The cohomology of period domains for reductive groups over local fields

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Abstract. We compute the étale cohomology of period domains over local fields in the case of a basic isocrystal for quasi-split reductive groups. Period domains, which have been introduced by Rapoport and Zink [RZ], are open admissible rigid-analytic subsets of generalized flag varieties. They parametrize weakly admissible filtrations of a given isocrystal with additional structure of a reductive group.

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0 Introduction

In the classical setting the theory of period domains has been introduced by Griffiths [G]. They are certain open subsets of generalized flag varieties over the field of complex numbers. They parametrize polarized $\mathbb{R}$-Hodge structures of a given type. In the p-adic setting the notion of a period domain exists as well, representing an analog of the classical case. Here the notion of a weakly admissible isocrystal is an important ingredient, which has been established by Fontaine [Fo]. In order to explain this concept, let L be an algebraically closed field of characteristic $p > 0$. Denote by $W(L)$ the associated Witt ring with fraction field $K_0 = K_0(L)$. A filtered isocrystal $(V, \Phi, \mathcal{F})$ is an isocrystal (V, Φ) over L (see section 1 for the definition of an isocrystal) together with a $\mathbb{Z}$-filtration $\mathcal{F}$ of the vector space V, which is defined over a finite field extension K of K_0. The isocrystal $(V, \Phi, \mathcal{F})$ is called weakly admissible if

$$\sum_i i \cdot \dim gr^i_{\mathcal{F}'}(V' \otimes_{K_0} K) \leq \mathrm{ord}_p \det(\Phi')$$

for all subisocrystals (V', Φ') of (V, Φ) and equality for $(V', \Phi') = (V, \Phi)$. Here $\mathcal{F}'$ denotes the filtration on $V' \otimes_{K_0} K$ induced by $\mathcal{F}$. Fix an isocrystal (V, Φ) over L. Considering only filtrations of a specified type on V, the weakly admissible ones form a rigid-analytic variety $\mathscr{F}^{wa}$ over K_0. This space is an open rigid-analytic subset of a generalized flag manifold $\mathscr{F}$. It is called the period domain of that specified data. By applying the machinery of Tannaka formalism, we may extend the theory above - the $GL(V)$-case - to arbitrary reductive groups G over $\mathbb{Q}_p$. For a detailed description see [RZ]. The most prominent example of a period domain is the Drinfeld upper half plane $\Omega(V)$ which is attained by a trivial isocrystal and the projective space $\mathcal{F} = \mathbb{P}(V)$. The Drinfeld upper half plane is just the complement of all $\mathbb{Q}_p$-rational hyperplanes in the $\mathbb{P}(V)$, i.e.,

$$\Omega(V) = \mathbb{P}(V) \setminus \bigcup_{\substack{H \subsetneq V \\ H \text{ is } \mathbb{Q}_p\text{-rat}}} \mathbb{P}(H).$$

This example is discussed extensively in the paper [SS] by Schneider and Stuhler.

A natural problem which arises with period domains is the determination of their cohomology, in this case the étale cohomology with compact support. The cohomology groups are equipped in a natural way with smooth actions of a p-adic group $J(\mathbb{Q}_p)$ and the Galois group $Gal(\overline{E}_s/E_s)$ of some local field E_s. Here J denotes the isomorphism group of the given isocrystal, which is an inner form of a Levi subgroup of G. Further, E_s is the field of definition of $\mathcal{F}^{wa}$. The étale cohomology with torsion coefficients of $\Omega(V)$ has been computed in [SS]. For period domains where the considered isocrystal is basic (i.e., J is an inner form of G), there exists a formula for the continuous ℓ-adic Euler-Poincaré characteristic in the Grothendieck-group of smooth $J(\mathbb{Q}_p) \times Gal(\overline{E}_s/E_s)$-representations due to Kottwitz and Rapoport [R1]-[R3]. Period domains are also definable over finite base fields $\mathbb{F}_q$, where much more is known. Instead of filtered isocrystals one considers filtered vector spaces $(V, \mathcal{F})$, where V is a finite-dimensional vector space over $\mathbb{F}_q$ and $\mathcal{F}$ is a filtration of V defined over some finite extension of $\mathbb{F}_q$. Subisocrystals are replaced by rational subspaces of V. In contrast to the p-adic case the corresponding period domain has the structure of a Zariski open subvariety of $\mathcal{F}$. The reason is that there are only finitely many rational subspaces of V. A precise formula for the ℓ-adic cohomology in the $GL(V)$-case has been computed in [O1]. A generalization of this computation to arbitrary reductive groups G over $\mathbb{F}_q$ is given in [O2] due to an idea of B. Totaro.

The aim of this paper is to compute the étale cohomology of $\mathcal{F}^{wa}$ for a basic isocrystal with coefficients in $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$ is a suitable chosen number with $(n,p) = 1$ (compare Theorem 1.1). The computation is carried out for a quasi-split reductive group G over $\mathbb{Q}_p$. The proof is based on the idea in [O1], [O2], which works as follows. In loc.cit. there has been constructed an acyclic complex of étale sheaves on the complement Y of the period domain, which is in that case a closed subvariety of $\mathcal{F}$. The index set of this complex corresponds to the Tits-building of the finite group of Lie type $G(\mathbb{F}_q)$. The étale sheaf associated to a facet is just the constant sheaf on the closed subvariety consisting of points where each vertex of the facet damages the weak admissibility. The resulting spectral sequence degenerates in E_2 and computes the cohomology of Y. If one tries to adapt this idea to the p-adic case, one is confronted with two difficulties. The first one is the fact that the complement of $\mathcal{F}^{wa}$ in $\mathcal{F}$ is in general not a rigid-analytic variety. This problem is solved by working in the bigger category of adic spaces defined by R. Huber [H1]. In the language of adic spaces, the complement induces a closed pseudo-adic subspace Y^{ad} of the adic flag manifold $\mathcal{F}^{ad}$. The second problem consists of having infinitely many subobjects of our fixed isocrystal, where the adapted complex is not defined. Here the solution is to define a complex on Y^{ad}, similar to the case of a finite field, but where the sheaves involved give rise to locally constant sections with respect to the p-adic topology on the Tits building. This complex is acyclic as well. Therefore, we get a spectral sequence degenerating in the E_2-term and which converges towards the cohomology of Y^{ad}. In contrast to the finite field situation, where the Frobenius automorphism acts semi-simply on representations, we use this time the vanishing of certain Ext-groups (see [D] resp. [O3])

$$Ext^1_{J(\mathbb{Q}_p)}(v_P^J, v_Q^J)$$

to conclude that the canonical filtration on $E_2 = E_\infty$ splits. Here P respectively Q are $\mathbb{Q}_p$-parabolic subgroups of J and v_P^J respectively v_Q^J denote the corresponding generalized Steinberg representations (see Section 1 for their definition). Applying the long exact cohomology sequence to the triple $(\mathcal{F}^{wa}, \mathcal{F}^{ad}, Y^{ad})$ we obtain finally the cohomology of $\mathcal{F}^{wa}$.

Now we come to the content of this paper. The main result is formulated in Section 1. Section 2 deals with the connection to Geometric Invariant Theory, preparing the foundation for the acyclicity of the fundamental complex introduced in section 3. At the end of the third part we give the proof of its acyclicity. The fourth section deals with the computation of some cohomology groups needed for the evaluation of the induced spectral sequence. The evaluation is done in the fifth section. Finally, in the last section we give a concluding example in the case $G = GL_4$.

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1 The main result

Let L be an algebraically closed field of characteristic $p > 0$. We denote by

$$K_0 = \mathrm{Quot}(W(L))$$

the corresponding fraction field of the ring of Witt vectors and by $\sigma \in \mathrm{Aut}(K_0/\mathbb{Q}_p)$ the Frobenius homomorphism. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ with Galois group $\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Recall that an isocrystal over L is a pair (V, Φ) consisting of a finite-dimensional vector space V over K_0 together with a σ-linear bijective endomorphism Φ of V.

Let G be a connected quasi-split reductive group over $\mathbb{Q}_p$. We repeat briefly the theory of isocrystals with G-structure [RR], which has been first introduced by Kottwitz [K1]. An isocrystal with G-structure on L is an exact faithful tensor functor

$$Rep_{\mathbb{Q}_p}(G) \longrightarrow Isoc(L)$$

from the category of finite-dimensional rational G-representations over $\mathbb{Q}_p$ into the category of isocrystals over L. Following [RR] Remark 3.5, every such isocrystal is induced by an element $b \in G(K_0)$ in the following way. For a finite-dimensional rational representation V of G, we put

$$N_b(V) := \Big(V \otimes_{\mathbb{Q}_p} K_0, b(id_V \otimes \sigma)\Big)$$

which defines an isocrystal over L. The map on the level of morphisms is the obvious one. This construction yields an isocrystal with G-structure N_b, where two elements $b, b' \in G(K_0)$ define isomorphic isocrystals if and only if they are σ-conjugated, i.e., if there exists a point $g \in G(K_0)$ with $b' = gb\sigma(g)^{-1}$.

Let $b \in G(K_0)$. Consider the tensor functor

$$\begin{array}{ccc} Rep_{\mathbb{Q}_p}(G) & \longrightarrow & Grad(Vec_{K_0}, \mathbb{Q}) \\ V & \longmapsto & \bigoplus_{i \in \mathbb{Q}} V_i \end{array}$$

from the category $Rep_{\mathbb{Q}_p}(G)$ into the category of $\mathbb{Q}$-graded vector spaces over K_0 which is given by the slope-grading of the isocrystal N_b. Using the Tannaka formalism, we obtain a rational 1-PS

$$\nu_b : \mathbb{D} \longrightarrow G_{K_0}$$

defined over K_0, which induces this tensor functor. It is called the slope homomorphism of N_b. Here $\mathbb{D}$ is the algebraic pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. We consider the multiplicative group $\mathbb{G}_m$, via the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ of their character groups, as a quotient of $\mathbb{D}$. If $b' \in G(K_0)$ is σ-conjugated to b by an element $g \in G(K_0)$, then we get $\nu_{b'} = Int(g) \circ \nu_b$. For the remainder of this paper, we fix a decent element $b \in G(K_0)$, i.e., its slope homomorphism ν_b satisfies an equation

$$(b\sigma)^s = s\nu_b(p)\sigma^s$$

in the semi-direct product $G(K_0) \rtimes \langle \sigma \rangle$ for some integer $s \in \mathbb{N}$, such that $s\nu_b : \mathbb{D} \longrightarrow G_{K_0}$ factors through $\mathbb{G}_m$. This is not really a restriction, since in any σ-conjugacy class there is a decent element (see [K1] 4.3) due to the algebraically closeness of L. It follows that $b \in G(\mathbb{Q}_{p^s})$ and that $\nu := \nu_b$ is defined over $\mathbb{Q}_{p^s}$.

Suppose that there is given a 1-PS

$$\lambda : \mathbb{G}_m \longrightarrow G_K$$

of G defined over a field extension K of K_0. We obtain for every finite-dimensional rational representation V of G a $\mathbb{Z}$-graded vector space

$$V_K = \bigoplus_{i \in \mathbb{Z}} V_i^{\lambda}$$

and a decreasing $\mathbb{Z}$-filtration $\mathcal{F}_\lambda(V)^\bullet$ on V_K defined by

$$\mathcal{F}_\lambda(V)^i = \bigoplus_{j \geq i} V_j^{\lambda},\ i \in \mathbb{Z}.$$

Thus, the pair (b, λ) yields a tensor functor from $Rep_{\mathbb{Q}_p}(G)$ into the category of filtered isocrystals over K. Following [RZ] the pair (b, λ) is called weakly admissible if for all faithful finite-dimensional G-representations V, the filtered isocrystal $(N_b(V), \mathcal{F}^\bullet_\lambda(V))$ is weakly admissible, i.e., if

$$\sum_i \dim gr^i_{\mathcal{F}^\bullet_\lambda(V)}(N' \otimes_{K_0} K) \leq \mathrm{ord}_p \det(b(\sigma \otimes id_V)|N')$$

for every subisocrystal N' of $N_b(V)$ and equality for $N' = N_b(V)$. It follows from the tensor product theorem of Faltings [F] respectively Totaro [T], which says that the tensor product of two weakly admissible filtered isocrystals is again weakly admissible, that it suffices to check the weak admissibility for a single faithful representation. Although Colmez and Fontaine have proved in [CF] that weak admissibility is the same as admissibility, we will use the former term in the sequel.

We fix a conjugacy class

$$\{\mu\} \subset X_*(G)$$

of one-parameter subgroups of G over $\overline{\mathbb{Q}}_p$. Denote by $Stab_{\Gamma_{\mathbb{Q}_p}}(\{\mu\}) \subset \Gamma_{\mathbb{Q}_p}$ the fixed group of $\{\mu\}$. Let

$$E = \Big\{ x \in \overline{\mathbb{Q}}_p;\ \tau(x) = x \ \ \forall \tau \in Stab_{\Gamma_{\mathbb{Q}_p}}(\{\mu\}) \Big\}$$

be the Shimura field of $\{\mu\}$, a finite intermediate field of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$. Since G is quasi-split, we may apply a lemma of Kottwitz ([K2] Lemma 1.1.3) which guarantees the existence of a 1-PS $\mu \in \{\mu\}$ that is defined over E. Hence, the conjugacy class $\{\mu\}$ defines a flag variety

$$\mathcal{F} := \mathcal{F}(G, \{\mu\}) := G_E/P(\mu)$$

over E. Here we denote for a 1-PS $\lambda \in X_*(G)$ defined over some finite field extension $F/\mathbb{Q}_p$, by $P(\lambda)$ the parabolic subgroup of G over F, whose $\overline{\mathbb{Q}}_p$−valued points are given by

$$P(\lambda)(\overline{\mathbb{Q}}_p) = \Big\{ g \in G(\overline{\mathbb{Q}}_p);\ \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{exists in } G(\overline{\mathbb{Q}}_p) \Big\}.$$

Notice that the geometric points of $\mathcal{F}$ coincide with the set

$$\{\mu\}/\sim,$$

where $\lambda_1, \lambda_2 \in \{\mu\}$ are equivalent, written $\lambda_1 \sim \lambda_2$, if they define the same filtration on $Rep_{\mathbb{Q}_p}(G)$. In the following, we suppose for trivial reasons that μ does not factor through the center of G.

Let $\mathbb{C}_p := \hat{\overline{\mathbb{Q}}}_p$ be the p-adic completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. This field is algebraically closed again. In the sequel we will often identify $\mathcal{F}$ with its $\mathbb{C}_p$-valued points. Put

$$E_s := E.\mathbb{Q}_{p^s}$$

and

$$\Gamma_{E_s} := Gal(\overline{E}_s/E_s).$$

Following [RZ] (Proposition 1.36), the set of weakly admissible filtrations

$$\mathcal{F}_b^{wa} := \{x \in \mathcal{F}; (b,x) \text{ is weakly admissible } \}$$

in $\mathcal{F}$ with respect to b has a natural structure of an admissible open rigid-analytic subset of $(\mathcal{F} \otimes_E E_s)^{rig}$, which is called the period domain of the triple $(G, b, \{\mu\})$. In his paper [K1], Kottwitz defines an algebraic group J over $\mathbb{Q}_p$ - the automorphism group of N_b - whose $\mathbb{Q}_p$-valued points are given by

$$J(\mathbb{Q}_p) = \{g \in G(K_0); g(b\sigma) = (b\sigma)g\}.$$

It can be shown that J is an inner form of a Levi subgroup of G, which is therefore a reductive group. Further, the period domain $\mathcal{F}_b^{wa}$ is stable under the action of $J(\mathbb{Q}_p)$. If $\lambda : \mathbb{G}_m \longrightarrow J$ is a 1-PS of J, then we denote by $P^J(\lambda)$ the parabolic subgroup of J with

$$P^J(\lambda)(\overline{\mathbb{Q}}_p) = P(\lambda)(\overline{\mathbb{Q}}_p) \cap J(\overline{\mathbb{Q}}_p).$$

Before we can state the main result of this paper, concerning the cohomology of $\mathcal{F}_b^{wa}$, we have to introduce a few more notations. Choose an invariant inner product on G. I.e., we have for all maximal tori T in G a non-degenerate positive definite pairing $(\ ,\ )$ on $X_*(T)_\mathbb{Q} := X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$, such that the natural maps

$$X_*(T)_\mathbb{Q} \longrightarrow X_*(T^g)_\mathbb{Q}$$

induced by conjugating with $g \in G(\overline{\mathbb{Q}}_p)$ and

$$X_*(T)_\mathbb{Q} \longrightarrow X_*(T^\tau)_\mathbb{Q}$$

induced by $\tau \in \Gamma_{\mathbb{Q}_p}$, are isometries for all $g \in G(\overline{\mathbb{Q}}_p), \tau \in \Gamma_{\mathbb{Q}_p}$. Here $T^g = gTg^{-1}$ is the conjugated torus and $T^\tau = \tau \cdot T$ is the image of T under the morphism $\tau : G \rightarrow G$ given by τ. The invariant inner product together with the natural pairing

$$\langle\ ,\ \rangle : X_*(T)_\mathbb{Q} \times X^*(T)_\mathbb{Q} \longrightarrow \mathbb{Q}\,,$$

give isomorphisms

$$\begin{array}{ccc} X_*(T)_\mathbb{Q} & \longrightarrow & X^*(T)_\mathbb{Q} \\ \lambda & \longmapsto & \lambda^* \end{array}$$

for all maximal tori T in G. We call λ^* the dual character of λ. Finally, we remark that the invariant inner product on G gives rise to one on J.

For the remainder of this paper, we assume that the element b is basic. By definition (see [K1]), the algebraic group J is then an inner form of G, or equivalently, the slope homomorphism ν factors through the center of G. It follows that μ is already defined over $\mathbb{Q}_p$ (see loc.cit. 5.1). Let T be a maximal torus of G such that $\mu, \nu \in X_*(T)_\mathbb{Q} \cong \mathrm{Hom}_{K_0}(\mathbb{D}, T)$. Denote by

$$W = N(T)/T$$

the Weyl group of G. Inside T we fix a maximal $\mathbb{Q}_p$-split torus S of the derived group J_{der} of J. Let

$$d := \mathrm{rk}(S)$$

be the semi-simple $\mathbb{Q}_p$-rank of J. Fix a minimal parabolic subgroup P_0 of J defined over $\mathbb{Q}_p$ such that $S \subset P_0$ and denote by

$$\Delta = \{\alpha_1, \ldots, \alpha_d\} \subset X^*(S)$$

the corresponding set of relative simple roots. Let

$$\{\omega_\alpha; \alpha \in \Delta\} \subset X_*(S)_{\mathbb{Q}}$$

be the dual basis of Δ, i.e., we have

$$\langle \omega_\alpha, \beta \rangle = \delta_{\alpha,\beta} \text{ (Kronecker delta) } \forall \alpha, \beta \in \Delta.$$

The parabolic subgroups $P^J(\omega_\alpha)$ are exactly the maximal $\mathbb{Q}_p$-parabolic subgroups of J that contain P_0. Fix a Borel subgroup B of G such that

$$B \subset P(\omega_\alpha) \; \forall \alpha \in \Delta. \tag{1}$$

Replacing μ by a conjugated element under W, we may suppose that μ lies in the positive Weyl chamber with respect to B. Furthermore, we may assume that Δ is given by restriction of a root basis of G with respect to $B \supset T$. Put

$$\bar{\mu} := \frac{1}{|Gal(E/\mathbb{Q}_p)|} \sum_{\gamma \in Gal(E/\mathbb{Q}_p)} \gamma\mu.$$

By [FR] we know that $\mathscr{F}_b^{wa}$ is non-empty if and only if $\bar{\mu}$ is greater or equal to ν with respect to the dominance order on $X_*(T)_{\mathbb{Q}}$ induced by B. In our situation, the latter condition simply means that $\bar{\mu} - \nu \in X_*(T_{der})$, where $T_{der} \subset T$ is the maximal torus of the derived group G_{der} of G. In the sequel we assume that these equivalent conditions are satisfied.

Let W_μ be the stabilizer of μ with respect to the action of W on $X_*(T)$. We denote by W^μ the set of Kostant-representatives with respect to W/W_μ. These are by definition the representatives of shortest length in their cosets. Consider the action of Γ_{E_s} on W. Since μ is defined over E_s, this action preserves W^μ. Denote the corresponding set of orbits by W^μ/Γ_{E_s} and its elements by $[w]$, where w is in W^μ. Clearly the length of an element in W only depends on its orbit. So, the symbol $l([w])$ makes sense. Fix an integer $n \in \mathbb{N}$ which is prime to p. For any orbit $[w] \in W^\mu/\Gamma_{E_s}$, we define the induced representation

$$ind_{[w]} := Ind_{Stab_{\Gamma_{E_s}}(w)}^{\Gamma_{E_s}}(\mathbb{Z}/n\mathbb{Z})$$

of Γ_{E_s}, where we consider the trivial action of $Stab_{\Gamma_{E_s}}(w)$ on $\mathbb{Z}/n\mathbb{Z}$. This induced representation is independent of the specified representative. For any subset $I \subset \Delta$, we set

$$\Omega_I := \Big\{[w] \in W^\mu/\Gamma_{E_s};\ (w\mu, \omega_\alpha) > (\nu, \omega_\alpha) \; \forall \alpha \notin I\Big\}.$$

We have the following inclusion relation

$$I \subset J \Rightarrow \Omega_I \subset \Omega_J.$$

Furthermore, we denote for $[w] \in W^\mu/\Gamma_{E_s}$, by $I_{[w]}$ the minimal subset of Δ such that $[w]$ is contained in $\Omega_{I_{[w]}}$. Obviously we have

$$I_{[w]} \subset I \;\Rightarrow\; [w] \in \Omega_I$$

and thus

$$I_{[w]} = \Big\{\alpha \in \Delta; (w\mu, \omega_\alpha) \leq (\nu, \omega_\alpha)\Big\}.$$

For a parabolic subgroup $P \subset J$ defined over $\mathbb{Q}_p$, we consider the trivial representation of $P(\mathbb{Q}_p)$ on $\mathbb{Z}/n\mathbb{Z}$. We denote by

$$i_P^J = Ind_{P(\mathbb{Q}_p)}^{J(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z}) = C^\infty(J(\mathbb{Q}_p)/P(\mathbb{Q}_p), \mathbb{Z}/n\mathbb{Z})$$

the induced representation of $J(\mathbb{Q}_p)$ consisting of locally constant functions on $(J/P)(\mathbb{Q}_p)$ $= J(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with values in $\mathbb{Z}/n\mathbb{Z}$. If $Q \supset P$ is another parabolic subgroup, then we have an injection $i_Q^J \hookrightarrow i_P^J$ which is induced by the surjection $J(\mathbb{Q}_p)/P(\mathbb{Q}_p) \to J(\mathbb{Q}_p)/Q(\mathbb{Q}_p)$. We set

$$v_P^J = i_P^J / \sum_{P \subsetneq Q} i_Q^J$$

and call v_P^J the generalized Steinberg representation with respect to P. Finally, if $I \subset \Delta$ we put

$$P_I := \bigcap_{\alpha \notin I} P^J(\omega_\alpha),$$

which is a standard-parabolic subgroup of J defined over $\mathbb{Q}_p$. As extreme cases we have $P_\Delta = J$ and $P_\emptyset = P_0$.

In [O3] (resp. [D] in the split case) it is shown that

$$Ext^1_{J(\mathbb{Q}_p)}(v_{P_I}^J, v_{P_{I'}}^J) = 0$$

in the category of smooth $J(\mathbb{Q}_p)$-representations with values in $\mathbb{Z}/n\mathbb{Z}$ for

$$|(I \cup I') \setminus (I \cap I')| \neq 1$$

and for a suitable choice of $n \in \mathbb{N}$. Those extensions appear in a spectral sequence given by the fundamental complex (see section 3) used in the proof of our main theorem. 'Suitable' means here that

- the pro-order of $J(\mathbb{Q}_p)$ is prime to n
- the natural injective homomorphisms

$$X^*(M)_{\mathbb{Q}_p} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow Hom(M(\mathbb{Q}_p)/{}^0M(\mathbb{Q}_p), \mathbb{Z}/n\mathbb{Z})$$

  induced by the composition of the valuation $val : \mathbb{Q}_p \to \mathbb{Z}$ and the natural homomorphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ are surjective for all Levi subgroups M of J. Here ${}^0M(\mathbb{Q}_p)$ is the normal open subgroup of $M(\mathbb{Q}_p)$ generated by all compact subgroups in $M(\mathbb{Q}_p)$.

- Let
$$\rho = \det Ad_{Lie(U)}|S \in X^*(S)$$
be the character given by the determinant of the adjoint representation of P_0 on $Lie(U)$ restricted to S, where U is the unipotent radical of P_0. Write ρ in the shape
$$\rho = \sum_{\alpha\in\Delta} n_\alpha \alpha,$$
where $n_\alpha \in \mathbb{N}$. Following the definition of a *bon* algebraically closed field by J.-F. Dat [D], we assume that
$$\prod_{r\leq sup\{n_\alpha;\, \alpha\in\Delta\}} (1-p^r)$$
is prime to n.

- Let $k/\mathbb{Q}_p$ be a finite splitting field of J. Then we further suppose that $|Gal(k/\mathbb{Q}_p)|$ is prime to n.

Now we can state the main result. It describes the étale cohomology with compact support of the period domain $\mathcal{F}_b^{wa}$ with values in $\mathbb{Z}/n\mathbb{Z}$ as representation of the locally profinite topological group $J(\mathbb{Q}_p) \times \Gamma_{E_s}$.

Theorem 1.1 *Let $b \in G(K_0)$ be a basic element such that $\mathcal{F}_b^{wa} \neq \emptyset$. Under the assumptions above on $n \in \mathbb{N}$, we have*

$$H_c^*(\mathcal{F}_b^{wa}, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{[w]\in W^\mu/\Gamma_{E_s}} v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w]) - |\Delta \setminus I_{[w]}|].$$

Here the symbol $(m), m \in \mathbb{N}$, indicates the m-th Tate twist and $[-m], m \in \mathbb{N}$, symbolizes that the corresponding module is shifted into degree m of the graded cohomology ring.

2 The relationship of period domains to GIT

In this section we want to explain the relationship between period domains and Geometric Invariant Theory. For details we refer to the papers [T] resp. [R2].

Let
$$M := P(\mu)/R_u(P(\mu))$$
be the Levi-quotient of $P(\mu)$ with center Z_M. Then μ defines an element of $X_*(Z_M)$. Let T_M be a maximal torus in M. Then we have $Z_M \subset T_M$ and T_M is the isomorphic image of a maximal torus in G. Thus, we get an invariant inner product on M. Consider the dual character
$$\mu^* \in X^*(T_M)_{\mathbb{Q}}.$$

As μ belongs to $X_*(Z_M)$, the dual character μ^* is contained in

$$X^*(M_{ab})_{\mathbb{Q}} \cong Hom(P(\mu), \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The inverse character $-\mu^*$ induces a homogeneous line bundle $\mathcal{L}_\mu := \mathcal{L}_{-\mu^*}$ on $\mathcal{F}$. The reason for the sign is that this line bundle is ample. Applying the above machinery to the inverse $\lambda_b := -\nu_b$ of our slope homomorphism, we get an ample line bundle $\mathcal{L}_b := \mathcal{L}_{-\lambda_b^*}$ on the flag variety $\mathcal{F}^b := G_{\mathbb{Q}_{p^s}}/P(\lambda_b)$. Consider the closed embedding

$$\mathcal{F}_{E_s} \hookrightarrow \mathcal{F}_{E_s} \times \mathcal{F}^b_{E_s},$$

given by the identity on the first factor and by the $\mathbb{Q}_{p^s}$-rational point λ_b of $\mathcal{F}^b$ on the second factor. Let $\mathcal{L}$ be the restriction - via this embedding - of the line bundle $\mathcal{L}_\mu \times \mathcal{L}_b$ to $\mathcal{F}_{E_s}$. We consider $\mathcal{L}$ as a J_{E_s}-equivariant line bundle.

For any point $x \in \mathcal{F}$ and any 1-PS $\lambda : \mathbb{G}_m \to J$, we denote by $\mu^{\mathcal{L}}(x, \lambda)$ the slope - in the sense of Geometric Invariant Theory - of x relative to λ and the line bundle $\mathcal{L}$ (see [M] Def. 2.2). The following theorem of Totaro (see [T] Theorem 3) describes the relation between the notion of weak admissibility and semi-stability in Geometric Invariant Theory.

Theorem 2.1 *(Totaro) Let x be a point of $\mathcal{F}$. Then x is weakly admissible if and only if $\mu^{\mathcal{L}}(x, \lambda) \geq 0$ for all 1-PS λ of J_{der} defined over $\mathbb{Q}_p$.*

Remark: If $b \in G(K_0)$ was not basic, then we would have to replace J_{der} by the algebraic group $J \cap G_{der}$, in order that the theorem above remains valid. Furthermore, we may consider the derived group of J, since we have made the assumption that $\mathcal{F}_b^{wa} \neq \emptyset$ (see [RZ] 1.51/1.52).

In the case we have chosen a faithful rational representation V of G, we can compute the slope of a point explicitly. If $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ are two filtrations on a finite-dimensional vector space V, we set

$$(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = \sum_{\alpha,\beta} \alpha \cdot \beta \cdot \dim gr^{\alpha}_{\mathcal{F}_1^\bullet}(gr^{\beta}_{\mathcal{F}_2^\bullet}(V)).$$

The following lemma is proved in [O2] in the case of a finite field, but the case treated here is proved similarly. It also follows from the results in [T] Lemma 6.

Lemma 2.2 *Let V be a faithful rational representation of G.*

(i) Let $x \in \mathcal{F}$ and $\lambda \in X_*(J)$. Denote their filtrations on $V_{\mathbb{C}_p} := V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ by $\mathcal{F}_x^\bullet$ respectively by $\mathcal{F}_\lambda^\bullet$. Furthermore, let $\mathcal{F}_b^\bullet$ be the filtration on $V_{\mathbb{C}_p}$ which is induced by λ_b. Then we have

$$\mu^{\mathcal{L}}(x, \lambda) = -\Big((\mathcal{F}_x^\bullet, \mathcal{F}_\lambda^\bullet) + (\mathcal{F}_b^\bullet, \mathcal{F}_\lambda^\bullet)\Big).$$

(ii) Let $T \subset G$ be a maximal torus and let $\lambda, \lambda' \in X_*(T)_{\mathbb{Q}}$. Assume that the inner invariant product on G is induced by the standard inner invariant product on $GL(V)$. Then we have

$$(\lambda, \lambda') = (\mathcal{F}_\lambda^\bullet, \mathcal{F}_{\lambda'}^\bullet).$$

In order to investigate the GIT-semi-stability of points on varieties, it is useful to consider the spherical building of the given group. Let $B(J_{der})_{\mathbb{Q}_p}$ be the real $\mathbb{Q}_p$-rational spherical building of the derived group J_{der} (see [CLT]). It is well-known that the space $B(J_{der})_{\mathbb{Q}_p}$ is homeomorphic to the geometric realisation of the combinatorial building of J (see [CLT], 6.1). Thus, we have a simplicial structure on $B(J_{der})_{\mathbb{Q}_p}$ which is defined as follows. For a $\mathbb{Q}_p$-rational parabolic subgroup $P \subset J$, we let

$$D(P) := \{x \in B(J_{der})_{\mathbb{Q}_p}; P(x) \supset P\}$$

be the facet corresponding to P. If P is a minimal $\mathbb{Q}_p$-parabolic subgroup, then $D(P)$ is called a chamber of $B(J_{der})_{\mathbb{Q}_p}$. If in contrast P is a proper maximal $\mathbb{Q}_p$-parabolic subgroup, then $D(P)$ is called a vertex. Consider the rational 1-PS $\omega_\alpha \in X_*(S)_{\mathbb{Q}}, \alpha \in \Delta$, introduced in the previous section. These 1-PS correspond to the vertices of the base chamber $D_0 := D(P_0)$, since the $P^J(\omega_\alpha), \alpha \in \Delta$, are the maximal $\mathbb{Q}_p$-rational parabolic subgroups that contain P_0. For any other chamber $D = D(P)$ in $B(J_{der})_{\mathbb{Q}_p}$, there exists an element $g \in J(\mathbb{Q}_p)$, such that the conjugated 1-PS $Int(g) \circ \omega_\alpha, \alpha \in \Delta$, correspond to the vertices of D. The element g is unique up to multiplication by an element of $P_0(\mathbb{Q}_p)$ from the right.

For every chamber D in $B(J_{der})_{\mathbb{Q}_p}$, we define the simplex

$$\tilde{D} := \Big\{ \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha \,;\, 0 \leq r_\alpha \leq 1, \sum_{\alpha \in \Delta} r_\alpha = 1 \Big\} \subset X_*(\tilde{S})_{\mathbb{R}},$$

where the 1-PS $\lambda_\alpha \in X_*(\tilde{S})_{\mathbb{R}}, \alpha \in \Delta$, for some maximal split torus $\tilde{S} \subset J_{der}$, represent the vertices of D. The topological spaces D and $\tilde{D}$ are obviously homeomorphic. We can extend $\mu^{\mathcal{L}}(x, \cdot\,)$ in a well-known way to a function on $X_*(\tilde{S})_{\mathbb{R}}$ for every maximal $\mathbb{Q}_p$-split torus $\tilde{S}$ in J. Notice that the slope function $\mu^{\mathcal{L}}(x, \cdot\,)$ is not defined on D but on $\tilde{D}$. In spite of this fact, we we say that $\mu^{\mathcal{L}}(x, \cdot\,)$ is affine on D, if it is affine on $\tilde{D}$, i.e., if the following equality holds:

$$\mu^{\mathcal{L}}(x, \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha) = \sum_{\alpha \in \Delta} r_\alpha \mu^{\mathcal{L}}(x, \lambda_\alpha) \quad \text{for all} \sum_{\alpha \in \Delta} r_a \lambda_\alpha \in \tilde{D}.$$

This definition does not depend on the choice of the 1-PS $\lambda_\alpha,\ \alpha \in \Delta$.

The proof of the next proposition, which is the same as in the case of a finite field [O2], follows from Lemma 2.2.

Proposition 2.3 *Let $x \in \mathcal{F}$ be an arbitrary point. The slope function $\mu^{\mathcal{L}}(x, \cdot\,)$ is affine on each chamber of $B(J_{der})_{\mathbb{Q}_p}$.*

Corollary 2.4 *Let x be a point in $\mathcal{F}$. Then x is not weakly admissible if and only if there exists an element $g \in J(\mathbb{Q}_p)$ and a simple root $\alpha \in \Delta$ such that $\mu^{\mathcal{L}}(x, Int(g) \circ \omega_\alpha) < 0$.*

3 The fundamental complex

In the case of a finite base field [O1], [O2], there was constructed an acyclic complex of étale sheaves on the complement of the period domain, which computes the cohomology of it. In this case the complement is a closed subvariety of $\mathcal{F}$. Unfortunately in our situation, the set $Y := \mathcal{F}^{rig} \setminus \mathcal{F}_b^{wa}$ is not in general an object of our category, i.e., a rigid-analytic variety. This can be already seen in the simplest case where

$$\mathcal{F}_b^{wa} = \Omega^2 = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p).$$

Therefore, we can't simply adopt the proof of the finite field case. The idea to avoid this problem is to work in the category of Huber's adic spaces [H1]. He has defined for every complete non-archimedean field k a fully faithful functor

$$\begin{array}{rcl} {}^{ad} : \{k\text{-rigid-analytic varieties}\} & \longrightarrow & \{k\text{-adic spaces}\}, \\ X & \longmapsto & X^{ad} \end{array}$$

where the category on the right hand side is a full subcategory of the category of topologically and locally ringed spaces with an equivalence class of valuations on each stalk of the corresponding structure sheaf. This functor respects open embeddings and converts open admissible coverings $X = \bigcup_{i\in I} X_i$ of a rigid-analytic variety X into coverings by open subsets $X^{ad} = \bigcup_{i\in I} X_i^{ad}$. Furthermore, it induces for a fixed rigid-analytic variety X a bijection between the quasi-compact open subsets of X and the quasi-compact open subsets of X^{ad}. If X is an algebraic variety over k, then we denote the adic space $(X^{rig})^{ad}$ simply by X^{ad}.

For both a rigid-analytic variety and an adic space X, we denote by $X_{ét}$ its étale site (see [H1], [JP]). Let X be a rigid-analytic variety. Then the above functor induces a morphism of sites

$$X_{ét} \to X^{ad}_{ét}$$

giving an equivalence of the associated topoi

$$\begin{array}{rcl} S(X_{ét}) & \longrightarrow & S(X^{ad}_{ét}) \\ F & \longmapsto & F^{ad}. \end{array}$$

Hence, we have for every rigid-analytic variety X and for every étale sheaf on X, functorial isomorphisms

$$H^p_{ét}(X, F) \xrightarrow{\sim} H^p_{ét}(X^{ad}, F^{ad}).$$

To deal with adic spaces in our situation has the advantage that this time, the complement

$$Y^{ad} := \mathcal{F}^{ad} \setminus (\mathcal{F}_b^{wa})^{ad}$$

has a reasonable topological structure. In fact, it is a so-called closed pseudo-adic subspace of $\mathcal{F}^{ad}$ (see Lemma 3.2). This means that it is a locally pro-constructible subset for the adic topology and convex with respect to the specializing order of points ([H1] 1.10.3). For these spaces, Huber has defined an étale site and hence a topos as well ([H1] 1.16).

But besides of the phenomenon above, we have another difficulty in order to apply the construction of the fundamental complex in [O1], [O2]. In our case we must deal with infinitely many subobjects of our fixed isocrystal. Thus, the fundamental complex with its summands is not well-defined. The first thought of substituting these summands by products turns out to fail. The reason is that infinite products of sheaves do not behave well with respect to localisation. The solution is to define a mixture between these two kind of sheaves, which we explain in the following.

For a rational 1-PS subgroup $\lambda \in X_*(J)_{\mathbb{Q}}$ of J, we let

$$Y_\lambda := \{x \in \mathcal{F}; \mu^{\mathcal{L}}(x, \lambda) < 0\}$$

be the closed subvariety of $\mathcal{F}$ consisting of points where λ damages the semi-stability condition. For a subset $I \subsetneq \Delta$, we put

$$Y_I := \bigcap_{\alpha \notin I} Y_{\omega_\alpha}$$

which is a closed subvariety of $\mathcal{F}$. The following statements are proved in loc.cit. in the case of a finite base field. The proof of the case considered here is the same.

Lemma 3.1 *Let $I \subsetneq \Delta$. The variety Y_I is defined over E_s. The natural action of $J(\mathbb{Q}_p)$ on $\mathcal{F}$ restricts to an action of $P_I(\mathbb{Q}_p)$ on Y_I.*

Fix a subset $I \subset \Delta$. Let g be a point in $J/P_I(\mathbb{Q}_p)$. Choose a representative $g' \in J(\mathbb{Q}_p)$ of g. By the lemma above we see that the image $g'Y_I$ of Y_I under the natural translation morphism induced by g' does not depend on the chosen representative. For this reason, we set $gY_I := g'Y_I$. Consider the closed adic subspace gY_I^{ad} of $\mathcal{F}^{ad}$. For any subset $W \subset J/P_I(\mathbb{Q}_p)$, we put

$$Z_I^W := \bigcup_{g \in W} gY_I^{ad}.$$

We consider it as an prepseudo-adic subspace of $\mathcal{F}^{ad}$, i.e., simply as a subset of $\mathcal{F}^{ad}$ (see [H1] 1.10.1). Applying Corollary 2.4 it follows that

$$Y^{ad} = \bigcup_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} Z_I^{J/P_I(\mathbb{Q}_p)}.$$

Lemma 3.2 *The subset Z_I^W is a closed pseudo-adic subspace of $\mathcal{F}^{ad}$ for every compact open subset $W \subset J/P_I(\mathbb{Q}_p)$.*

Proof: We follow the construction of [RZ] 1.3.2. Let $\mathcal{H}_I$ be the closed subvariety of $J/P_I \times \mathcal{F}$ defined over E_s consisting of (geometric) points $(t, x) \in J/P_I \times \mathcal{F}$ such that $x \in tY_I$. Thus, as a set Z_I^W is nothing but the union

$$Z_I^W = \bigcup_{t \in W} (\mathcal{H}_I^{ad})_t,$$

where $(\mathcal{H}_I^{ad})_t \subset \mathcal{F}^{ad}$ denotes the fibre through the point $t \in J/P_I(\mathbb{Q}_p)$. Let

$$f_\alpha(X_0, \ldots, X_n; T_0, \ldots, T_m) \in O[X_0, \ldots, X_n, T_0, \ldots, T_m],\ \alpha \in A$$

be a system of equations of $\mathcal{H}_I$ in some projective space $\mathbb{P}^n \times \mathbb{P}^m$, where O is the ring of integers in E_s. For a real number $\epsilon > 0$ and a point $t \in J/P_I(\mathbb{Q}_p)$, we denote by $(\mathcal{H}_I^{ad})_t(\epsilon)$ the closed epsilon tube around $(\mathcal{H}_I^{ad})_t$, i.e.,

$$(\mathcal{H}_I^{ad})_t(\epsilon) = \{x \in \mathcal{F}^{ad}; |f_\alpha^t(x)| < \epsilon\ ,\ \forall \alpha \in A\},$$

where

$$f_\alpha^t := f_\alpha(X_0, \ldots, X_n; t) \in O[X_0, \ldots, X_n]$$

are the equations of the variety tY_I (compare [H2]). As in Lemma 1.33 [RZ], we conclude the existence of a $\delta > 0$ such that

$$(\mathcal{H}_I^{ad})_t(\epsilon) = (\mathcal{H}_I^{ad})_{t'}(\epsilon)$$

for $||t' - t|| < \delta$ (the maximum norm). From the compactness of W we deduce for a fixed $\epsilon > 0$, the existence of a finite subset $S \subset W$ with

$$\bigcup_{t \in W} (\mathcal{H}_I^{ad})_t(\epsilon) = \bigcup_{t \in S} (\mathcal{H}_I^{ad})_t(\epsilon).$$

Thus, the subset $\bigcup_{t \in W} (\mathcal{H}_I^{ad})_t(\epsilon)$ is closed in $\mathcal{H}_I^{ad}$. Consider for a fixed $x \in \mathcal{F}^{ad}$ the function

$$\begin{array}{rcl} W & \longrightarrow & \mathbb{R}_{\geq 0}. \\ t & \longmapsto & \min\{\epsilon \in \mathbb{R}_{\geq 0}; x \in (\mathcal{H}_I^{ad})_t(\epsilon)\} \end{array}$$

This function is continuous by the argument above. So it assumes its minimum since W is compact. We conclude that

$$Z_I^W = \bigcap_{\epsilon > 0} \Big(\bigcup_{t \in W} (\mathcal{H}_I^{ad})_t(\epsilon) \Big). \qquad (2)$$

Thus, Z_I^W is closed in $\mathcal{F}^{ad}$. Further, we see by (2) that it is locally pro-constructible and convex. Hence, it is a closed pseudo-adic subspace of $\mathcal{F}^{ad}$. $\square$

For a compact open subset $W \subset J/P_I(\mathbb{Q}_p)$, $I \subset \Delta$, and a point $g \in W$, we denote by

$$\Phi_{g,I} : gY_I^{ad} \longrightarrow Y^{ad}$$

resp.

$$\tilde{\Phi}_{g,I,W} : gY_I^{ad} \longrightarrow Z_I^W$$

resp.

$$\Psi_{I,W} : Z_I^W \longrightarrow Y^{ad}$$

the natural closed embeddings of pseudo-adic spaces. Let $\mathbb{Z}/n\mathbb{Z}$ be the constant étale sheaf on Y^{ad}, where $n \in \mathbb{N}$ is prime to p. Put

$$(\mathbb{Z}/n\mathbb{Z})_{g,I} := (\Phi_{g,I})_*(\Phi_{g,I}^*(\mathbb{Z}/n\mathbb{Z}))$$

resp.

$$(\mathbb{Z}/n\mathbb{Z})_{Z_I^W} := (\Psi_{I,W})_*(\Psi_{I,W}^*(\mathbb{Z}/n\mathbb{Z}))$$

and let

$$\tilde{\Phi}^{\#}_{g,I,W} : (\mathbb{Z}/n\mathbb{Z})_{Z_I^W} \longrightarrow (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

be the adjunction homomorphism given by restriction. We denote by

$$\prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

the subsheaf of $\prod_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$, which is defined as the sheaf associated to the following presheaf ${}^P\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$. For any element $U \to Y^{ad}$ of the étale site $Y^{ad}_{ét}$, we put

$$\Big({}^P\prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I}\Big)(U) \;:=\; \Big\{(s_g)_g \in \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}(U);\ \text{there exists a (finite)}$$

$$\text{disjoint covering } J/P_I(\mathbb{Q}_p) = \dot{\bigcup_{j\in A}} W_j \text{ by compact open}$$

$$\text{subsets and sections } s_j \in (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}}(U),\ j\in A, \text{ such}$$

$$\text{that } \tilde{\Phi}^{\#}_{g,I,W_j}(s_j) = s_g \ \text{ for all } g \in W_j\Big\}.$$

If we work with the restricted étale site $Y^{ad}_{ét,f.p.}$ consisting of objects U in $Y^{ad}_{ét}$ where the structure morphism $U \to Y^{ad}$ is quasi-compact and quasi-separated, it is easy to see that the presheaf ${}^P\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$ is already a sheaf. In fact, since Y^{ad} is quasi-compact the site $Y^{ad}_{ét,f.p.}$ is noetherian (thus every étale covering can be refined into a finite étale covering). Notice that the topoi of $Y^{ad}_{ét,f.p.}$ and $Y^{ad}_{ét}$ are equivalent [H1] 2.3.12. Another description of the sheaf above is given by viewing it as an inductive limit of sheaves. To explain this, let $\mathcal{C}_I$ be the category of compact open disjoint coverings of $J/P_I(\mathbb{Q}_p)$ ordered in the usual way, i.e., by refinement. Then we may write

$$\prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I} = \varinjlim_{c\in\mathcal{C}_I} G_c, \tag{3}$$

where, for a covering $c = (W_j)_{j\in A} \in \mathcal{C}_I$, G_c is the sheaf defined by

$$G_c(U) \;:=\; \Big\{(s_g)_g \in \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}(U);\ \text{ there are sections } s_j \in (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}}(U),$$

$$j\in A,\ \text{such that } \tilde{\Phi}^{\#}_{g,I,W_j}(s_j) = s_g \text{ for all } g\in W_j\Big\}$$

for any element $U \to Y^{ad}$ of the étale site $Y^{ad}_{ét}$. Notice that G_c is just the image of the natural morphism of sheaves

$$\bigoplus_{j\in A} (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}} \hookrightarrow \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}.$$

We call $\prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$ the subsheaf of locally constant sections of $\prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$.

Let $\bar{x}$ be a geometric point of Y^{ad} with underlying point $x \in Y^{ad}$ (see [H1] 2.5.1/2 for the definition of a geometric point). Then we have

$$\Big(\prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}\Big)_{\bar{x}} = \prod'_{g\in J/P_I(\mathbb{Q}_p)} ((\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}} \subset \prod_{g\in J/P_I(\mathbb{Q}_p)} ((\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}}, \tag{4}$$

where the term in the middle is defined for abelian groups similarly as in the sheaf case:

$$\begin{aligned}\prod'_{g\in J/P_I(\mathbb{Q}_p)} ((\mathbb{Z}/n\mathbb{Z})_{g,I}))_{\bar{x}} \;:=\; \Big\{ & (s_g)_g \in \prod_{g\in J/P_I(\mathbb{Q}_p)} ((\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}};\ \text{there is a (finite) disjoint} \\ & \text{covering } J/P_I(\mathbb{Q}_p) = \dot{\bigcup_{j\in A}} W_j \text{ by compact open subsets} \\ & \text{and } s_j \in ((\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}})_{\bar{x}},\ j\in A,\ \text{such that } s_{j|gY_I} = s_g \ \text{ for} \\ & \text{all } g \in W_j \Big\}.\end{aligned}$$

Thus, we may identify the stalk of $\prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$ at $\bar{x}$ with the set of locally constant $\mathbb{Z}/n\mathbb{Z}$ -valued functions on the topological space $\{g \in J/P_I(\mathbb{Q}_p); x \in gY_I^{ad}\}$.

Now we are able to construct the fundamental complex of sheaves on Y^{ad} which is defined analogously as in [O1],[O2]. Let $I \subset I'$ be two subsets of Δ. We get canonically a homomorphism

$$p_{I,I'} : \prod'_{h\in J/P_{I'}(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{h,I'} \longrightarrow \prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

which is induced by the closed embeddings $gY_I^{ad} \to hY_{I'}^{ad}$, for $g \in J/P_I(\mathbb{Q}_p), h \in J/P_{I'}(\mathbb{Q}_p)$, such that g is mapped onto h via the projection $J/P_I(\mathbb{Q}_p) \to J/P_{I'}(\mathbb{Q}_p)$. Choose an order on the set Δ. For two subsets $I, I' \subset \Delta$ with $|I'| - |I| = 1$ and $I' = \{\beta_1 < \ldots < \beta_r\}$, we put

$$d_{I,I'} = \begin{cases} (-1)^i p_{I,I'} & : \quad I' = I \cup \{\beta_i\} \\ 0 & : \quad I \not\subset I' \end{cases}.$$

We thus get in the usual way a complex of sheaves on $Y^{ad}_{\acute{e}t}$ with boundary morphisms induced by the homomorphisms $d_{I,I'}$:

$$(*) : 0 \to \mathbb{Z}/n\mathbb{Z} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=1}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=2}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I} \to$$

$$\cdots \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=d-1}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I} \to \prod_{g\in J/P_\emptyset(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,\emptyset} \to 0.$$

Theorem 3.3 *The complex $(*)$ is acyclic.*

Before proving this theorem we remind the reader of the definition of an overconvergent sheaf on an adic space X. Following [H1] Def. 8.2.1, an étale sheaf F on X is called overconvergent, if for all geometric points $\bar{x}, \bar{y}$ of X such that x is a specialising point of y, the resulting specialising homomorphism $F_{\bar{x}} \to F_{\bar{y}}$ is bijective.

Lemma 3.4 *All the sheaves in the complex $(*)$ are overconvergent.*

Proof: Since every constant sheaf is overconvergent and the morphisms $\Phi_{g,I}$ are quasi-compact, we may conclude from [H1] Prop. 8.2.3 resp. [JP] 3.5 that the étale sheaves $(\mathbb{Z}/n\mathbb{Z})_{g,I}$ are overconvergent as well. Applying of (4) yields the statement for the sheaf $\prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$. Alternatively, one might use the identity (3). Obviously the direct sum of overconvergent sheaves is overconvergent again. Thus, the claim is proved. $\square$

Proof of Theorem 3.3: Since the complex $(*)$ consists of overconvergent sheaves, it is enough to show the acyclicity of $(*)$ in the maximal geometric points (with respect to the order given by specialising) of Y^{ad}. So, let ξ be such a maximal geometric point. By definition, this is just a morphism of adic spaces (see [H1] 2.5.1/2)

$$Spa(F_\xi) \longrightarrow \mathscr{F}^{ad}$$

for some separably closed affinoid field $F_\xi = (F^\triangleright, F^\circ)$, which factors through Y^{ad}. Here F° denotes the rank-one valuation ring consisting of power bounded elements in $F^\triangleright$ with respect to the valuation on $F^\triangleright$. This morphism corresponds to a flag $\mathcal{F}_\xi \in \mathscr{F}(F^\triangleright)$, such that $\mathcal{F}_\xi \in Y^{ad}\hat{\otimes}F^\triangleright$ (Compare also [JP] for a description of these maximal points). Localizing the above complex in ξ yields a chain complex with values in $\mathbb{Z}/n\mathbb{Z}$. The chain complex corresponds to a subcomplex of the combinatorial Tits building of J, whose simplices are given by

$$\{gP_Ig^{-1}; g \in J, \mathcal{F}_\xi \in gY_I(F^\triangleright),\ I \subsetneq \Delta\}.$$

Let R_ξ be the canonical realisation of this subcomplex in $\mathcal{B}(J_{der})$. In [O2] it was shown by using Proposition 2.3, that the space R_ξ is contractible. In our case we may apply exactly the same arguments. Thus, the acyclicity follows, since the complex above is just the homology version of locally constant functions of R_ξ (Compare also the remark on p. 66 [SS]). $\square$

4 Some cohomology groups

In this chapter we determine the cohomology groups which appear in the evaluation of the spectral sequence induced by $(*)$. We fix a subset $I \subsetneq \Delta$.

Proposition 4.1 *We have the following description of the closed varieties Y_I in terms of the Bruhat cells of G with respect to $P(\mu)$:*

$$Y_I \;=\; \bigcup_{\substack{w \in W^\mu \\ [w] \in \Omega_I}} BwP(\mu)/P(\mu). \tag{5}$$

Proof: Compare also Proposition 4.1 in [O2]. It is enough to show the assertion in the case $I = \Delta \setminus \{\alpha\}$, $\alpha \in \Delta$, since the sets Ω_I and Y_I are compatible with forming intersections with respect to the sets $I \subset \Delta$, i.e.,

$$\Omega_{I \cap J} = \Omega_I \cap \Omega_J$$

and

$$Y_{I \cap J} = Y_I \cap Y_J, \ \forall I, J \subset \Delta.$$

Denote by $\check{\mu}$ the point of $\mathcal{F}$ which is induced by the 1-PS μ. Let b be a point of the Borel subgroup B of G. Then we have

$$\mu^{\mathcal{L}}(bw\check{\mu}, \omega_\alpha) = \mu^{\mathcal{L}}(w\check{\mu}, \omega_\alpha)$$

since $B \subset P(\omega_\alpha)$ (see [M] Prop. 2.7). The notion of semi-stability does not depend on the chosen invariant inner product [T] Theorem 3. Thus we may assume by Lemma 2.2 that

$$\mu^{\mathcal{L}}(w\check{\mu}, \omega_\alpha) = -((w\mu, \omega_\alpha) - (\nu, \omega_\alpha)).$$

It follows that ω_α damages the weak admissibility if and only if $(w\mu, \omega_\alpha) > (\nu, \omega_\alpha)$, i.e., $[w] \in \Omega_I$. $\square$

The cell decomposition (5) allows us to compute the cohomology of the varieties Y_I. The computation is the same as for Schubert varieties.

Proposition 4.2 *We have*

$$H^*_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{[w] \in \Omega_I} ind_{[w]}(-l[w])[-2l([w])] \tag{6}$$

Proof: See Proposition 4.2 [O2]. $\square$

Next, we want to compute the cohomology of the sheaves $\prod'_{g \in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}$.

Proposition 4.3 *We have*

$$H^i_{\acute{e}t}(Y^{ad}, \prod_{g\in J/P_I(\mathbb{Q}_p)}' (\mathbb{Z}/n\mathbb{Z})_{g,I}) = C^\infty(J/P_I(\mathbb{Q}_p), H^i_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z}))$$

for all $i \in \mathbb{N}$. Here $C^\infty(J/P_I(\mathbb{Q}_p), H^i_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z})) = i^J_{P_I} \otimes H^i_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z})$ denotes the space of locally constant functions with values in $H^i_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z})$.

Proof: We have $\prod'_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I} = \varinjlim_{c\in\mathcal{C}_I} G_c$ (see (3)). Since Y^{ad} is quasi-compact and $\mathcal{C}_I$ is pseudo-filtered, we get ([H1] 2.3.13)

$$H^i_{\acute{e}t}(Y^{ad}, \prod_{g\in J/P_I(\mathbb{Q}_p)}' (\mathbb{Z}/n\mathbb{Z})_{g,I}) = \varinjlim_{c\in\mathcal{C}_I} H^i_{\acute{e}t}(Y^{ad}, G_c).$$

But $G_c \cong \bigoplus_{W\in c} (\mathbb{Z}/n\mathbb{Z})_{Z^W_I}$. Thus, we obtain

$$H^i_{\acute{e}t}(Y^{ad}, G_c) = \bigoplus_{W\in c} H^i_{\acute{e}t}(Z^W_I, \mathbb{Z}/n\mathbb{Z}).$$

Let $(W_s)_{s\in\mathbb{N}}$ be a family of compact open neighbourhoods of the base point $1 \cdot P_I$ in $J/P_I(\mathbb{Q}_p)$ such that

$$\bigcap_{s\in\mathbb{N}} W_s = \{1 \cdot P_I\} \subset J/P_I(\mathbb{Q}_p).$$

For instance, we may choose for W_s the open tube around the base point in $J/P_I(\mathbb{Q}_p)$ of radius $|p^s| = \frac{1}{p^s}$, i.e.,

$$W_s = \{t \in J/P_I(\mathbb{Q}_p);\ |g_\alpha(t)| \leq |p^s|\ \forall \alpha \in A\},$$

where $(g_\alpha)_{\alpha\in A} \in \mathbb{Z}_p[T_0, \ldots, T_N]$ is a system of equations for the vanishing ideal of $\{1 \cdot P_I\}$ in some standard projective space $\mathbb{P}^N$ containing J/P_I. We need the following lemma.

Lemma 4.4

$$\bigcap_{s\in\mathbb{N}} Z^{W_s}_I = Y^{ad}_I.$$

Proof: By (2) it is enough to show that for all $\epsilon > 0$, there exists an integer $s > 0$ such that $Z^{W_s}_I \subset (\mathcal{H}^{ad}_I)_{1\cdot P_I}(\epsilon)$. But this follows from the considerations in the proof of Lemma 3.2. □

By applying [H1] 2.4.6 we get

$$\varinjlim_{s\in\mathbb{N}} H^i_{\acute{e}t}(Z^{W_s}_I, \mathbb{Z}/n\mathbb{Z}) = H^i_{\acute{e}t}(Y_I, \mathbb{Z}/n\mathbb{Z}).$$

Using the transition maps $Y_I \to tY_I$, $t \in J/P_I(\mathbb{Q}_p)$, the corresponding statement is also true for the spaces $tY_I \cong Y_I$. Combining these facts we get

$$\varinjlim_{c \in \mathcal{C}_I} H^i_{ét}(Y^{ad}, G_c) = \varinjlim_{c \in \mathcal{C}_I} \bigoplus_{W \in c} H^i_{ét}(Z_I^W, \mathbb{Z}/n\mathbb{Z}) = C^\infty(J/P_I(\mathbb{Q}_p), H^i_{ét}(Y_I, \mathbb{Z}/n\mathbb{Z})).$$

Thus, the proposition is proved. □

We finish this chapter with a result which is also needed for evaluating the spectral sequence. We will construct a complex of smooth $J(\mathbb{Q}_p)$-representations in analogy to the complex $(*)$. Let $I \subset \Delta$ be a subset. Recall that $i^J_{P_I} = C^\infty(J/P_I(\mathbb{Q}_p), \mathbb{Z}/n\mathbb{Z})$. For subsets $I \subset I' \subset \Delta$ with $|I' \setminus I| = 1$, we get a homomorphism

$$p_{I,I'} : i^J_{P_{I'}} \longrightarrow i^J_{P_I},$$

which is induced by the projection $(J/P_I)(\mathbb{Q}_p) \longrightarrow (J/P_{I'})(\mathbb{Q}_p)$. For arbitrary subsets $I, I' \subset \Delta$ with $|I'| - |I| = 1$, we define the homomorphism $d_{I,I'}$ as in the case of the complex $(*)$. Thus, we get for every $I \subset \Delta$, a $\mathbb{Z}$-indexed complex

$$K_I^\bullet : 0 \to i^J_J \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 1}} i^J_{P_K} \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 2}} i^J_{P_K} \to \cdots \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |K \setminus I| = 1}} i^J_{P_K} \to i^J_{P_I} \to v^J_{P_I} \to 0$$

where the differentials are induced by the $d_{I,I'}$ and the map $i^J_{P_I} \to v^J_{P_I}$ is the natural one. The component i^J_J is in degree -1.

Proposition 4.5 *The complex $K_I^\bullet$ is acyclic.*

Proof: See [BS] Cor. 3.3, [SS] §6 Prop. 13 resp. [O3] Proposition 11. □

5 The proof of Theorem 1.1

In this section we evaluate the spectral sequence which is induced by the acyclic complex $(*)$. This computation ends with the determination of the cohomology of Y^{ad}. The proof of Theorem 1.1 then follows from the long exact cohomology sequence with respect to $(\mathcal{F}^{wa}_b)^{ad} \subset \mathcal{F}^{ad} \supset Y^{ad}$.

Consider the spectral sequence

$$E_1^{p,q} = H^q_{ét}(Y^{ad}, \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = p+1}} \prod_{g \in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}) \Longrightarrow H^{p+q}_{ét}(Y^{ad}, \mathbb{Z}/n\mathbb{Z})$$

given by the acyclic complex $(*)$. By Proposition 4.3 we have

$$E_1^{p,q} = \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = p+1}} C^\infty(J/P_I(\mathbb{Q}_p), H^q(Y_I, \mathbb{Z}/n\mathbb{Z})).$$

As in the finite field case [O1], [O2], we obtain by the cell-decomposition (5) of the varieties Y_I, $I \subset \Delta$, a decomposition

$$E_1 = \bigoplus_{[w] \in W^\mu / \Gamma_{E_s}} E_{1,[w]}$$

into subcomplexes, where $E_{1,[w]}$ is the complex

$$\Big(\bigoplus_{\substack{I_{[w]} \subset I \\ |\Delta \setminus I| = 1}} i^J_{P_I} \otimes ind_{[w]}(-l([w])) \to \bigoplus_{\substack{I_{[w]} \subset I \\ |\Delta \setminus I| = 2}} i^J_{P_I} \otimes ind_{[w]}(-l([w])) \to \cdots$$

$$\cdots \to i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \Big) [-2l([w])].$$

By Proposition 4.5 we get for $E_2 = \bigoplus_{[w] \in W^\mu / \Gamma_{E_s}} E_{2,[w]}$ the following terms:

$$\begin{aligned}
I_{[w]} = \Delta \ &: \ E^{p,q}_{2,[w]} &&= 0 \ \ p \geq 0, q \geq 0 \\
|\Delta \setminus I_{[w]}| = 1 \ &: \ E^{0,2l([w])}_{2,[w]} &&= i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \\
& \ E^{p,q}_{2,[w]} &&= 0 \ \ (p,q) \neq (0, 2l([w])) \\
|\Delta \setminus I_{[w]}| > 1 \ &: \ E^{0,2l([w])}_{2,[w]} &&= i^J_J \otimes ind_{[w]}(-l([w])) \\
& \ E^{j,2l([w])}_{2,[w]} &&= 0, \ \ j = 1, \ldots, |\Delta \setminus I_{[w]}| - 2 \\
& \ E^{j,2l([w]))}_{2,[w]} &&= v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])), \ \ j = |\Delta \setminus I_{[w]}| - 1 \\
& \ E^{p,q}_{2,[w]} &&= 0, \ \ q \neq 2l([w]) \text{ or } p > |\Delta \setminus I_{[w]}| - 1.
\end{aligned}$$

Since E_s is a local field, we conclude by weight arguments that $E_2 = E_\infty$. In fact, the argument which was pointed out to me by B. Totaro is the following. By the Chinese remainder theorem we may assume that $n = \ell^a, a > 0$, ℓ a prime number with $(\ell, p) = 1$. All the objects appearing in E_2 are free modules over their base ([SS] Cor. 5 §4) and are given by the tensor product of a $J(\mathbb{Q}_p)$-module and a Γ_{E_s}-module. Furthermore, we have $E_2(\mathbb{Z}/\ell^b\mathbb{Z}) \otimes_{\mathbb{Z}/\ell^b\mathbb{Z}} \mathbb{Z}/\ell^a\mathbb{Z} = E_2(\mathbb{Z}/\ell^a\mathbb{Z})$ for any integer $b > a$. Thus, we are reduced to showing that any homomorphism $d_a : \mu_{\ell^a}^{\otimes m} \to \mu_{\ell^a}^{\otimes n}, m \neq n$, of Γ_{E_s}-modules which sits in a compatible projective system of such objects vanishes. For an integer $b > a$, we consider the following commutative diagram of Γ_{E_s}-modules

$$\begin{array}{ccc}
\mu_{\ell^b}^{\otimes m} & \xrightarrow{d_b} & \mu_{\ell^b}^{\otimes n} \\
\downarrow & & \downarrow \\
\mu_{\ell^a}^{\otimes m} & \xrightarrow{d_a} & \mu_{\ell^a}^{\otimes n},
\end{array}$$

where the vertical homomorphisms are the natural ones. Let $\mathbb{F}_q$ be the residue field of E_s and let $\gamma \in \Gamma_{E_s}$ be an automorphisms which reduces to the arithmetic Frobenius on $\overline{\mathbb{F}}_q$. Since E_s is a local field, we can find a ℓ^b-root of unity $\xi \in \overline{E}_s \setminus E_s$ for some $b \gg a$. It follows that $d_b(\xi)^{q^m-q^n} = 1$. If ℓ is prime to $q^m - q^n$, we conclude that $d(\xi) = 1$. Otherwise we may enlarge b such that the composition of $d_b : \mu_{\ell^b}^{\otimes m} \to \mu_{\ell^b}^{\otimes n}$ with the map $\mu_{\ell^b}^{\otimes n} \to \mu_{\ell^a}^{\otimes n}$ vanishes. By the surjectivity of the latter homomorphism, we deduce that d_a must vanish. Thus, we have for $p, r \in \mathbb{N}$,

$$gr^p(H^r_{\acute{e}t}(Y^{ad}, \mathbb{Z}/n\mathbb{Z})) = E_\infty^{p,r-p} = E_2^{p,r-p} = \bigoplus_{[w]\in W^\mu} E_{2,[w]}^{p,r-p}$$

$$= \begin{cases} \displaystyle\bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ |\Delta\setminus I_{[w]}|=1 \\ 2l([w])=r}} i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ |\Delta\setminus I_{[w]}|>1 \\ 2l([w])=r}} i^J_J \otimes ind_{[w]}(-l([w])) & : \ p = 0 \\ \displaystyle\bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ 2l([w])+|\Delta\setminus I_{[w]}|-1=r \\ p=|\Delta\setminus I_{[w]}|-1}} v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) & : \ p > 0 \end{cases}$$

In order to show that the canonical filtration on E_∞ splits, we proceed as follows. By results of Berkovich [B] it is known that the cohomology groups $H^r_{\acute{e}t}(Y^{ad}, \mathbb{Z}/n\mathbb{Z})$ are smooth $J(\mathbb{Q}_p)$-modules. In [O3] (resp. [D] in the split case) it is shown that in the category of smooth representations (with coefficients in $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$ is suitable chosen as in the the introduction) we have

$$Ext^i_{J(\mathbb{Q}_p)}(v^J_{P_I}, v^J_{P_{I'}}) = \begin{cases} (\mathbb{Z}/n\mathbb{Z})^{\binom{r}{j}} & : \ i = |I \cup I'| - |I \cap I'| + j \\ 0 & : \ else \end{cases},$$

where $r \in \mathbb{N}$ is the $\mathbb{Q}_p$-rank of the center of J. Consider the equation

$$2l([w]) + |\Delta \setminus I_{[w]}| - 1 = r = 2l([w']) + |\Delta \setminus I_{[w']}| - 1$$

with $[w], [w'] \in W^\mu/\Gamma_{E_s}$. If $l([w]) \neq l([w'])$ then $|\Delta \setminus I_{[w]}|$ and $|\Delta \setminus I_{[w']}|$ differ at least by two. Hence $|I_{[w]}|$ and $|I_{[w']}|$ differ at least by two, so that

$$Ext^1_{J(\mathbb{Q}_p)}(v^J_{P_{I_{[w]}}}, v^J_{P_{I_{[w']}}}) = 0.$$

Thus, we obtain

$$H^r_{\acute{e}t}(Y^{ad}) \ \cong \ \bigoplus_{p\in\mathbb{N}} gr^p(H^r_{\acute{e}t}(Y^{ad})) = \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ |\Delta\setminus I_{[w]}|=1 \\ 2l([w])=r}} i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))$$

$$\oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s}\\ |\Delta\setminus I_{[w]}|>1\\ 2l([w])=r}} i_J^J \otimes ind_{[w]}(-l([w])) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s}\\ 2l([w])+|\Delta\setminus I_{[w]}|-1=r\\ p=|\Delta\setminus I_{[w]}|-1}} v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])).$$

Summarizing the discussion above, we get:

Theorem 5.1 *The spectral sequence E_1 degenerates in the E_2-term and we get*

$$H^*_{ét}(Y^{ad}, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s}\\ |\Delta\setminus I_{[w]}|=1}} \Big(i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w])]\Big) \oplus$$

$$\bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s}\\ |\Delta\setminus I_{[w]}|>1}} \Bigg(\Big(i^J_J \otimes ind_{[w]}(-l([w]))[-2l([w])]\Big) \oplus \Big(v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w])-|\Delta\setminus I_{[w]}|+1]\Big)\Bigg).$$

As a consequence we are able to prove the main theorem.

Proof of Theorem 1.1: The computation of $H^*_c(\mathcal{F}^{wa}_b, \mathbb{Z}/n\mathbb{Z})$ and hence the proof of Theorem 1.1 is shown by applying the long exact sequence

$$\cdots \to H^p_c(\mathcal{F}^{wa}_b, \mathbb{Z}/n\mathbb{Z}) \to H^p_{ét}(\mathcal{F}^{ad}, \mathbb{Z}/n\mathbb{Z}) \to H^p_{ét}(Y^{ad}, \mathbb{Z}/n\mathbb{Z}) \to H^{p+1}_c(\mathcal{F}^{wa}, \mathbb{Z}/n\mathbb{Z}) \to \cdots$$

to the triple $(\mathcal{F}^{wa}_b, \mathcal{F}^{ad}, Y^{ad})$. This is done in [O1], [O2]. Thus, Theorem 1.1 is proved. □

Let $n = \ell$ be a prime fulfilling the assumptions at the end of the introduction. As a corollary of Theorem 1.1, we want to treat the ordinary ℓ-adic cohomology $H^{*,ord}_c(\mathcal{F}^{wa}_b, \mathbb{Q}_\ell)$ of period domains. By ordinary ℓ-adic cohomology, we mean that the cohomology groups $H^{*,ord}_c(\mathcal{F}^{wa}_b, \mathbb{Q}_\ell)$ are defined in the usual way, i.e, by

$$H^{*,ord}_c(\mathcal{F}^{wa}_b, \mathbb{Q}_\ell) = \varprojlim_m H^*_c(\mathcal{F}^{wa}_b, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes \mathbb{Q}_\ell.$$

For a parabolic subgroup $P \subset J$, we denote by $C^{cont}(J/P(\mathbb{Q}_p), \mathbb{Q}_\ell)$ the $J(\mathbb{Q}_p)$-representation consisting of continuous functions $J/P(\mathbb{Q}_p) \to \mathbb{Q}_\ell$. Furthermore, let $v^{J,cont}_P$ be the corresponding continuous generalized Steinberg representation, i.e.,

$$v^{J,cont}_P = C^{cont}(J/P(\mathbb{Q}_p), \mathbb{Q}_\ell) / \sum_{P \subsetneq Q} C^{cont}(J/Q(\mathbb{Q}_p), \mathbb{Q}_\ell).$$

As it was first pointed out by R.Huber in the Drinfeld case (compare [H3] Example 2.7), we get for the ordinary ℓ-adic cohomology the following result.

Corollary 5.2

$$H^{*,ord}_c(\mathcal{F}^{wa}_b, \mathbb{Q}_\ell) = \bigoplus_{[w]\in W^\mu/\Gamma_{E_s}} v^{J,cont}_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w]) - |\Delta \setminus I_{[w]}|].$$

Remark: In order to get smooth $J(\mathbb{Q}_p)$-representations with values in $\mathbb{Q}_\ell$, one has to use the continuous ℓ-adic cohomology (compare loc.cit.) which will be done in an upcoming paper by the author.

6 A concluding example

We finish this paper with an example.

Example 6.1 *Let $G = GL(\mathbb{Q}_p^4)$. Identify rational characters resp. cocharacters of the diagonal torus T in G with tuples in $\mathbb{Q}^4$ in the usual way. Let $\mu = (1,1,0,0) \in \mathbb{Q}^4$ and choose $b \in G(K_0)$ such that $\nu_b = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbb{Q}^4$. Then $\mathcal{F}$ coincides with the Grassmannian $Gr_2(\mathbb{Q}_p^4)$ of degree two. We then have*

- $\mathcal{F}_b^{wa} = \{W \in Gr_2(\mathbb{C}_p^4); W \neq N \textit{ for all subisocrystals } N \subset N_b(\mathbb{Q}_p^4)\}$
- $J = GL_2(B)$, *where B is a quaternion division algebra over $\mathbb{Q}_p$*
- $S = \{diag(t \cdot 1_B, t^{-1} \cdot 1_B); t \in \mathbb{G}_m\} \subset J_{der} = SL_2(B)$
- $E_s = \mathbb{Q}_{p^2}$.

Let $\alpha = (0,1,-1,0) \in X^*(T)$ and denote by $\alpha_{|S}$ its restriction to S. Then we have

- $\Delta = \{\alpha_{|S}\}$
- $\omega_{\alpha_{|S}} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in X_*(S)_{\mathbb{Q}}$
- $W_\mu = S_2 \times S_2$
- $W^\mu = \{w_1 = 1, w_2 = (2,3), w_3 = (1,2,3), w_4 = (2,4,3), w_5 = (1,2,4,3), w_6 = (1,3)(2,4)\}$
- *The action of $\Gamma_{\mathbb{Q}^2}$ on W is trivial.*

We get $I_{w_1} = \emptyset$ and $I_{w_i} = \{\alpha\}$ for $i \geq 1$. Thus we have

$$H_c^*(\mathcal{F}_b^{wa}) = v_{P_0}^J[-1] \;\oplus\; v_J^J(-1)[-2] \;\oplus\; (v_J^J(-2)[-4])^2 \oplus\; v_J^J(-3)[-6] \;\oplus\; v_J^J(-4)[-8].$$

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