\mathfrak{W}$ be a class of connected sets being a union of cylinders of the same rank belonging to the same cylinder of previous rank. The class $\mathfrak{W}$ consists of the sets of the following type:

$$1)\ \Delta^S_{c_1c_2\ldots c_m}, \quad 2)\ \bigcup_{i=n}^{\infty} \Delta^S_{c_1c_2\ldots c_m i}, \quad 3)\ \bigcup_{i=1}^{n} \Delta^S_{c_1c_2\ldots c_m i}, \quad 4)\ \bigcup_{i=k}^{n} \Delta^S_{c_1c_2\ldots c_m i}$$

for positive integers k, m, n and tuples of positive integers $(c_1, c_2, \ldots, c_m)$.

Theorem 2. *The class $\mathfrak{W}$ is sufficient for definition of Hausdorff–Besicovitch dimension of any Borel set $E \subset [0,1]$, i.e.,*

$$\alpha_0(E, \mathfrak{W}) = \alpha_0(E).$$

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ASYMPTOTICS OF THE CHARACTERISTIC FUNCTION OF A RANDOM VARIABLE WITH INDEPENDENT BINARY DIGITS

OLEG MAKARCHUK

Let (ψ_k) be a sequence of independent random variables taking the values 0, 1 with probabilities p_{0k}, p_{1k} respectively. The random variable $\psi = \sum\limits_{k=1}^{\infty} \psi_k 2^{-k}$ is called a random variable with independent binary digits.

Characteristic function of a random variable ξ is $f_\xi(t) = \mathsf{M}(e^{it\xi})$, where $\mathsf{M}(\xi)$ is an expectation of a random variable ξ. Consider the value of $L_\xi = \limsup\limits_{|t|\to\infty} |f_\xi(t)|$. If random variable ξ has 1) a discrete distribution, then $L_\xi = 1$; 2) an absolutely continuous distribution, then $L_\xi = 0$; 3) a singular distribution, then $L_\xi \in [0; 1]$.

In the paper [1], the necessary and sufficient condition for $L_\psi = 0$ is obtained, namely: $L_\psi = 0$ if and only if $\lim\limits_{k\to\infty} p_{0k} = \frac{1}{2}$.

This paper presents the necessary and sufficient conditions for the equality $L_\psi = 1$.

Theorem. *The equality $L_\psi = 1$ holds if and only if for any $\varepsilon > 0$ there exists positive integer k such that*

$$\sum_{j=0}^{\infty} p_{0(k+j)} p_{1(k+j)} \sin^2 \frac{\pi}{2^{j+1}} < \varepsilon.$$

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SUPERFRACTALITY OF THE SET OF Q_∞-ESSENTIALLY NON-NORMAL NUMBERS

ROMAN NIKIFOROV AND GRYGORIY TORBIN

Let $\Delta^{Q_\infty}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}$ be the Q_∞-expansion [1] of the number $x \in [0,1]$, and let $N_i(x,n)$ be the number of digits "i" among the first n digits of the Q_∞-expansion of x. If the limit $\lim\limits_{n\to\infty} \frac{N_i(x,n)}{n}$ exists, then its value $\nu_i^{Q_\infty}(x)$ is said to be the asymptotic frequency of the digit "i" in the Q_∞-expansion of x.

Definition. The set

$$L(Q_\infty) = \left\{ x\colon \lim_{k\to\infty} \frac{N_i(x,k)}{k} \text{ does not exist, } \forall\, i \in \mathbb{N}_0 \right\}$$

is said to be the set of Q_∞-essentially non-normal numbers.

Theorem. *Let a stochastic vector Q_∞ satisfy*

$$\sum_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < +\infty.$$

Then the set $L(Q_\infty)$ of Q_∞-essentially non-normal numbers is of full Hausdorff dimension:

$$\dim_H(L(Q_\infty)) = 1.$$

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FRACTAL PROPERTIES OF THE SET OF INCOMPLETE SUMS OF POSITIVE SERIES WITH SOME CONDITION OF HOMOGENEITY

IGOR SAVCHENKO

We consider the convergent positive series

$$s = \sum_{n=1}^{\infty} a_n, \tag{1}$$

which satisfies the following condition of homogeneity:

$$a_n + a_{n+1} = \frac{1-t}{t} \cdot (a_{n+2} + a_{n+3} + \ldots) \ \Leftrightarrow\ a_{n+2} = t \cdot a_n,$$

where $\mathbb{R} \ni t \in (0,1)$ and the sequence $\{a_n\}$ is monotonically nonincreasing, i.e., $a_n \geq a_{n+1}$ for any $n \in \mathbb{N}$.

Definition. The set $\Delta' = \Big\{x\colon x = \sum\limits_{n=1}^{\infty} x_n \cdot a_n,\ x_n \in \{0,1\}\Big\}$ is called *the set of incomplete sums* of the series (1).

We study the topological, metric and fractal properties of the set Δ'.

Theorem 1. *The set Δ' has the following properties:*

1) *if $t \in \big(0, \max\{\frac{a_1-a_2}{2a_1}, \frac{a_2}{a_1+2a_2}\}\big)$, then it is a nowhere dense set; if $t \in (0, \frac{1}{4})$, then it is of zero Lebesgue measure;*

2) *if $t \in \big[\max\{\frac{a_1-a_2}{2a_1}, \frac{a_2}{a_1+2a_2}\}, 1\big)$, then it is a segment $[0, \frac{a_1+a_2}{1-t}]$;*

3) *if $t \in \big(0, \min\{\frac{a_1-a_2}{2a_1}, \frac{a_2}{a_1+2a_2}\}\big]$, then its Hausdorff–Besicovitch dimension is equal to $\alpha_0(\Delta') = -\log_t 4$.*

Theorem 2. *If $t = \frac{a_2}{a_1} \in [\frac{1}{4}, \frac{a_1-a_2}{2a_1})$, then Δ' is of zero Lebesgue measure and its Hausdorff–Besicovitch dimension is equal to $\alpha_0(\Delta') = -\log_t 3$.*

Theorem 3. *If $t = \frac{a_1-a_2}{a_1+a_2} \in [\frac{1}{4}, \frac{a_2}{a_1+2a_2})$, then Δ' is of zero Lebesgue measure.*

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ON ASYMPTOTIC PROPERTIES OF THE FOURIER–STIELTJES TRANSFORMS FOR SOME FAMILIES OF PROBABILITY MEASURES

LILIYA SINELNYK

Let

$$\begin{aligned} A &= \{k\colon k = 2^{n+1} - n, k = 2^{n+1} - n + 1, \ldots, k = 2^{n+1}, n \in \mathbb{N}\} \\ &= \{3, 4, 6, 7, 8, 13, 14, 15, 16, \ldots\} \subset \mathbb{N}, \end{aligned}$$

and let us consider the following subset of the unit interval:

$$L = \left\{x\colon x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{2^k}\right\},$$

where

$$\alpha_k(x) = \begin{cases} 0, & \text{if } k \notin A, \\ 0 \text{ or } 1, & \text{if } k \in A. \end{cases}$$

This means that any number x from L has the following binary expansion:

$$x = 0{,}00\alpha_3\alpha_4 0\alpha_6\alpha_7\alpha_8 0000\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}0\ldots 0\alpha_{2^{n+1}-n}\ldots\alpha_{2^{n+1}-1}\alpha_{2^{n+1}}0\ldots$$

It is easy to see that set L has the continuum cardinality.

Let L^* be a set created from L by deleting of countable set of points having zero at period.

Let a be an arbitrary real number from L^*. Then, for a given number

$$a = 0{,}00\alpha_3\alpha_4 0\alpha_6\alpha_7\alpha_8 0000\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}0\ldots 0\alpha_{2^{n+1}-n}\ldots\alpha_{2^{n+1}-1}\alpha_{2^{n+1}}0\ldots,$$

we construct the following sequence $\{a_n\}$ of real numbers:

$$\begin{aligned} a_1 &= 0, 00000\alpha_6\alpha_7\alpha_8 0000\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}0\ldots, \\ a_2 &= 0, 000000000000\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}0\ldots, \\ &\ldots \\ a_n &= \frac{\{a \cdot 2^{2^{n+1}}\}}{2^{2^{n+1}}}, \\ &\ldots \end{aligned}$$

Let us consider a family of generalized Bernoulli convolutions

$$\xi = \xi(a) = \sum_{k=1}^{\infty} \xi_k a_k,$$

where ξ_k are independent random variables taking values -1 and 1 with probabilities p_{0k} and p_{1k} correspondingly.

From [1] it follows that the distribution μ_ξ is singular w.r.t. Lebesgue measure and its spectrum S_ξ is of zero Hausdorff–Besicovitch dimension.

Let $L_\xi := \overline{\lim\limits_{|t|\to\infty}} |f_\xi(t)|$, where $f_\xi(t)$ is the Fourier–Stieltjes transform of the probability measure μ_ξ, i.e.,

$$f_\xi(t) = \int_{-\infty}^{\infty} e^{it\xi} \left(dF_\xi(x)\right) = \mathsf{M}\left(e^{it\xi}\right).$$

Theorem. *For any number a from the set L^* the value*

$$L_\xi = \overline{\lim_{|t|\to\infty}} |f_\xi(t)| = 1.$$

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PACKING DIMENSION AND PACKING DIMENSION PRESERVING TRANSFORMATIONS

ALEXANDER V. SLUTSKIY

We study packing dimension ($\dim_P(\cdot)$) and packing dimension preservations (PDP). One can find a definition and basic properties of $\dim_P(\cdot)$ in [1] and [2]. Note that $\dim_P(\cdot)$ is countably stable, i.e.,

$$\dim_P\Big(\bigcup_i E_i\Big) = \sup_i \dim_P(E_i).$$

Definition 1. Let (X, ρ) be a metric space. An automorphism f of X is called a PDP-transformation if

$$\dim_P(f(E)) = \dim_P(E) \quad \text{for any} \quad E \subset X.$$

Basic properties of PDP-transformations in metric spaces:

(1) PDP-transformations form a group with respect to operation "composition".
(2) Isometries are PDP-transformations.
(3) Similarity transformations are PDP-transformations.
(4) Bi-Lipschitz transformations are PDP-transformations. So, any affine transformation is in PDP-group.

Theorem 1. *Any projective transformation in $\mathbb{R}^1$ preserves the Hausdorff–Besicovitch dimension* [3] *and packing dimension.*

Theorem 2. *If E is a compact self-similar set in $\mathbb{R}^n$ that satisfying an open set condition then its packing dimension is equal to its self-similar dimension.*

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FRACTAL PROPERTIES OF DENSE SUBSPACES OF BESICOVITCH METRIC SPACES

VITALIY SUSHCHANSKY

In the talk we describe some fractal properties of special dense subspaces in Besicovitch metric spaces of infinite sequences over finite alphabets. Every such subspace is defined as a direct limit of finite Hamming spaces over a fixed alphabet.

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MODIFIED $\bar{Q}_3^*$-REPRESENTATION, FEATURES AND CRITERIA OF RATIONALITY (IRRATIONALITY) FOR REPRESENTATION OF REAL NUMBERS

IRYNA V. ZAMRIY

For the obvious reason, each two-character coding system of real numbers, including the classic binary numeral system, deserves a special interest. However, the three-character system (in particular, classic ternary numeral system) deserves particular attention. Studies of the three-character system allow us to generalize it to an s-character coding system. Moreover, the ternary system is the most economical among the systems with integer basis [3]. It can be generalized to Q_3-representation, and Q_3-representation can be generalized to Q_3^*-representation [1,2,5,6]. However, these systems have several disadvantages, including bulkiness of representation of numbers. This can be partially avoided by writing series of identical sequences of numbers more compactly that would lead to a new way of coding with an infinite alphabet.

Let q_{0k}, q_{1k}, q_{2k} be sequences of positive real numbers, such that

$$q_{0k} + q_{1k} + q_{2k} = 1; \quad \beta_{0k} = 0, \quad \beta_{1k} = q_{0k}, \quad \beta_{2k} = q_{0k} + q_{1k}, \quad k \in \mathbb{N}.$$

Theorem 1 ([1]). *For any number $x \in [0,1]$, there exists a sequence (a_n), $a_n \in \{0,1,2\}$, such that*

$$x = \beta_{a_1 1} + \sum_{k=2}^{\infty} \left(\beta_{a_k k} \prod_{i=1}^{k-1} q_{a_i i} \right) \equiv \Delta^{Q_3^*}_{a_1 a_2 \ldots a_n \ldots}. \tag{1}$$

Definition 1 ([1]). Presentation of a number x by series (1) is called the Q_3^*-expansion, and its symbolic notation $x = \Delta^{Q_3^*}_{a_1 a_2 \ldots a_n \ldots}$ is called the Q_3^*-representation.

If $q_{ik} = q_i$, we get Q_3-representation of real number.

Note that this representation is quite natural and useful for studying of probability measures with complicated local structure, since the probability distribution function of a random variable with independent ternary digits has a Q_3^*-representation.

If all elements of matrix $\|q_{ik}\|$ are rational numbers, then Q_3^*-representation is called a rational Q_3^*-representation.

The criterion of rationality (irrationality) of a ternary number (a real number $x_0 \in [0,1]$ is rational if and only if its ternary representation is periodic) is not fulfilled in the case of Q_3-representation, its modifications and generalizations.

Theorem 2. *If an infinite stochastic matrix $\|q_{ik}\|$ is periodic and Q_3^*-representation of the number x is periodic, then x is rational.*

Definition 2. Representation $\bar{\Delta}^{Q_3^*}_{\tau_1\tau_2\ldots\tau_n\ldots}$ of real number

$$x = \Delta^{Q_3^*}_{\underbrace{0\ldots0}_{\tau_1}\underbrace{1\ldots1}_{\tau_2}\underbrace{2\ldots2}_{\tau_3}\ldots\underbrace{0\ldots0}_{\tau_{3n-2}}\underbrace{1\ldots1}_{\tau_{3n-1}}\underbrace{2\ldots2}_{\tau_{3n}}\ldots}, \quad \tau_n \in \mathbb{Z}_0,$$

is called the $\bar{Q}_3^*$-*representation* or *modified* $\bar{Q}_3^*$-*representation* of x.

Theorem 3. *If $\bar{Q}_3^*$-representation of a number x in $[0,1]$ and an infinite stochastic matrix $\|\tau_{ik}\|$ are periodic, then x is rational.*

Similar studies are presented in paper [4].

In the talk we offer various applications of the representation to series, functions, dynamical systems, random variables, etc.

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WACŁAW SIERPIŃSKI AT LVIV UNIVERSITY

YAROSLAV G. PRYTULA

Jósef Puzyna was the leading mathematician at Lviv University at the end of XIX–beginning of XX century. Starting from 1892, he headed the only department of mathematics at the university. Due to his efforts, the second department of mathematics was opened in 1900, headed by Jan Rajewskyi until 1906. At the beginning of 1908 J. Puzyna proposed W. Sierpiński to make habilitation at Lviv University. Habilitation took place in July, and starting from the fall W. Sierpiński began lecturing as a docent of the second department of mathematics [1]. At that time W. Sierpiński had the degree of the Candidate of Science from the Tsar's University of Warsaw and Ph.D. from the Jagiellonian University. These degrees were awarded for his works on number theory, where significant results have been obtained by his teacher Georgiy Voronyi. In 1907 W. Sierpiński became interested in the new areas of mathematics: set theory, then theory of functions of real variables, topology and foundations of mathematics. Possibly the first in the world, he started teaching courses on set theory in Lviv. Also, he taught courses on number theory, analytic number theory, higher algebra, irrationalities of the second order, theory of functions of real variables, notions of measure of point sets, Lebesgue integral and others. Lectures by W. Sierpiński were very popular at the Department of Philosophy (he had 89 students in his course on theory of infinite series, 49 students in course on applications of set theory, 85 students in course on critical analysis of fundamental concepts of mathematics). He presented at his seminars the newest works on theory of functions and set theory by G. Cantor, H. Lebesgue, É. Borel, R. Baire, U. Dini, É. Picard and others. O. Nikodym, S. Ruziewicz and M. Zarycki (Zarytskyi) were among his students that time. In 1913, under Sierpiński's supervisory, S. Mazurkiewicz defended his dissertation on topology and S. Ruziewicz on theory of functions. Also, in 1913 W. Sierpiński invited to Lviv Z. Janiszewski (who got doctoral degree in Paris in 1911). During Lviv period in his life Sierpiński elaborated and published the following textbooks: "Teorya liczb niewymiernych" (1910), "Zarys teoryi mnogośći" (1912), "Teorya liczb" (1914).

In September 1910 W. Sierpiński got the position of Extraordinary Professor. However, this decision of Lviv University was not supported by the ministry in Vienna in 1912. Sierpiński got this title in 1919 as a professor of Warsaw University [1].

At the beginning of the First World War W. Sierpiński lived on Russian territory and, as an Austrian citizen was deported to Viatka. A year after he got a permission to move to Moscow. Lviv University was trying to support him via American consulate in Vienna. In February 1918 Sierpiński returned to Lviv via Finland and Sweden and continued teaching there for one semester.

In the fall of 1918 W. Sierpiński was invited to Warsaw University. There, together with Z. Janiszewski and S. Mazurkiewicz, they established the journal "Fundamenta Mathematicae". Sierpiński continued to cooperate with mathematicians in Lviv, the most fruitful and friendly relations he had with S. Ruziewicz and S. Banach. In 1929 Lviv University granted W. Sierpiński the honorary doctorate Honoris Causa.

There are many signs of very respectful attitude of W. Sierpiński to his teacher, one of which was his talk on memory of Georgiy Voronyi published in "Wiadomości matematyczne" [2]. This talk was delivered on November 28, 1908, and it begins with the words "These days the number theory lost professor G. Voronyi, it's the most prominent representative". Then Sierpiński describes main results of Voronyi and his views on mathematical education. He was the first who presented Voronyi's teaching instructions. Sierpiński concluded: "... professor Voronyi was one of few professors of Russian university in Warsaw who was able to grow up the interest to the science and who forever left thankful memories".

W. Sierpiński, together with J. Puzyna, was a forerunner of ideas that blossomed in 20–30ies in Lviv mathematical school [3]. Grown on the ideas of Ukrainian mathematician G. Voronyi, he continued raising interest to mathematics in many young people, including M. Zarycki. In his doctoral dissertation [4] defended in 1930, M. Zarycki continued investigation of foundations of topology done by K. Kuratowski, the student of W. Sierpiński.

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[1]The explanations of Cyrillic abbreviations are on page iv.

STANDARDLESS ADAPTIVE SELF-CALIBRATION ARRAY PHOTODETECTORS AND ITS PERSPECTIVES IN THE LIGHT OF VORONOI IDEAS

VLADIMIR SAPTSIN

I learned about the personality of George Voronoi reading his teenage diary. It was several years ago.

The young scientist (Voronoi) demonstrated his unique mathematical gift and solved a number of problems of classical mathematics relevant at his time. He also received fundamentally new results ahead of his time for almost a century. So, he did not attract the author's interest. As it turned out, the world is small. In the early '70s as a student and a member of a climbing club of the MSU, the author repeatedly went to the forests around Moscow, and sitting by the campfire he listened the stories on mathematics and mathematicians by B. N. Delone, one of the brightest followers of G. F. Voronoi. A couple of his statements would be appropriate to remind, "A mathematician is the same as a musician, you cannot just become one, they can only be born" and "Contrary to a popular belief that the peak manifestation of mathematical talent is between the ages of 20–30, creative longevity in mathematics is not uncommon". These statements can be fully attributed to the Voronoi, who died aged 40, on the rise of his mathematical genius.

Since the beginning of the '80s of the twentieth century, the author became a researcher at the Lebedev Physics Institute (Kuibyshev branch) and was engaged in reaserch in the digital thermovision and its applications.

At the best thermal cameras of that time a single photodetector elements (PD) were used. They "viewed" row by row the entire frame with the help of optical-mechanical scanning.

The very first line of the PD use showed some problems associated with the spread of sensitivity of the PD and the relevant picture distortion.

The traditional approach to solution of this problem based on preliminary calibration is to use external standard emitters being uniform in the area, and it is used in modern digital cameras with array photodetectors [1].

In the late '80s the author realized the fundamental nature of the problems associated with the inhomogeneity of the PD matrix, as well as with the limited capability of its traditional calibration. He suggested and substantiated theoretically a self-calibration method without the use of standards, from the scene signals directly [2].

The method is based on the use of two-dimensional low-amplitude matrix scan – per pixel (picture element) in the horizontal and vertical directions, and it allows comparing of signals of any pair of neighboring elements during their sighting on the same element of the scene.

From a geometrical point of view, the PD matrix is the simplest (two-dimensional) case of partitioning of the surface into parallelohedra – squares, and its single-pixel scans (displacements) vertically and horizontally are the corresponding symmetry transformations.

The case of the partition of the photosensitive area into regular hexagons – primitive two-dimensional parallelohedra – is interesting, but it have not been investigated yet.

The idea of standardless self-calibration of spatially distributed scanning "sensors", dynamically adapting to changes in their parameters and the observed "scene" is valuable not only for technical applications. This idea is realized in the visual system of man and the higher mammals, in other natural, biological and socio-organized systems.

It is of interest to generalize the proposed calibration method to an n-dimensional case. The ideas and results of Voronoi on the algebraic representation of multi-dimensional spatial structures should be used. These tasks are waiting to be explored.

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SOME NEW DOCUMENTS ABOUT GEORGES VORONOÏ'S PARENTS FOUND IN KYIV'S ARCHIVES

HALYNA SYTA

Father of Georges Voronoï (Voronyi) – Theodosiy Yakovych Voronyi (1837–1910) was a philologist, from 1857 to 1861 he studied at the Historical-Philological Department of St. Volodymyr Kyiv University, in 1861–1862 – at the teaching courses at the University [1].

Theodosiy was a socially active person. In the archives [2] there exists some information about creation in 1859–62 of several Sunday schools in Kyiv for young workers first initiated by Th. Voronyi. The archives [3] hold the original of the collective gratitude letter by 68 Kyiv Sunday school teachers (and Th. Voronyi was among them) to the poet Taras Shevchenko who sent 50 copies of his poetry book "Kobzar" to Kyiv Sunday schools. The list of signatories included several teachers of Kyiv Podil female Sunday school. In particular, here we find the names of Mary Lychkova, Sophia Lychkova, Elizabeth Lychkova and Cleopatra Lychkova. No doubt that Cleopatra Lychkova, Theodosiy Voronyi's wife, and the young lady from the Kyiv Podil female Sunday school who had signed this letter was the same person. Thus, using some more archive documents we learned that Georges Voronoï's mother came from a family of hereditary honorary citizens of Kyiv, whose representatives took part in the cultural life of the city and in some charity events. No wonder that Georges Voronoï in his letter to professor of Kyiv University Boris Ya. Bukreyev (this letter is held in [4]) mentioned Kyiv as a city, familiar and close to him since his childhood years.

Theodosiy Voronyi student's work "Necessary Conditions for Preparation of Bibliographic Textbook on Russian History for Sunday Schools" is held in [1]. In it he presented his attitude to Sunday school, stressed importance of dissemination of the historical knowledge and awareness of the social sciences. On his view, "success in political and social life is impossible without people's knowledge about moral sciences".

After graduation from his courses, Theodosiy Voronyi received an appointment as a teacher to Nemyriv Gymnasium, where he worked in 1863–1864. In [2] there is some information about Nemyriv Gymnasium at that time. Some Polish-born students of this high school organized a secret society, and they offered Theodosiy to join it, but he refused because he did not share their views. The conspirators were afraid of getting caught by police and decided to kill him. But the student who was charged with the attempt to commit to do it respected and loved Theodosiy, he decided to tell to the police himself. After this incident, in 1864, Theodosiy Voronyi was transferred to Nizhyn Prince Bezborodko Lyceum as a professor of Russian literature, and he worked there until 1872. Thus, Georges Voronoï's family during his early childhood (in 1868–1872) probably lived in Nizhyn, the beautiful ancient city, and they spent summer vacations in Zhuravka, as their native small town Zhuravka was quite near to Nizhyn. Then Theodosiy Voronyi was appointed the Director to Kyshyniv Gymnasium and later on, from 1875 to 1881 he headed the Berdyansk Gymnasium.

In published documents one can find a number of positive references about Th. Voronyi from famous public persons who knew him (Olena Pchilka, Mykhailo Drahomanov, Ivan Nechuy-Levytskyi etc.). Theodosiy Voronyi was acquianted with Hryhoriy Galagan (1819–1888), a renowned public person and philanthropist. Galagan's family had an estate in the village Sokyryntsi near Zhuravka. Archives [4] keep several letters from Theodosiy Voronyi to Hryhoriy Galagan about pedagogy and selection of school teachers. In September 1874, the 4-class high school was opened in Pryluky, a town quite close to Zhuravka, and the Ministry of Education approved H. Galagan to be an honorary trustee of this school. Thus, perhaps it was H. Galagan who helped to transfer Theodosiy Voronyi to Pryluky. He was the director of this school from 1882 to 1887 when he retired.

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[1]The explanations of Cyrillic abbreviations are on page iv.

Index of Authors

R

S

T

U

V

W

Y

Z

Наукове видання

Fifth International Conference on Analytic Number Theory and Spatial Tessellations

Abstracts

П'ята міжнародна конференція з аналітичної теорії чисел і просторових мозаїк

Тези доповідей

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