A specialized mathematician: Julius Hurwitz, and an application of his complex continued fraction.

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What happened on July 13, 1895?
Who was J. Hurwitz?

"A special kind of continued fraction expansion of complex numbers."
What happened on July 13, 1895?
Who was J. Hurwitz?
What happened on 13 July 1895?
The detours before.
The continued fraction expansion of a complex number

\[ z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_n + \cfrac{1}{\ldots}}}} \in \mathbb{C} \]

is based on partial quotients \( a_i \in \mathbb{C}, i = 0, \ldots, n \).

The algebraic structure of the partial quotients influences arithmetical characteristics of the continued fraction.
A. Hurwitz considered the set of the Gaussian integers

\[ \mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z} \]

as possible partial quotients.

(Ueber die Entwicklung Complexer Größen in Kettenbrüche, 1887)
J. Hurwitz considered the subset

$$(1 + i)\mathbb{Z}[i] = (1 + i)\mathbb{Z} + (1 - i)\mathbb{Z}$$

of the Gaussian integers $\mathbb{Z}[i]$ as possible partial quotients.

(Ueber eine besondere Art der Kettenbruch-Entwicklung
Complexer Größen, 1895)
Let $\Delta$ be the volume of the fundamental domain and let $I$ be set of possible partial quotient. Then we know for the approximation quality:

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For any $z \in \mathbb{C} \setminus \mathbb{Q}(I)$ there exist infinitely many $p, q \in I$ such that

$$\left| z - \frac{p}{q} \right| < \frac{\sqrt{6}\Delta}{\pi|q|^2}.$$
Salomon Hurwitz and Sons 
(1813 - 1885) 

Childhood in Hildesheim
Family.

Salomon Hurwitz and Sons
(1813 - 1885)


"Moreover he set a high value that the young boys started smoking since he could not imagine a proper man without cigar [...]"

(Biographical dossier of Ida Hurwitz, Archive ETH Zürich.)
Childhood in Hildesheim
Education.

School certificate,
April 07, 1870
(Municipal archive of Hildesheim)
Childhood in Hildesheim

Education.

School certificate, April 07, 1870

(Municipal archive of Hildesheim)
Once [Julius] H. visited (unauthorized) a tavern. No other impertinent behaviour is noted.
"All of the three brothers showed a particular talent for mathematics, which was documented at a very early stage [...] concerning Adolf."
Childhood in Hildesheim
Education: Schubert.

Hermann Caesar Hannibal Schubert
(1848 - 1911)
Childhood in Hildesheim
Education: Schubert.

Hermann Caesar Hannibal Schubert
(1848 - 1911)

Dissertation in Halle, 1870
(Archive of the University of Halle)
"Schubert even visited the father to convince him to let both sons choose the studies of mathematics [...]. [Salomon] talked to a very wealthy and childless friend E. Edwards, who offered to bear the costs of the studies for one of the sons. Dr. Schubert elected Adolf."
Julius became banker in Nordhausen, Thuringia.

(Salomon to Julius, ETH archive)
Julius’ detours begin.
... in Hamburg and Hanover.

"[...] the second brother, Julius, lives in Hamburg working as an exporter."

(Adolf Hurwitz to Luigi Bianchi, October 30, 1885.)

"Since his uncle Adolph’s death Max and Julius were owning the bank ’Adolph M. Wertheimer’s Nachf.’ in Hanover, however, they felt uncomfortable in this business. [...]"
A professorship changes Julius’ life.

1884.

Salomon to Julius Hurwitz in Hanover on April 1, 1884
”It is an extraordinary event, and we cannot thank enough the destiny, that our Adolf is so gifted, so acclaimed and is already recognized by the most important mathematicians as excellent person. [...] So your youngest brother is professor by the age of 25 and after only two years of being private lecturer [in Göttingen]!”

(Salomon to Julius, April 1, 1884, ETH library.)
"[...] Hence, first Julius quitted in order to return to school at the age of 33 years and finished with the school examination [...]

Excerpts of the Abitur certificate of Julius Hurwitz

(Archive of the University of Halle.)
"[...] Afterwards, in 1890, at the age of 33, he finally followed his brother to Königsberg."

"Warmest greetings to you and Hurwitz major and minor natu."

(H. Minkowski to D. Hilbert, February 9, 1892.)


Hilbert, David. Die eindeutigen Funktionen mit linearen Transformationen in sich, nach Vorlesungen in Königsberg 1892 SS ausgearbeitet von Julius Hurwitz.

(Archive of the ETH.)
In 1892 Adolf Hurwitz was offered the Georg Frobenius’ chair at the Eidgenössische Polytechnische Hochschule Zürich, a polytechnic, and Hermann Amandus Schwarz’s chair at the University of Göttingen. Adolf Hurwitz chose Zurich.

"[Also] his brother Julius followed him soon to Zurich, where he wrote his doctoral thesis for which he had received the subject from his brother."
Working on the dissertation.
Subject from his brother: my guess!

Theorem A. Hurwitz (1886)
The root of a quadratic (irreducible) equation with complex integer coefficients has a periodic continued fraction expansion.

(Mathematical diaries of A. Hurwitz, no.5., 1886, Archive ETH.)
Working on the dissertation.
Subject from his brother: my guess!

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Theorem A. Hurwitz (1886)
The root of a quadratic (irreducible) equation with complex integer coefficients has a periodic continued fraction expansion.

An assumption about Julius’ task: Transferring the theorem to the case that the set of possible partial quotients is restricted.
Julius as a doctoral student.
University of Halle.

New University of Halle, 1836
(Lithography of W. Breye.)
Working on the dissertation.
Help from the brother.

(Mathematical diaries of A. Hurwitz, no.9., November 04 1894, Archive ETH.)
Working on the dissertation.
Help from the brother.

(Mathematical diaries of A. Hurwitz, no.9., November 04 1894, Archive ETH.)
Es besteht demnach folgendes Gesetz für die Aufeinanderfolge der Teilnener:

\[(B) \begin{align*}
    r = 0 & \quad \Rightarrow \quad a_r = 1 + i, \quad \text{so} \quad a_{r+1} \text{ nicht Typen:} & \quad 2, \quad 1 - i, \quad -2i \\
    r = -1 + i: & \quad r = -1 - i: & \quad r = 1 - i: & \quad \text{Typen:} & \quad -2i, \quad -1 - i, \quad -2 \quad 2, \quad -1 + i, \quad 2i \quad 2i, \quad 1 + i, \quad 2
    \end{align*}\]

Für das Folgende ist es zweckmässig, die Entwicklungen erster Art derjenigen Grössen, welche auf einem der Bogen \(B_{1+i}, B_{1-i}, B_{-1+i}, B_{-1-i}\), oder auf einer der Geraden \(G_1^-, G_1^+, G_{-1}^-, G_{-1}^+\), liegen, von den übrigen gesondert zu betrachten.
Finally: the dissertation in June 1895.
Who was the real advisor?

"Eight days ago the news from my brother came that he passed the doctoral exam very good. The dissertation concerns continued fraction expansions of complex numbers. I determined the subject"

A. Hurwitz to D. Hilbert, June 19, 1895

(Collected letters from Adolf Hurwitz, Archive of the Mathematical Institute in Göttingen.)
Julius’ time in Halle.
A plausible answer: Wangerin.

Albert Wangerin (1844 - 1933)

Wangerin advised the remarkable number of 53 students to their dissertation, Julius Hurwitz being number 28 of them. The supervised topics range from calculus, in particular, differential equations, via analytic and differential geometry to mathematical physics; there are only two theses from number theory.
In 1985 the Japanese mathematician Shigeru Tanaka published (nearly) the same complex continued fraction considering a completely different point of view.

**Our task.**

On behalf of methods from ergodic theory we want to derive information of the approximation characteristics of Julius Hurwitz’s respectively Shigeru Tanaka’s complex continued fraction algorithm.
Our ingredients are a probability space \((X, \Sigma, \mu, T)\) with non-empty set \(X\), a \(\sigma\)-algebra \(\Sigma\) on the set \(X\), a probability measure \(\mu\) on \((X, \Sigma)\) and a measure preserving transformation \(T : X \to X\), i.e.

\[\mu(T^{-1}E) = \mu(E)\]

for \(E \in \Sigma\).

Notice! Here, we model a time-process and assume \(T\) as 'shift into the future' whereas \(T^{-1}\) can be considered as 'shift into the past'.
We consider the fundamental domain

\[ X := \{ z = (1 + i)x + (1 - i)y : \frac{-1}{2} \leq x, y \leq \frac{1}{2} \} \]

and the transformation

\[ T : X \to X, \quad Tz := \frac{1}{z} - \left[ \frac{1}{z} \right] T, \]

with \([ \cdot ]_T : \mathbb{C} \to (1 + i)\mathbb{Z}[i],\]

\[ [z]_T := \left[ x + \frac{1}{2} \right] (1 + i) + \left[ y + \frac{1}{2} \right] (1 - i). \]
Setting

\[ a_n := \left[ \frac{1}{T^{n-1}z} \right] \in (1 + i)\mathbb{Z}[i] \]

the transformation \( T \) serves as operator in Tanaka’s complex continued fraction. We receive

\[ z = [0; a_1, a_2, \ldots, a_n + T^n z]. \]

Here, \( T \) can be considered as a shift in the sequence of partial quotients.
This transformation $T$ is \textit{ergodic} which means that for each $\mu$-measurable, $T$-invariant set $E$ either

\[ \mu(E) = 0 \text{ or } \mu(E) = 1. \]

\textbf{Birkhoff, 1931}

Let $T$ be a measure preserving, ergodic transformation on a probability space $(X, \Sigma, \mu)$. If $f$ is integrable, then

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n \leq N} f(T^n x) = \int_X f \, d\mu \]

for almost all $x \in X$. 

\[ \]
Tools from ergodic theory.
Dual transformation.

We define a dual transformation \( S : Y \to Y \) by

\[
Sw = \frac{1}{w} - \left[ \frac{1}{w} \right]_S \quad \text{for } w \neq 0.
\]

With

\[
Y = \{ w \in \mathbb{C} : |w| \leq 1 \}
\]

and \( [w]_S = a \in (1 + i)\mathbb{Z}[i] \) its convergents \( \frac{r_n}{s_n} \) satisfy

\[
\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \ldots, a_1] = \frac{r_n}{s_n} =: V_n.
\]

Tanaka, Duality

The sequence of partial quotients \( a_1, \ldots, a_n \) is \( T \)-admissible if and only if the reverse sequence of partial quotients \( a_n, \ldots, a_1 \) is \( S \)-admissible.
We construct a *natural extension* $R$ containing information of $T$ as well as information of $S$. Thus, $R$ includes 'future' and 'past' of the sequence of partial quotients. We define $R : X \times Y \to X \times Y$ by

$$R(z, w) = \left( Tz, \frac{1}{a_1(z) + w} \right),$$

where $a_1 = \left[ \frac{1}{z} \right]_T$.

How does $R^n(z, w)$ look like?
Tanaka proved

**Natural Extension**

*R* is a natural extension; in particular *T* and *S* are ergodic and the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(z, w) = \frac{1}{|1 + zw|^4}$$

is the density function of an absolutely continuous $R$-invariant measure.
We examine

\[ \theta'_n := |q_n|^2 |z - \frac{p_n}{q_n}|, \]

an analogue to the so called *approximation coefficients* from real theory. Tanaka stated the equation

\[
\frac{\theta'_n}{|q_n|^2} = \left| z - \frac{p_n}{q_n} \right| = \frac{|2i(-1)^n T^nz|}{|q_n(q_n + q_{n-1} T^n z)|}. \]

Hence,

\[
\varphi_n = \frac{\theta'_n}{2} = \frac{|T^n z|}{|1 + V_n T^n z|} \leq \frac{1}{(1 - |V_n T^n z|)}. \quad (1)
\]
Our aim is to transfer the *Döblin-Lenstra-Conjecture* to the complex case:

**Complex Conjecture**

The limiting distribution function

\[ l(g) := \lim_{n \to \infty} \frac{1}{n} \left| \{ j; j \leq n, \varphi_j(z) \leq g \} \right| \]

exists.
For approximation coefficients, we know

\[ \varphi_j(z) \leq g \iff \left| \frac{T^j z}{1 + V_j T^j z} \right| \leq g, \quad (2) \]

and

\[ R(z, y) = \left( Tz, \frac{1}{a_1 + y} \right), \]

where \( a_1 \coloneqq \left[ \frac{1}{z} \right]_T \) is the first partial quotient of \( z \). We have

\[ R^i(z, y) = (T^i z, [0; a_i, a_{i-1}, \ldots, a_2, a_1 + y]), \]

and in particular

\[ R^i(z, 0) = (T^i z, V_i). \]
Because of (2), this leads to a new equivalence:
We have \( \varphi_i(z) \leq g \) if and only if

\[
R^i(z, 0) \in A_g := \{(u, v) \in X \times Y : \left| \frac{u}{1 + uv} \right| \leq g \},
\]
respectively

\[
R^i(z, 0) \in A_g = \{(u, v) \in X \times Y : \left| \frac{1}{u} + v \right| \geq \frac{1}{g} \}.
\]

As shown by Tanaka, there exists a probability measure \( \mu \) with density function \( h(z, y) = \frac{1}{|1+zy|^4} \), such that the quadrupel \((X \times Y, \Sigma, \mu, R)\) forms an ergodic system with \( A_g \in \Sigma \).
Applying the Ergodic Theorem of Birkhof leads to

**Theorem 1**

For almost all $z$ the distribution function exists and is given by

$$l(g) = \lim_{n \to \infty} \frac{1}{n} \sum_{i \leq n} \mathbb{I}_{A_g}(R^i(z, 0)) = \mu(A_g) = \frac{1}{G} \int \int_{A_g} \frac{d\lambda(u, v)}{|1 + uv|^4},$$

where $G := \int \int_{X \times Y} \frac{d\lambda(u, v)}{|1 + uv|^4}$. 
Döblin-Lenstra-Conjecture.
Explicit Calculation.

Just some remarks about the explicit calculation of

$$G := \int \int \mathcal{X} \times \mathcal{Y} \frac{d\lambda(u,v)}{|1 + uv|^4} :$$

- We simplify $|1 + uv|^2 = (1 + uv)(1 + \overline{uv})$. 


Döblin-Lenstra-Conjecture.
Explicit Calculation.

Just some remarks about the explicit calculation of
\[ G := \int \int_{X \times Y} \frac{d\lambda(u,v)}{|1+uv|^4} : \]

- We simplify \(|1 + uv|^2 = (1 + uv)(1 + \overline{uv})\).
- The formula \(\frac{1}{(1-x)^2} = \sum_{m \geq 1} mx^{m-1}\), being valid for \(|x| < 1\), is used forward and backward.
Just some remarks about the explicit calculation of $G := \int \int_{X \times Y} \frac{d\lambda(u,v)}{|1+uv|^4}$:

- We simplify $|1 + uv|^2 = (1 + uv)(1 + \overline{uv})$.
- The formula $\frac{1}{(1-x)^2} = \sum_{m \geq 1} mx^{m-1}$, being valid for $|x| < 1$, is used forward and backward.
- To prevent difficulties from singularities, we firstly consider subdomains $X_r$ and $Y_r$ inheriting the symmetric structure of $X$ and $Y$. 
Döblin–Lenstra–Conjecture.
Explicit Calculation.

Just some remarks about the explicit calculation of
\[ G := \int \int_{X \times Y} \frac{d\lambda(u,v)}{|1 + uv|^4}. \]

- We simplify \(|1 + uv|^2 = (1 + uv)(1 + \overline{uv})\).
- The formula \(\frac{1}{(1-x)^2} = \sum_{m \geq 1} mx^{m-1}\), being valid for \(|x| < 1\), is used forward and backward.
- To prevent difficulties from singularities, we firstly consider subdomains \(X_r\) and \(Y_r\) inheriting the symmetric structure of \(X\) and \(Y\).
- We split \(X\) into four domains of equal sizes along the axes and facilitate the calculation of the integral over \(X\).
Finally, we receive

Theorem 2

The normalizing constant $G$ is given by

$$G = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2},$$

where $\sum \frac{(-1)^n}{(2n+1)^2}$ is known as Catalan constant.

Remark: The explicit calculation of $\int \int_{A_g}$ is still in its infancy.
Thank you...  
... for your attention.

Thank you for your attention!
Announcement.
... excursion and barbecue!

This afternoon, there will be an excursion and in the evening we will have BBQ at the Schönstattzentrum!