EQUIVARIANT VECTOR BUNDLES ON DRINFELD'S UPPER HALF SPACE

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Abstract. Let $\mathcal{X} \subset \mathbb{P}^d_K$ be Drinfeld's upper half space over a finite extension K of $\mathbb{Q}_p$. We construct for every GL_{d+1}-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$, a $\mathrm{GL}_{d+1}(K)$-equivariant filtration by closed subspaces on the K-Fréchet $H^0(\mathcal{X}, \mathcal{F})$. This gives rise by duality to a filtration by locally analytic $\mathrm{GL}_{d+1}(K)$-representations on the strong dual $H^0(\mathcal{X}, \mathcal{F})'$. The graded pieces of this filtration are locally analytic induced representations from locally algebraic ones with respect to maximal parabolic subgroups. This paper generalizes the cases of the canonical bundle due to Schneider and Teitelbaum [ST1] and that of the structure sheaf by Pohlkamp [P].

Contents

Introduction

Let K be a finite extension of $\mathbb{Q}_p$. We denote by $\mathcal{X}$ Drinfeld's upper half space of dimension $d \geq 1$ over K. Explicitly, $\mathcal{X}$ is the complement of all K-rational hyperplanes in projective space $\mathbb{P}^d_K$, i.e.,

$$\mathcal{X} = \mathbb{P}^d_K \setminus \bigcup\nolimits_{H \subsetneq K^{d+1}} \mathbb{P}(H).$$

The interest for studying the rigid analytic variety $\mathcal{X}$ is manifold. The first is its connection to formal groups, since it is the generic fibre of a formal scheme classifying certain p-divisible groups due to Drinfeld [D]. Another aspect is its role for the uniformization of certain Shimura varieties [RZ]. On the other hand, as conjectured by Drinfeld, the étale coverings of $\mathcal{X}$ realize the supercuspidal spectrum of the local Langlands correspondence by considering the ℓ-adic cohomology of these spaces. Our interest is connected with the latter aspect. In [S1] Schneider studies the cohomology of local systems on projective varieties which are uniformized by $\mathcal{X}$. For this purpose, he defines the notion of a p-adic holomorphic discrete series representation. These representations can be realized by the space of rigid analytic holomorphic sections $\mathcal{F}(\mathcal{X})$ of GL_{d+1}-equivariant vector bundles $\mathcal{F}$ on $\mathbb{P}^d_K$. They appear naturally in the theory of p-adic modular forms [T1], [T2]. The dual of a holomorphic discrete series representation is a locally analytic $\mathrm{GL}_{d+1}(K)$-representation in the sense of Schneider and Teitelbaum [ST3]. Those representations come up in the p-adic Langlands theory of Breuil and Schneider [BS] as the locally analytic part of certain Banach space representations. Our aim in this paper is a description of p-adic holomorphic discrete series representations by considering a $\mathrm{GL}_{d+1}(K)$-equivariant filtration on them. We determine the graded pieces in terms of locally analytic $\mathrm{GL}_{d+1}(K)$-representations.

In [SS] Schneider and Stuhler computed the étale and the de Rham cohomology of Drinfeld's upper half space. The cohomology is equipped with an action of the general linear group $G = \mathrm{GL}_{d+1}(K)$, which is induced by the natural action on $\mathcal{X}$. It turns out that the cohomology groups are duals of certain smooth elliptic representations having Iwahori fix vectors. In particular, the top cohomology group is the dual of the Steinberg representation. In contrast, the space of holomorphic sections $\Omega^d(\mathcal{X}) = H^0(\mathcal{X}, \Omega^d)$ of the canonical bundle Ω^d on $\mathbb{P}^d_K$ is a much bigger object, it is a reflexive K-Fréchet space with a continuous G-action. Its strong dual $\Omega^d(\mathcal{X})'$ is a locally analytic G-representation. In order to describe this latter space, Schneider and Teitelbaum construct in [ST1] a G-equivariant decreasing filtration by closed K-Fréchet spaces

$$\Omega^d(\mathcal{X})^0 \supset \Omega^d(\mathcal{X})^1 \supset \cdots \supset \Omega^d(\mathcal{X})^{d-1} \supset \Omega^d(\mathcal{X})^d \supset \Omega^d(\mathcal{X})^{d+1} = \{0\}$$

on $\Omega^d(\mathcal{X})^0 = \Omega^d(\mathcal{X})$. The definition of the filtration involves the geometry of $\mathcal{X}$ being the complement of an hyperplane arrangement. Further they construct isomorphisms

$$I^{[j]} : (\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1})' \stackrel{\sim}{\longrightarrow} C^{an}(G, P_{\underline{j}}; V_j')^{\mathfrak{d}_{\underline{j}}=0}$$

of locally analytic G-representations, which they call (partial) boundary value maps. Here, $P_{\underline{j}}{=}P_{(j,d+1-j)} \subset G$ is the (lower) standard-parabolic subgroup attached to the decomposition $(j, d+1-j)$ of $d+1$. The right hand side is a locally analytic induced representation. The $P_{\underline{j}}$-representation V_j' is a locally algebraic representation. It is isomorphic to the tensor product $\mathrm{Sym}^j(K^{d+1-j}) \otimes \mathrm{St}_{d+1-j}$ of the irreducible algebraic GL_{d+1-j}-representation $\mathrm{Sym}^j(K^{d+1-j})$ and the Steinberg representation St_{d+1-j} of $\mathrm{GL}_{d+1-j}(K)$. Here the factor $\mathrm{GL}_{d+1-j}(K)$ of the Levi subgroup $L_{(j,d+1-j)} = \mathrm{GL}_j(K) \times \mathrm{GL}_{d+1-j}(K)$ acts through the way just described. The action of $\mathrm{GL}_j(K)$ is given by the inverse of the determinant character. The operation of the unipotent radical of $P_{\underline{j}}$ on V_j' is trivial. Finally, $\mathfrak{d}_{\underline{j}}$ denotes a system of differential equations which is here a submodule of a generalized Verma module. In particular, the case $j = 0$, i.e., the first subquotient of the above filtration is isomorphic to $H^d_{dR}(\mathcal{X})$ and yields the Steinberg representation of G. Their paper presents consequently in a sense a generalization of the computation [SS] since it computes not only the top cohomology of $\mathcal{X}$. Further, it generalizes pioneering work by Morita (e.g. [Mo2]) who considered such representations in the SL_2-case. We refer to the introduction of [ST1] for a more comprehensive background on this topic.

Pohlkamp [P] considers the other extreme, that of the structure sheaf $\Omega^0 = \mathcal{O}$ on $\mathbb{P}^d_K$. Again, by using a similar construction, Pohlkamp defines a G-equivariant increasing filtration by closed K-Fréchet spaces

$$K = \mathcal{O}(\mathcal{X})_0 \subset \mathcal{O}(\mathcal{X})_1 \subset \cdots \subset \mathcal{O}(\mathcal{X})_{d-1} \subset \mathcal{O}(\mathcal{X})_d$$

on $\mathcal{O}(\mathcal{X})_d = H^0(\mathcal{X}, \mathcal{O})$ together with isomorphisms

$$(\mathcal{O}(\mathcal{X})_j)/\mathcal{O}(\mathcal{X})_{j-1})' \stackrel{\sim}{\longrightarrow} C^{an}(G, P_{\underline{d+1-j}}; W_j')^{\mathfrak{d}_{\overline{j}}=0}$$

of locally analytic G-representations. Analogously to the above case, the $P_{\underline{d+1-j}}$-representation W'_j is a tensor product of a Steinberg representation and an irreducible algebraic representation.

Our goal in this paper is to construct a decreasing G-equivariant filtration on $\mathcal{F}(\mathcal{X})$ for all G-equivariant vector bundles $\mathcal{F}$ on $\mathcal{X}$, which are induced by restriction of a homogeneous vector bundle on $\mathbb{P}^d_K \cong G/P_{(1,d)}$. The latter objects are defined by finite-dimensional algebraic representations of the parabolic subgroup $P_{(1,d)}$. Our approach is different from [ST1], [P]. We use local cohomology of coherent sheaves on rigid analytic varieties as a technical ingredient. In fact, $\mathcal{F}(\mathcal{X}) = H^0(\mathcal{X},\mathcal{F})$ appears in an exact sequence

$$0 \to H^0(\mathbb{P}^d_K,\mathcal{F}) \to H^0(\mathcal{X},\mathcal{F}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F}) \to H^1(\mathbb{P}^d_K,\mathcal{F}) \to 0.$$

We consider the K-Fréchet space $H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F})$, where $\mathcal{Y} \subset \mathbb{P}^d_K$ is the "closed" complement of $\mathcal{X}$ in $\mathbb{P}^d_K$. By a technique used in [O3], we are able to compute this latter module as a G-representation. Here, we use an acyclic resolution of the constant sheaf $\mathbb{Z}$ on $\mathcal{Y}^{ad}$, where $\mathcal{Y}^{ad} \stackrel{i}{\hookrightarrow} (\mathbb{P}^d_K)^{ad}$ is the closed complement of the adic space $\mathcal{X}^{ad}$ in $(\mathbb{P}^d_K)^{ad}$. By applying the functor $\mathrm{Hom}(i_*(\ -\ ),\mathcal{F})$ to this complex, we get a spectral sequence converging to $H^1_{\mathcal{Y}^{ad}}((\mathbb{P}^d_K)^{ad},\mathcal{F}^{ad}) = H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F})$. The canonical filtration on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F})$ coming from this spectral sequence gives rise to a decreasing filtration by closed K-Fréchet spaces

$$\mathcal{F}(\mathcal{X})^0 \supset \mathcal{F}(\mathcal{X})^1 \supset \cdots \supset \mathcal{F}(\mathcal{X})^{d-1} \supset \mathcal{F}(\mathcal{X})^d = H^0(\mathbb{P}^d,\mathcal{F})$$

on $\mathcal{F}(\mathcal{X})^0 = H^0(\mathcal{X},\mathcal{F})$. Our first main theorem is:

Theorem 1: *Let $\mathcal{F}$ be a homogeneous vector bundle on $\mathbb{P}^d_K$. For $j = 0,\ldots,d-1$, there are extensions of locally analytic G-representations*

$$0 \to v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K,\mathcal{F})') \to (\mathcal{F}(\mathcal{X})^j/\mathcal{F}(X)^{j+1})' \to C^{an}(G,P_{\underline{j+1}};U'_j)^{\mathfrak{d}_j=0} \to 0.$$

Here the module $v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K,\mathcal{F})')$ is a generalized Steinberg representation with coefficients in the finite-dimensional algebraic G-module $H^{d-j}(\mathbb{P}^d_K,\mathcal{F})'$. The $P_{\underline{j+1}}$-representation U'_j is a tensor product $N'_j \otimes \mathrm{St}_{d-j}$ of an algebraic $P_{\underline{j+1}}$-representation N'_j and the Steinberg representation St_{d-j}. The symbol $\mathfrak{d}_j$ indicates again a system of differential equations depending on N_j. Indeed, the representation N_j is not uniquely determined. It is characterized by the property that it generates the kernel of the natural homomorphism $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F}) \to H^{d-j}(\mathbb{P}^d_K,\mathcal{F})$ as a module with respect to the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra of G. By enlarging the module N_j we have to enlarge $\mathfrak{d}_j$, as well. In either case, the locally analytic G-representations $C^{an}(G, P_{\underline{j+1}}; U_j')^{\mathfrak{d}_j}$ remains the same.

In the case where $\mathcal{F}$ arises from an irreducible representation of the Levi subgroup $L_{(1,d)}$, we can make our result more precise. Let $\lambda' = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ be a dominant integral weight of GL_d and a let $\lambda_0 \in \mathbb{Z}$. Set $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^{d+1}$. Denote by $\mathcal{F}_\lambda$ the homogeneous vector bundle on $\mathbb{P}^d_K$ such that its fibre in the base point is the irreducible algebraic $L_{(1,d)}$-representation corresponding to λ. Put $w_j := s_j \cdots s_1$, where $s_i \in W$ is the (standard) simple reflection in the Weyl group $W \cong S_{d+1}$ of G. Then the above representation N_j can be characterized as follows. By Bott [Bo] we know that there is at most one integer $i \geq 0$ with $H^i(\mathbb{P}^d_K, \mathcal{F}) \neq 0$. Denote this integer by i_0 if it exists. Otherwise, there is an $i_0 \leq d-1$ with $w_{i_0} * \lambda = w_{i_0+1} * \lambda$, where $*$ is the dot operator of W on the set of weights. For $j = 1, \ldots, d$, we set

$$\mu_{j,\lambda} := \begin{cases} w_{j-1} * \lambda & : j \leq i_0 \\ w_j * \lambda & : j > i_0 \end{cases} .$$

Write $\mu_{j,\lambda} = (\mu', \mu'')$ with $\mu' \in \mathbb{Z}^j$ and $\mu'' \in \mathbb{Z}^{d-j+1}$. For $j = 1, \ldots, d$, let

$$\Psi_{j,\lambda} = \bigcup_{k=0}^{|\mu''|} \Big\{ \big(\mu'' + (c_1, \ldots, c_{d-j+1}), \mu' - (d_j, \ldots, d_1)\big) \mid \textstyle\sum_l c_l = \sum_l d_l = k, c_1 = 0$$
$$\text{or } d_1 = 0,\ c_{l+1} \leq \mu''_l - \mu''_{l+1},\ l = 1, \ldots, d-j,\ d_{l+1} \leq \mu'_{j-l} - \mu'_{j-l+1},\ l = 1, \ldots, j-1 \Big\}.$$

Here $|\mu''| = \mu''_1 - \mu''_{d-j+1}$. The elements in the finite set $\Psi_{j,\lambda}$ are dominant with respect to the Levi subgroup $L_{(d-j+1,j)}$ and (μ'', μ') is its highest weight. Hence, for $\mu \in \Psi_{j,\lambda}$, we may consider the irreducible algebraic $L_{(d-j+1,j)}$-representation V_μ attached to it.

Theorem 2: *Let $\mathcal{F} = \mathcal{F}_\lambda$ be the homogeneous vector bundle on $\mathbb{P}^d_K$ with respect to the dominant integral weight $\lambda \in \mathbb{Z}^{d+1}$ of $L_{(1,d)}$. Then we can choose N_j to be a quotient of $\bigoplus_{\mu \in \Psi_{d-j,\lambda}} V_\mu$.*

We want to point out that for some weights λ, it happens that all irreducible constituents in the direct sum apart from the module V_μ with $\mu = (\mu'', \mu')$ vanish under the corresponding quotient map. But also the other extreme is possible, i.e., the quotient map can be an isomorphism, as well.

Our filtration coincides with the filtrations in [ST1], [P]. More precisely, in the case of the structure sheaf $\mathcal{F} = \mathcal{O}$ the increasing filtration of Pohlkamp is related to our decreasing filtration by $\mathcal{F}(\mathcal{X})^j = \mathcal{O}(\mathcal{X})_{d-j},\ j = 0, \ldots, d$. In the case $\mathcal{F} = \Omega^d$

the filtration of Schneider and Teitelbaum is related to ours by a shift, i.e., we have $\mathcal{F}(\mathcal{X})^i = \Omega^d(\mathcal{X})^{i+1}$ for $i \geq 1$. For $i = 0$, we get an extension

$$0 \to \Omega^d(\mathcal{X})^1/\Omega^d(\mathcal{X})^2 \to \mathcal{F}(\mathcal{X})^0/\mathcal{F}(\mathcal{X})^1 \to \Omega^d(\mathcal{X})^0/\Omega^d(\mathcal{X})^1 \to 0.$$

The dual sequence coincides with the corresponding one of Theorem 1.

The content of this paper is organized as follows. The first part deals with algebraic and analytic local cohomology of equivariant vector bundles $\mathcal{F}$ on $\mathbb{P}^d_K$. In the first section we recall some facts on the restriction of these bundles to $\mathcal{X}$. Amongst other things, we explain the structure of the strong dual $H^0(\mathcal{X}, \mathcal{F})'$ as a locally analytic G-representation. In the following section we treat the algebraic local cohomology of G-equivariant vector bundles on $\mathbb{P}^d_K$. We study the cohomology groups $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ as representations of $U(\mathfrak{g})$ and $P_{\underline{j+1}}$. In Section 1.3 we turn to the analytic local cohomology groups $H^{d-j}_{\mathbb{P}^j_K(\epsilon)}(\mathbb{P}^d_K, \mathcal{F})$ with support in the rigid analytic tube $\mathbb{P}^j_K(\epsilon)$. These groups are naturally equipped with a topology of a locally convex K-vector space. One of our focal points is to see that they are Hausdorff. Further, we prove a local duality theorem which is similar to a result obtained by Morita [Mo3]. It describes the dual of $\ker(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{F}))$ for $\epsilon \to 0$, by means of analytic functions on certain polydiscs. In the final section of the first part we compute this kernel as representation of $U(\mathfrak{g})$ and $P_{\underline{j+1}}$ when $\mathcal{F} = \mathcal{F}_\lambda$ is defined by a dominant integral weight $\lambda \in \mathbb{Z}^{d+1}$ of $L_{(1,d)}$. Here we make use of the Grothendieck-Cousin complex with respect to the covering by Schubert cells. The second part of this paper deals with the computation of $H^0(\mathcal{X}, \mathcal{F})$ as G-representation. First we repeat the construction of an acyclic resolution of the constant sheaf $\mathbb{Z}$ on the closed complement $\mathcal{Y}^{ad}$ [O3]. In Section 2.2 we evaluate the spectral sequence obtained by applying the functor $\mathrm{Hom}(i_*(\ -\ ), \mathcal{F})$ to this acyclic complex. In Part 3 we compare our result to that of [ST1] and [P]. Further, we provide with the cotangent bundle Ω^1 another example for our computation. Finally, in the Appendix we present an alternative way for the computation avoiding adic spaces. It is based purely on rigid analytic varieties.

Notation: We denote by p a prime, by $K \supset \mathbb{Q}_p$ a finite extension of the field of p-adic integers $\mathbb{Q}_p$, by O_K its ring of integers and by π a uniformizer of K. Let $|\ \ | : K \to \mathbb{R}$ be the normalized norm, i.e., $|\pi| = \#(O_K/(\pi))^{-1}$. We denote by $\mathbb{C}_p$ the completion of an algebraic closure $\overline{K}$ of K. Let $S := K[X_0, \ldots, X_d]$ be the polynomial ring in $d+1$ indeterminates and denote by $\mathbb{P}^d_K := \mathrm{Proj}(S)$ the projective space over K. If $Y \subset \mathbb{P}^d_K$ is a closed algebraic K-subvariety and $\mathcal{F}$ is a sheaf on

$\mathbb{P}^d_K$ we write $H^*_Y(\mathbb{P}^d_K, \mathcal{F})$ for the corresponding local cohomology. If Y is a rigid analytic subvariety (resp. pseudo-adic subspace) of $(\mathbb{P}^d_K)^{rig}$ (resp. $(\mathbb{P}^d_K)^{ad}$) we also write $H^*_Y(\mathbb{P}^d_K, \mathcal{F})$ instead of $H^*_Y((\mathbb{P}^d_K)^{rig}, \mathcal{F}^{rig})$ (resp. $H^*_Y((\mathbb{P}^d_K)^{ad}, \mathcal{F}^{ad})$) to simplify matters. For a locally convex K-vector space V, we denote by V' its strong dual, i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence.

We use bold letters $\mathbf{G}, \mathbf{P}, \ldots$ to denote algebraic group schemes over K, whereas we use normal letters $G, P, \ldots$ for their K-valued points of p-adic groups. We use Gothic letters $\mathfrak{g}, \mathfrak{p}, \ldots$ for their Lie algebras. The corresponding enveloping algebras are denoted as usual by $U(\mathfrak{g}), U(\mathfrak{p}), \ldots$. Finally, we set $\mathbf{G} := \mathbf{GL_{d+1}}$. If $\mathbf{H} \subset \mathbf{G}$ is any closed linear algebraic subgroup and R is a O_K-algebra, then we denote for simplicity by $\mathbf{H}(R)$ the set of R-valued points of the schematic closure of $\mathbf{H}$ in $\mathbf{GL_{d+1,\mathrm{O_K}}}$.

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1. Local cohomology of equivariant vector bundles on $\mathbb{P}^d_K$

1.1. **The rigid analytic variety $\mathcal{X}$.** In this section we recall some geometric properties of Drinfeld's upper half space $\mathcal{X}$. We explain its rigid analytic structure making it into a Stein space. Furthermore, we treat briefly G-equivariant vector bundles on $\mathcal{X}$ which are induced by homogeneous vector bundles on $\mathbb{P}^d_K$. We discuss how we can associate to to such a sheaf a locally analytic G-representation in the sense of [ST3]. In what follows, we denote for a variety X over K by X^{rig} the rigid analytic variety attached to X [BGR].

Let $\epsilon \in \bigcup_{n\in\mathbb{N}} \sqrt[n]{|K^\times|} = |\overline{K^\times}|$ be a n-th square root of some absolute value in $|K^\times|$. Recall the definition of an open respectively closed ϵ-neighborhood of a closed K-subvariety $Y \subset \mathbb{P}^d_K$. Let $f_1, \ldots, f_r \in S = K[X_0, \ldots, X_d]$ be finitely many homogeneous polynomials with integral coefficients generating the vanishing ideal of Y. We suppose that each polynomial has at least one coefficient in $O_K^\times$. Let $|\ \ |$ be the unique extension of our fixed norm $|\ \ |$ on K to $\mathbb{C}_p$. A tuple $(z_0, \ldots, z_d) \in \mathbb{A}^{d+1}_K(\mathbb{C}_p)$ is called unimodular if $|z_i| \leq 1$ for $i = 0, \ldots, d$, and $|z_i| = 1$ for at least one i with $0 \leq i \leq d$. The open ϵ-neighborhood of Y is defined by

$$Y(\epsilon) = \Big\{ z \in (\mathbb{P}^d_K)^{rig} \mid \text{ for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| \leq \epsilon \text{ for all } 1 \leq j \leq r \Big\}.$$

This definition is independent of the chosen unimodular representatives, so it is well-defined. By using the standard covering $(D_+(X_i))_{i=0\ldots,d}$ of $\mathbb{P}^d_K$, one verifies that $Y(\epsilon)$ is a finite union of K-affinoid spaces, cf. [BGR] 7.2. In particular, it is a quasi-compact open rigid analytic subspace of $(\mathbb{P}^d_K)^{rig}$. On the other hand, the set

$$Y^-(\epsilon) = \Big\{ z \in (\mathbb{P}^d_K)^{rig} \mid \text{ for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| < \epsilon \text{ for all } 1 \leq j \leq r \Big\}$$

is called the closed ϵ-neighborhood of Y. Again, it is an admissible open subset of $(\mathbb{P}^d_K)^{rig}$, but which is in general not quasi-compact.

Remark 1.1.1. We use the terminology open respectively closed, since the corresponding neighborhoods for adic spaces [H] are open respectively closed in the adic space $(\mathbb{P}^d_K)^{ad}$. □

For a non-trivial linear K-subspace $U \subsetneq K^{d+1} = V$, let Y_U be the closed linear K-subvariety $\mathbb{P}(U)$ of $\mathbb{P}^d_K$. Set $\epsilon_n := |\pi^n|, n \in \mathbb{N}$.

Proposition 1.1.2. *For every* $n \in \mathbb{N}$, *both*

$$(\mathbb{P}_K^d)^{rig} \setminus Y_U(\epsilon_n), \text{ for } U \subsetneq V,$$

and

$$\mathcal{Y}_n := \bigcup_{U \subsetneq V} Y_U(\epsilon_n) \text{ resp. } \mathcal{X}_n^- := (\mathbb{P}_K^d)^{rig} \setminus \mathcal{Y}_n$$

are admissible open subsets of $(\mathbb{P}_K^d)^{rig}$, *where* $\mathcal{Y}_n$ *is quasi-compact. The covering*

$$\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n^-$$

is admissible open and $\mathcal{X}$ *is defined over* K.

Proof. See [SS] Proposition 1. □

As in [ST1] we also work with the closed ϵ-neighborhoods $Y_U^-(\epsilon)$. Similar to the above proposition, we have for these spaces, the following statements.

Proposition 1.1.3. *For every* $n \in \mathbb{N}$, *both*

$$(\mathbb{P}_K^d)^{rig} \setminus Y_U^-(\epsilon_n), \text{ for } U \subsetneq V,$$

and

$$\mathcal{Y}_n^- := \bigcup_{U \subsetneq V} Y_U^-(\epsilon_n) \text{ resp. } \mathcal{X}_n := (\mathbb{P}_K^d)^{rig} \setminus \mathcal{Y}_n^-$$

are admissible open subsets of $(\mathbb{P}_K^d)^{rig}$, *where* $\mathcal{X}_n$ *is quasi-compact. The covering*

$$\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n$$

is admissible open. Furthermore, this covering induces on $\mathcal{X}$ *the structure of a Stein space. The* K*-algebra of global sections* $\mathcal{O}(\mathcal{X})$ *is a* K*-Fréchet space. More precisely, we have* $\mathcal{O}(\mathcal{X}) = \varprojlim_{n \in \mathbb{N}} \mathcal{O}(\mathcal{X}_n)$, *where the* K*-algebras* $\mathcal{O}(\mathcal{X}_n)$ *are* K*-Banach spaces.*

Proof. See [SS] Proposition 4 resp. [ST1] chapter 1. □

We follow the convention in [ST1] and consider the algebraic action $m : \mathbf{G} \times \mathbb{P}_K^d \to \mathbb{P}_K^d$ of $\mathbf{G}$ on $\mathbb{P}_K^d$ given by

$$g \cdot [q_0 : \cdots : q_d] := m(g, [q_0 : \cdots : q_d]) := [q_0 : \cdots : q_d] g^{-1}.$$

Let $\mathcal{F}$ be a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}_K^d$. This is a vector bundle $\mathcal{F}$ on $\mathbb{P}_K^d$ together with a $\mathbf{G}$-linearization, i.e., an isomorphism of sheaves

$$m^*(\mathcal{F}) \xrightarrow{\sim} pr^*(\mathcal{F}) \tag{1.1}$$

on $\mathbf{G} \times \mathbb{P}^d_K$ satisfying a certain cocycle condition, cf. [MFK] Definition 1.6. Here $pr : \mathbf{G} \times \mathbb{P}^d_K \to \mathbb{P}^d_K$ is the projection map onto the second factor. We get by functoriality an induced $\mathbf{G}^{rig}$-equivariant vector bundle on $(\mathbb{P}^d_K)^{rig}$, which we denote for simplicity by $\mathcal{F}$, as well.

Alternatively, there is the following description of $\mathbf{G}$-equivariant vector bundles on $\mathbb{P}^d_K$, cf. [Bo], [Ja] where they are called homogeneous vector bundles. Denote by $\mathbf{P_{(1,d)}}$ the stabilizer of the base point $[1 : 0 : \cdots : 0] \in \mathbb{P}^d_K(K)$, which is a parabolic subgroup of $\mathbf{G}$. Let

$$\pi : \mathbf{G} \to \mathbf{G}/\mathbf{P_{(1,d)}}$$

be the projection map and identify $\mathbf{G}/\mathbf{P_{(1,d)}}$ with $\mathbb{P}^d_K$. Let V be a finite-dimensional algebraic representation of $\mathbf{P_{(1,d)}}$. For a Zariski open subset $U \subset \mathbb{P}^d_K$, put

$$\mathcal{F}_V(U) := \Big\{ \text{ algebraic morphisms } f : \pi^{-1}(U) \to V \mid f(gp) = p^{-1}f(g) \text{ for all } g \in \mathbf{G}(\overline{K}), p \in \mathbf{P_{(1,d)}}(\overline{K}) \Big\}.$$

Then $\mathcal{F}_V$ defines a homogeneous vector bundle on $\mathbb{P}^d_K$ with fibre V. If $\mathcal{F}$ is a $\mathbf{G}$-equivariant vector bundle with fibre V in the base point, one has a natural identification $\mathcal{F} \cong \mathcal{F}_V$.

Our p-adic group G stabilizes $\mathcal{X}$. Therefore, we obtain an induced action of G on the K-vector space of rigid analytic holomorphic sections $\mathcal{F}(\mathcal{X})$. Let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^d_K$. Since $\mathcal{X}$ is contained in the rigid analytic variety attached to affine scheme $D_+(X_0) \cong \mathbf{A}^d_K$, we may choose a K-linear isomorphism

$$\mathcal{O}(\mathcal{X})^n \xrightarrow{\sim} \mathcal{F}(\mathcal{X}). \tag{1.2}$$

Here the integer $n = \mathrm{rk}(\mathcal{F}) \in \mathbb{N}$ is the rank of $\mathcal{F}$. We transfer the natural topology of the former one onto $\mathcal{F}(\mathcal{X})$. The topology on $\mathcal{F}(\mathcal{X})$ is independent of the chosen isomorphism. Thus $\mathcal{F}(\mathcal{X})$ inherits the structure of a K-Fréchet space. Similarly, the sets $\mathcal{F}(\mathcal{X}_n)$ are K-Banach spaces and we get[1]

$$\mathcal{F}(\mathcal{X}) = \varprojlim_n \mathcal{F}(\mathcal{X}_n).$$

Applying the same arguments as in [ST1] Lemma 1.3, Proposition 1.4 and Proposition 2.1, we conclude that $\mathcal{F}(\mathcal{X})$ is a reflexive K-Fréchet space and its strong dual

$$\mathcal{F}(\mathcal{X})' = \varinjlim_{n \in \mathbb{N}} \mathcal{F}(\mathcal{X}_n)'$$

[1]Hence the $\mathcal{O}(\mathcal{X})$-module $\mathcal{F}(\mathcal{X})$ is coadmissible in the sense of Schneider and Teitelbaum [ST2], cf. p. 152.

is a locally convex inductive limit of duals of K-Banach spaces. Furthermore, the action

$$G \times \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$$

is continuous and the orbit maps

$$\begin{array}{rcl} G & \to & \mathcal{F}(\mathcal{X})' \\ g & \mapsto & g \cdot f \end{array}$$

are locally analytic for $f \in \mathcal{F}(\mathcal{X})'$. Thus, the strong dual $\mathcal{F}(\mathcal{X})'$ is a locally analytic G-representation in the sense of Schneider and Teitelbaum [ST3]. By definition it is a barrelled locally convex Hausdorff K-vector space together with a continuous action of G such that the orbit maps are locally analytic functions on G.

1.2. **Algebraic local cohomology I.** This section deals with the algebraic local cohomology $H^*_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ of $\mathbf{G}$-equivariant vector bundles $\mathcal{F}$ on $\mathbb{P}^d_K$. We will study these K-vector spaces as representations of $\mathfrak{g}$ and of the parabolic subgroup fixing $\mathbb{P}^j_K$.

Denote by $\mathbf{B} \subset \mathbf{G}$ the Borel subgroup of lower triangular matrices and let $\mathbf{U}$ be its unipotent radical. Let $\mathbf{T} \subset \mathbf{G}$ be the diagonal torus and denote by $\overline{\mathbf{T}}$ its image in $\mathbf{PGL}_{d+1}$. For $0 \leq i \leq d$, let $\epsilon_i : \mathbf{T} \to \mathbb{G}_\mathbf{m}$ be the character defined by $\epsilon_i(\mathrm{diag}(t_1, \ldots, t_d)) = t_i$. Put $\alpha_{i,j} := \epsilon_i - \epsilon_j$ for $i \neq j$, and $\alpha_i := \alpha_{i+1,i}$ for $0 \leq i \leq d-1$. Then

$$\Delta := \{\alpha_i \mid 0 \leq i \leq d-1\}$$

are the simple roots and

$$\Phi := \{\alpha_{i,j} \mid 0 \leq i \neq j \leq d-1\}$$

are the roots of $\mathbf{G}$ with respect to $\mathbf{T} \subset \mathbf{B}$. For a decomposition $(i_1, \ldots, i_r)$ of $d+1$, let $\mathbf{P}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ be the corresponding standard-parabolic subgroup of $\mathbf{G}$, $\mathbf{U}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ its unipotent radical, $\mathbf{U}^+_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ its opposite unipotent radical and $\mathbf{L}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ its Levi component.

Fix a $\mathbf{G}$-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$. Let F be a graded $\mathbf{G}$-module which is projective and of finite type over $S = K[X_0, \ldots, X_d]$, such that its associated sheaf on $\mathbb{P}^d_K$ is just $\mathcal{F}$, cf. [H] ch. 2, §5. Then $\mathcal{F}$ is naturally a $\mathfrak{g}$-module, i.e., there is a homomorphism of Lie algebras

$$\mathfrak{g} \to \mathrm{End}(\mathcal{F}) \tag{1.3}$$

defined in the following way. Restrict the linearization (1.1) to $G^{(1)} \times \mathbb{P}^d_K$, where $G^{(1)}$ is the first infinitesimal neighborhood of the identity. Let $\mathfrak{x} \in \mathfrak{g}$ and let $f \in \mathcal{F}(U)$ be a section for a Zariski open subset $U \subset \mathbb{P}^d_K$. Then

$$\mathfrak{x} \cdot f := \frac{d}{dT}((1 + T\mathfrak{x}) \cdot f)_{T=0}.$$

Further, there is the following Leibniz rule concerning the multiplication with functions $\Xi \in \mathcal{O}_{\mathbb{P}^d_K}(U)$,

$$\mathfrak{x} \cdot (\Xi \cdot f) = \Xi \cdot (\mathfrak{x} \cdot f) + (\mathfrak{x} \cdot \Xi) \cdot f. \tag{1.4}$$

Here we consider the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{P}^d_K}$ with its natural $\mathbf{G}$-linearization. In this case we can specify the action of $\mathfrak{g}$ on $\mathcal{O}$. Indeed, for a root $\alpha = \alpha_{i,j} \in \Phi$, let

$$L_\alpha := L_{(i,j)} \in \mathfrak{g}_\alpha$$

be the standard generator of the weight space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$. Let $\mu \in X^*(\overline{\mathbf{T}})$ be a character of the torus $\overline{\mathbf{T}}$. Write μ in the shape $\mu = \sum_{i=0}^d m_i \epsilon_i$ with $\sum_{i=0}^d m_i = 0$. Define $\Xi_\mu \in \mathcal{O}(\mathcal{X})$ by

$$\Xi_\mu(q_0, \ldots, q_d) = q_0^{m_0} \cdots q_d^{m_d}.$$

For these functions, the action of $\mathfrak{g}$ is given by

$$L_{(i,j)} \cdot \Xi_\mu = m_j \cdot \Xi_{\mu + \alpha_{i,j}} \tag{1.5}$$

and

$$t \cdot \Xi_\mu = (\sum\nolimits_i m_i t_i) \cdot \Xi_\mu, \ t \in \mathfrak{t}.$$

Fix an integer $0 \leq j \leq d-1$. Let

$$\mathbb{P}^j_K = V(X_{j+1}, \ldots, X_d) \subset \mathbb{P}^d_K$$

be the closed K-subvariety defined by the vanishing of the coordinates $X_{j+1}, \ldots, X_d$. The algebraic local cohomology modules $H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}),\ i \in \mathbb{N}$, sit in a long exact sequence

$$\cdots \to H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F}) \to H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) \to H^i(\mathbb{P}^d_K, \mathcal{F}) \to H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F}) \to \cdots$$

Let $\mathcal{H}^*_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ be the local cohomology sheaf with support in the closed subvariety $\mathbb{P}^j_K$. It is related to the local cohomology groups by a spectral sequence (cf. [SGA2], Theorem 2.6.)

$$E_2^{p,q} = H^p(\mathbb{P}^d_K, \mathcal{H}^q_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})) \Longrightarrow H^{p+q}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}).$$

Since $\mathbb{P}^j_K$ and $\mathbb{P}^d_K$ are both smooth, the local cohomology groups $\mathcal{H}^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ vanish for $i \neq d-j$ [SGA2]. It follows that

$$H^p(\mathbb{P}^d_K, \mathcal{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})) \cong H^{p+d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$$

for all $p \in \mathbb{N}$. In particular,

$$H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) = 0 \text{ for } i < d-j.$$

On the other hand, the cohomology groups $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F})$ can be computed by the Čech complex

$$\begin{aligned} &\bigoplus_{j+1 \leq k \leq d} \mathcal{F}(D_+(X_k)) \to \bigoplus_{j+1 \leq k_1 < k_2 \leq d} \mathcal{F}(D_+(X_{k_1} \cdot X_{k_2})) \to \cdots \to \mathcal{F}(D_+(X_{j+1} \cdots X_d)) \\ &= \bigoplus_{j+1 \leq k \leq d} (F_{X_k})^0 \to \bigoplus_{j+1 \leq k_1 < k_2 \leq d} (F_{X_{k_1} \cdot X_{k_2}})^0 \to \cdots \to (F_{X_{j+1} \cdots X_d})^0. \end{aligned} \tag{1.6}$$

Here for a homogeneous polynomial $f \in S$, the set $D_+(f) \subset \mathbb{P}^d_K$ denotes as usual the Zariski open subset of $\mathbb{P}^d_K$, where f does not vanish. The symbol ^{0} indicates the degree zero contribution of a graded module.

Alternatively, we may compute $H^{d-j-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F})$ for $j \leq d-2$, by the inductive limit

$$\varinjlim_{n \in \mathbb{N}} (F/(X^n_{j+1}, \ldots, X^n_d)F)^0, \tag{1.7}$$

cf. [EGAIII] Prop. 2.1.5. In the case $j = d-1$, we have merely an exact sequence

$$0 \to H^0(\mathbb{P}^d_K, \mathcal{F}) \to H^0(\mathbb{P}^d_K \setminus \mathbb{P}^{d-1}_K, \mathcal{F}) \to \varinjlim_{n \in \mathbb{N}} (F/X^n_d F)^0 \to 0.$$

Here, the (twisted) degree of a coset $[f] \in F/(X^n_{j+1}, \ldots, X^n_d)$ is by definition

$$\deg([f]) - n \cdot (d-j),$$

where $\deg([f])$ is the ordinary degree of $[f]$, cf. loc.cit. 2.1. Sometimes, when we write down elements of $\varinjlim_{n \in \mathbb{N}} F/(X^n_{j+1}, \ldots, X^n_d)F$, we use generalized fractions. This terminology arises from the natural embedding of K-vector spaces

$$\varinjlim_{n \in \mathbb{N}} (F/(X^n_{j+1}, \ldots, X^n_d)F)^0 \hookrightarrow (F_{X_{j+1} \cdot \ldots \cdot X_d})^0.$$

For $[f] \in (F/(X^n_{j+1}, \ldots, X^n_d)F)^0$, we use the symbol

$$\begin{bmatrix} [f] \\ X^n_{j+1} \cdot \ldots \cdot X^n_d \end{bmatrix}$$

for its image in $(F_{X_{j+1}\cdot\ldots\cdot X_d})^0$. The transition maps are then simply given by

$$\begin{bmatrix} [f] \\ X_{j+1}^n\cdot\ldots\cdot X_d^n \end{bmatrix} \mapsto \begin{bmatrix} [X_{j+1}\cdot\ldots\cdot X_d\cdot f] \\ X_{j+1}^{n+1}\cdot\ldots\cdot X_d^{n+1} \end{bmatrix}.$$

We deduce from (1.6) that $H^*(\mathbb{P}^d_K\setminus\mathbb{P}^j_K,\mathcal{F})=0$ for $i\geq d-j$ and consequently

$$H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})=H^i(\mathbb{P}^d_K,\mathcal{F})$$

for $i>d-j$. In the case $i=d-j$, we get an exact sequence

$$\begin{aligned} 0\to\ & H^{d-j-1}(\mathbb{P}^d_K,\mathcal{F}) &&\to H^{d-j-1}(\mathbb{P}^d_K\setminus\mathbb{P}^j_K,\mathcal{F})\to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F}) \\ \to\ & H^{d-j}(\mathbb{P}^d_K,\mathcal{F}) &&\to 0. \end{aligned} \tag{1.8}$$

There is a natural algebraic action of $\mathbf{P_{(j+1,d-j)}}$ on each of the entries in this sequence, since the parabolic subgroup stabilizes $\mathbb{P}^j_K$. In particular, this sequence is equivariant with respect to this action. Furthermore, by (1.3) the Lie algebra $\mathfrak{g}$ acts by functoriality on all the cohomology groups, so that the sequence is equivariant for $\mathfrak{g}$, too. We set

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F}):=\ker\left(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})\to H^{d-j}(\mathbb{P}^d_K,\mathcal{F})\right)$$

which is consequently a $\mathbf{P_{(j+1,d-j)}}\ltimes U(\mathfrak{g})$-module. Here the semi-direct product is defined via the adjoint action of $\mathbf{P_{(j+1,d-j)}}$ on $\mathfrak{g}$. Indeed, for a section f of $\mathcal{F}$ and $\mathfrak{z}\in\mathfrak{g}$ resp. $p\in P_{(j+1,d-j)}$, we compute

$$\frac{d}{dT}((1+\mathfrak{z}T)(p\cdot f))=\frac{d}{dT}(p\cdot(p^{-1}\cdot(1+\mathfrak{z}T)\cdot p)\cdot f))=p\cdot\frac{d}{dT}((1+(p^{-1}\cdot\mathfrak{z}\cdot p)T)\cdot f)).$$

This compatibility transfers by functoriality onto the cohomology groups.

Lemma 1.2.1. *There exists a finite-dimensional $\mathbf{P_{(j+1,d-j)}}$-invariant K-subspace*

$$N_j\subset\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$$

which generates $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$ as $U(\mathfrak{g})$-module.

Proof. Consider the formula (1.7). The parabolic subgroup $\mathbf{P_{(j+1,d-j)}}$ acts on each entry appearing in the inductive limit separately. Each entry is a finite-dimensional K-vector space. Let $S/(X_{j+1}\ldots X_d)\to S$ be the K-linear section of the projection, given by $X_i\mapsto X_i$ for $i\leq j$, and $X_i\mapsto 0$ for $i\geq j+1$. This map induces a K-linear section $F/(X_{j+1},\ldots,X_d)F\to F$. Denote by F' the image of this section which forms consequently a system of representatives of $F/(X_{j+1},\ldots,X_d)F$. Then $(F/(X_{j+1},\ldots,X_d)F)^0$ may be identified with the homogeneous elements $f\in F_1:=F'$

of degree $d-j$. Similarly, $(F/(X_{j+1}^2, \ldots, X_d^2)F)^0$ may be identified with the homogeneous elements

$$f \in F_2 := F' \oplus \bigoplus_{k \geq j+1}^{d} X_k \cdot F' \oplus \bigoplus_{\substack{k,l \geq j+1 \\ k \neq l}}^{d} X_k \cdot X_l \cdot F' \oplus \cdots \oplus X_{j+1} \cdots X_d \cdot F'$$

of degree $2(d-j)$, etc. Under this identification the outer term $X_{j+1} \cdots X_d \cdot F'$ coincides with the image of the first transition map in (1.7). Since F is a finitely generated graded S-module, there is an integer $n \in \mathbb{N}$, such that any homogeneous representative $f \in F'$ of degree $n(d-j)$ is divisible by some monomial $X_0^{k_0} \cdot X_1^{k_1} \cdot \cdots \cdot X_j^{k_j}$ of degree $d-j$. Set

$$N_j = \operatorname{im}\Big((F/(X_{j+1}^{n-1}, \ldots, X_d^{n-1}))^0 \longrightarrow (F_{X_{j+1} \cdots X_d})^0\Big).$$

Thus

$$N_j = \Big\{ \left[\frac{[f]}{X_{j+1}^{n-1} \cdot \ldots \cdot X_d^{n-1}}\right] \mid f \in F_{n-1} \text{ homogeneous of degree } (n-1)(d-j) \Big\}.$$

We claim that this finite-dimensional $\mathbf{P_{(j+1,d-j)}}$-invariant K-subspace satisfies the condition of our lemma. In fact, let $f \in F_n$ be a homogeneous element of degree $n(d-j)$. By assumption, we may assume that

$$f = X_0^{k_0} \cdot X_1^{k_1} \cdot \cdots \cdot X_j^{k_j} \cdot X_{j+1}^{k_{j+1}} \cdots X_d^{k_d} \cdot g$$

with $g \in F_{n-1}$ and $\sum_l k_l = d-j$. Further, we may assume that $k_i \geq 1$ for at least one $i \leq j$. Consider the identity

$$L_{(i,j+1)} \cdot \left[\frac{[g]}{X_{j+1}^{n-1} \cdot \ldots \cdot X_d^{n-1}}\right] = \left[\frac{[L_{(i,j+1)} \cdot g]}{X_{j+1}^{n-1} \cdot \ldots \cdot X_d^{n-1}}\right] - (n-1) \left[\frac{[X_i \cdot X_{j+2} \cdot \cdots \cdot X_d \cdot g]}{X_{j+1}^{n} \cdot \ldots \cdot X_d^{n}}\right].$$

The left hand side and the first summand are contained in $U(\mathfrak{g}) \cdot N_j$. It follows that $\left[\frac{[X_i \cdot X_{j+2} \cdot \cdots \cdot X_d \cdot g]}{X_{j+1}^{n} \cdot \ldots \cdot X_d^{n}}\right]$ is contained in $U(\mathfrak{g}) \cdot N_j$. By induction on the indices $l \leq j$ with $k_l \geq 1$, we see that $\left[\frac{[f]}{X_{j+1}^{n} \cdot X_{j+2}^{n} \cdot \ldots \cdot X_d^{n}}\right] \in U(\mathfrak{g}) \cdot N_j$. The case where $F \in F_{n'}$ with $n' > n$ follows inductively, as well. $\square$

Remark 1.2.2. Alternatively we can prove Lemma 1.2.1 by using Corollary 1.4.9 in section 1.4. This section is independent of the results in 1.2 and 1.3. In the case where $\mathbf{U} = \mathbf{U_{(1,d)}}$ acts trivially on the fibre V of $\mathcal{F}$, it produces an explicit candidate. In the general case, we know that the fix point set $V^{\mathbf{U}} \neq 0$ is non-trivial since $\mathbf{U}$ is unipotent, cf. [Ja] ch.I, 2.14 (8). Consider the exact sequence

$$0 \to V^{\mathbf{U}} \to V \to V/V^{\mathbf{U}} \to 0$$

of algebraic $\mathbf{P_{(1,d)}}$-modules, which induces an exact sequence of homogeneous vector bundles

$$0 \to \mathcal{F}_{V^{\mathbf{U}}} \to \mathcal{F}_V \to \mathcal{F}_{V/V^{\mathbf{U}}} \to 0$$

on $\mathbb{P}^d_K$. We get an equivariant long exact sequence

$$\begin{aligned} 0 \to \quad & H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_{V^{\mathbf{U}}}) \quad \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_V) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_{V/V^{\mathbf{U}}}) \\ \to \quad & H^{d-j+1}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_{V^{\mathbf{U}}}) \quad \to \cdots \end{aligned}$$

of $\mathbf{P_{(j+1,d-j)}} \ltimes U(\mathfrak{g})$-modules. The groups $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_W)$ and $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_W)$, $W \in \{V, V^{\mathbf{U}}, V/V^{\mathbf{U}}\}$, differ only by the finite-dimensional K-vector space $H^{d-j}(\mathbb{P}^d_K, \mathcal{F}_W)$. By Corollary 1.4.9 and by induction on the dimension of V, there are finite-dimensional $\mathbf{P_{(i+1,d-i)}}$-submodules of the outer terms generating them as $U(\mathfrak{g})$-modules. But then the statement is true for the middle term $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_V)$ of the exact sequence and thus for $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}_V)$. □

1.3. **Analytic local cohomology.** In the following we study the analytic local cohomology groups $H^{d-i}_{\mathbb{P}^i_K(\epsilon)}(\mathbb{P}^d_K, \mathcal{F})$ as topological K-vector spaces. We shall prove a local duality theorem which describes the topological dual of $\ker\big(H^{d-i}_{\mathbb{P}^i_K(\epsilon)}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-i}(\mathbb{P}^d_K, \mathcal{F})\big)$ for $\epsilon \to 0$, by means of analytic functions on certain polydiscs.

Let X be a rigid analytic variety over K and consider a coherent sheaf $\mathcal{G}$ on X. Let $U \subset X$ be an admissible open subset and let $Y := X \setminus U$ be its set theoretical complement. Then the local algebraic cohomology groups $H^*_Y(X, \mathcal{G})$ are defined by the right derived functors of

$$\ker\big(\Gamma(X, \mathcal{G}) \to \Gamma(U, \mathcal{G})\big).$$

If X is a separated rigid analytic variety of countable type one can equip these cohomology groups with a locally convex topology as follows, cf. [vP]. For a rigid analytic variety X of countable type, the space of global sections $\mathcal{G}(X) = \Gamma(X, \mathcal{G})$ has a natural structure of a K-Fréchet space. If X is an arbitrary separated rigid analytic variety of countable type with an admissible covering $X = \bigcup_i X_i$ by affinoids resp. by (quasi-) Stein spaces [K2], one considers the corresponding Čech complex

$$\prod_i \mathcal{G}(X_i) \to \prod_{i<j} \mathcal{G}(X_i \cap X_j) \to \cdots$$

computing $H^*(X, \mathcal{G})$. All contributions are K-Fréchet spaces, in particular, they are locally convex Hausdorff K-vector spaces. Hence they induce on the cohomology groups $H^*(X, \mathcal{G})$ in a natural way a locally convex topology. This topology does not depend on the covering $X = \bigcup X_i$, cf. [Ba] Lemma 1.32. We point out that the topology on the cohomology is in general not Hausdorff. Finally, we consider the long exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow\ & H^i_Y(X,\mathcal{G}) \longrightarrow H^i(X,\mathcal{G}) \longrightarrow H^i(U,\mathcal{G}) \\ \stackrel{\delta^i}{\longrightarrow}\ & H^{i+1}_Y(X,\mathcal{G}) \longrightarrow \cdots \end{aligned}$$

The cohomology groups $H^i_Y(X,\mathcal{G})$ are equipped with the finest locally convex topology such that the boundary maps δ^i become continuous. It turns out that the long exact cohomology sequence is then even topological[2] exact, cf. Lemma 5.1 in [S2].

We are interested in the analytic cohomology groups $H^*_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})$ where $\mathcal{F}$ is our fixed homogenous vector bundle. Recall that $\epsilon_n = |\pi^n|,\ n \in \mathbb{N}$. By GAGA (cf. [K2] §4), we know that

$$H^i(\mathbb{P}^d_K,\mathcal{F}) = H^i((\mathbb{P}^d_K)^{rig},\mathcal{F})$$

for $i \geq 0$. The cohomology group $H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})$ sits in the long exact cohomology sequence

$$\begin{aligned} \cdots \rightarrow\ & H^{i-1}((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n),\mathcal{F}) \rightarrow H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \rightarrow H^i(\mathbb{P}^d_K,\mathcal{F}) \\ \rightarrow\ & H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n),\mathcal{F}) \rightarrow \cdots . \end{aligned}$$

As for the computation of $H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n),\mathcal{F})$, we consider the Čech complex

$$\begin{aligned} \bigoplus_{j+1\leq k\leq d} \mathcal{F}(D_+(X_k)^-_{\epsilon_n}) \rightarrow \bigoplus_{j+1\leq k_1<k_2\leq d} \mathcal{F}(D_+(X_{k_1})^-_{\epsilon_n} \cap D_+(X_{k_2})^-_{\epsilon_n}) \rightarrow \cdots \\ \cdots \rightarrow \mathcal{F}(D_+(X_{j+1})^-_{\epsilon_n} \cap \cdots \cap D_+(X_d)^-_{\epsilon_n}) \end{aligned}$$

with respect to the covering of Stein spaces

$$(\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n) = \bigcup_{k=j+1}^{d} D_+(X_k)^-_{\epsilon_n},$$

where

$$D_+(X_k)^-_{\epsilon_n} := \Big\{[x_0:\ldots:x_d] \in (\mathbb{P}^d_K)^{rig} \mid |x_k| > |x_l|\cdot\epsilon_n \ \ \forall\, l\Big\}.$$

For $1 \geq \epsilon > 0$, we set

$$D_+(X_k)_\epsilon := \Big\{[x_0:\ldots:x_d] \in (\mathbb{P}^d_K)^{rig} \mid |x_k| \geq |x_l|\cdot\epsilon \ \ \forall\, l\Big\}.$$

[2] A topological exact sequence (or topological complex) of topological vector spaces is an algebraic exact sequence (complex) $\cdots \rightarrow E^{i-1} \rightarrow E^i \rightarrow E^{i+1} \rightarrow \cdots$, such that all homomorphisms are continuous.

These are affinoid rigid analytic varieties and we can write

$$(\mathbb{P}^d_K)^{rig} \setminus (\mathbb{P}^j_K)(\epsilon)^- = \bigcup_{k=j+1}^{d} D_+(X_k)_\epsilon.$$

Thus, we get an admissible covering

$$(\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n) = \bigcup_{\substack{\epsilon \to \epsilon_n \\ \epsilon_n < \epsilon \in |\overline{K^*}|}} (\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon)^-$$

by quasi-compact admissible open subsets. Consider the Čech complex computing $H^*((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon)^-, \mathcal{F})$:

$$(1.9) \quad \bigoplus_{j+1 \le k \le d} \mathcal{F}(D_+(X_k)_\epsilon) \to \bigoplus_{j+1 \le k_1 < k_2 \le d} \mathcal{F}(D_+(X_{k_1})_\epsilon \cap D_+(X_{k_2})_\epsilon) \to \cdots$$
$$\cdots \to \mathcal{F}(D_+(X_{j+1})_\epsilon \cap \cdots \cap D_+(X_d)_\epsilon)$$

Lemma 1.3.1. *The cohomology groups $H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon)^-, \mathcal{F})$ (resp. $H^i_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K, \mathcal{F})$), $i, j = 0, \ldots, d$, are K-Banach spaces in which the algebraic cohomology $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F})$ (resp. $H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$) is a dense subspace.*

Proof. First we treat the case $\mathcal{F} = \mathcal{O}$. Set $\epsilon' = \frac{1}{\epsilon}$. Consider the Gauss-Norm $|\ |_{\epsilon'}$ on the homogenous localization $(K[X_0, \ldots, X_d]_{X_0 \cdots X_d})^0 = \mathcal{O}(D_+(X_0 \cdot \ldots \cdot X_d))$ given as follows. Let $f \in K[X_0, \ldots, X_d]$ be a homogeneous polynomial of degree $n(d+1)$, $n \in \mathbb{N}$. Write

$$f = \sum_{i_0 + \cdots + i_d = n(d+1)} a_{i_0 \cdots i_d} X_0^{i_0} \cdot \ldots \cdot X_d^{i_d}.$$

Then

$$\left|\frac{f}{(X_0 \cdots X_d)^n}\right|_{\epsilon'} = \max_{i_0, \ldots, i_d} |a_{i_0 \cdots i_d}| (\epsilon')^{r(i_0, \cdots, i_d)}$$

Here $r(i_0, \cdots, i_d) = \sum_{i_j \ge n} (i_j - n)$. Then for every subset $\{i_1, \ldots, i_{r+1}\} \subset \{0, \ldots, d\}$, the restriction maps

$$\mathcal{O}(D_+(X_{i_1}) \cap \cdots \cap \widehat{D_+(X_{i_j})} \cap \cdots \cap D_+(X_{i_{r+1}})) \to \mathcal{O}(D_+(X_{i_1}) \cap \cdots \cap D_+(X_{i_{r+1}}))$$

are isometries. Here the symbol $\widehat{D_+(X_{i_j})}$ indicates that we omit the open subset $D_+(X_{i_j})$ from the intersection. The images of the differentials in (1.6) are closed, in particular the differentials are strict[3] . In fact, it suffices to show that the image of the maps

$$\bigoplus_{j=1}^{r+1} \mathcal{O}(D_+(X_{i_1}) \cap \cdots \cap \widehat{D_+(X_{i_j})} \cap \cdots \cap D_+(X_{i_{r+1}})) \xrightarrow{\delta} \mathcal{O}(D_+(X_{i_1}) \cap \cdots \cap D_+(X_{i_{r+1}}))$$

are closed. But

$$\mathrm{im}(\delta) = \Big\{ \sum_{i_0+\cdots+i_d=0} a_{i_0\cdots i_d} X_0^{i_0} \cdots X_d^{i_d} \mid a_{i_0\cdots i_d} = 0 \ \text{ if } i_k < 0 \text{ for some } k \notin \{i_1, \ldots, i_{r+1}\}$$

$$\text{resp. if } i_j < 0 \text{ for all } j = 1, \ldots, r+1 \Big\}.$$

Further, the completion of (1.6) with respect to the norm $|\ |_{\epsilon'}$ is exactly (1.9). By [BGR] section 1.2, Cor. 6, the completion functor is exact for strict homomorphism. Hence the statement of our lemma follows in the case $\mathcal{F} = \mathcal{O}$.

For an arbitrary vector bundle $\mathcal{F}$, we use the fact that it splits on the affine sets $D_+(X_i)$ as a direct sum of $\mathrm{rk}\,\mathcal{F}$ copies of $\mathcal{O}$. Again the complex (1.9) is the completion of (1.6) and the images of the differentials are closed. $\square$

Now we treat the situation of the open tubes $\mathbb{P}^d_K(\epsilon_n) \subset (\mathbb{P}^d_K)^{rig}$. We shall see that $H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n), \mathcal{F})$ respectively $H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$ are naturally K-Fréchet spaces in which the algebraic cohomology $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{F})$ respectively $H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ is a dense subset. More precisely, we can write these cohomology groups as projective limits of K-Banach spaces:

Lemma 1.3.2. *We have*

$$H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n), \mathcal{F}) = \varprojlim_{\substack{\epsilon \to \epsilon_n \\ \epsilon_n < \epsilon \in |\overline{K^*}|}} H^i((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon)^-, \mathcal{F})$$

respectively

$$H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) = \varprojlim_{\substack{\epsilon \to \epsilon_n \\ \epsilon_n < \epsilon \in |\overline{K^*}|}} H^i_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K, \mathcal{F}).$$

Proof. In fact, the compatibility with the projective limit follows from the following propositions. Here, the density condition follows from the previous lemma. $\square$

[3]Recall that a homomorphism $f : V \to W$ of topological vector spaces is strict if the induced homomorphism $V/\mathrm{ker} f \to \mathrm{im} f$ of topological vector spaces with the inherited topologies is a homeomorphism, cf. [BGR].

Proposition 1.3.3. *Let $\mathcal{G}$ be a coherent sheaf on a rigid analytic variety X. Consider a decreasing family of subsets $Y_1 \supset Y_2 \supset \cdots \supset Y_k \supset Y_{k+1} \supset \cdots$ in X, such that every subset $X \setminus Y_i$ is admissible open in X. Set $Y := \bigcap_{k\in\mathbb{N}} Y_k$ and assume that $X \setminus Y$ is admissible open in X, as well. Suppose that all cohomology groups $H^{i-1}_{Y_k}(X,\mathcal{G})$ are K-Fréchet spaces, such that the images of the transition maps $H^{i-1}_{Y_{k+1}}(X,\mathcal{G}) \to H^{i-1}_{Y_k}(X,\mathcal{G})$ are dense for $k \in \mathbb{N}$. Then there is a topological isomorphism*

$$\varprojlim_{k\in\mathbb{N}} H^i_{Y_k}(X,\mathcal{G}) \cong H^i_Y(X,\mathcal{G})$$

of K-Fréchet spaces.

Proof. By the same reasoning as in Proposition 4 on §2 of [SS] (cf. also Proposition 2.2.1), we have a short exact sequence

$$0 \to \varprojlim_k^{(1)} H^{i-1}_{Y_k}(X,\mathcal{G}) \to H^i_Y(X,\mathcal{G}) \to \varprojlim_k H^i_{Y_k}(X,\mathcal{G}) \to 0.$$

But the projective system $(H^{i-1}_{Y_k}(X,\mathcal{G}))_{k\in\mathbb{N}}$ of K-Fréchet spaces has the topological Mittag-Leffler property by our condition on the density, cf. [EGAIII] 13.2.4. Thus we get by loc.cit. 13.2.3 an algebraic isomorphism $p : H^i_Y(X,\mathcal{G}) \xrightarrow{\sim} \varprojlim_k H^i_{Y_k}(X,\mathcal{G})$. But p is continuous and $\varprojlim_k H^i_{Y_k}(X,\mathcal{G})$ is a K-Fréchet space. It follows from the bijectivity that $H^i_Y(X,\mathcal{G})$ has to be Hausdorff. Since it is a quotient of a K-Fréchet space it has to be a K-Fréchet space, as well. Now the claim follows from the open mapping theorem [S2] 8.6. □

Analogously one proves the "dual" version of this Proposition.

Proposition 1.3.4. *Let $\mathcal{G}$ be a coherent sheaf on a rigid analytic variety X. Let $U \subset X$ be an admissible open subset and consider an increasing family of open admissible subsets $U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots \subset U$ of U with $\bigcup_{k\in\mathbb{N}} U_k = U$. Suppose that all cohomology groups $H^{i-1}(U_k,\mathcal{G})$ are K-Fréchet spaces and that the images of the transition maps $H^{i-1}(U_{k+1},\mathcal{G}) \to H^{i-1}(U_k,\mathcal{G})$ are dense for $k \in \mathbb{N}$. Then there is a topological isomorphism*

$$\varprojlim_{k\in\mathbb{N}} H^i(U_k,\mathcal{G}) = H^i(U,\mathcal{G}).$$

Remark 1.3.5. In [SS] Corollary 5 the authors consider a similar question concerning the compatibility with projective limits. They deal with constant coefficients in which all the cohomology groups are finitely generated modules over an artinian ring, so that the usual Mittag-Leffler property holds. □

Remark 1.3.6. An alternative way for proving Lemma 1.3.2 is to apply the following Lemma to the Čech complex (1.9).

Lemma 1.3.7. *Let* $0 \to V_n^1 \to V_n^2 \to V_n^3 \to 0$, $n \in \mathbb{N}$, *be a projective system of topological exact sequences of* K*-Banach spaces (or more generally of* K*-Fréchet spaces). Suppose that the transition maps* $V_{n+1}^1 \to V_n^1$, $n \in \mathbb{N}$, *have dense image. Then the sequence*

$$0 \to \varprojlim_n V_n^1 \to \varprojlim_n V_n^2 \to \varprojlim_n V_n^3 \to 0$$

is topological exact, too.

Proof: The exactness follows from the topological Mittag-Leffler property, cf. [EGAIII], 13.2.4. □

□

It follows from Lemma 1.3.2 that

$$H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) = 0 \ \text{ for } i < d-j \tag{1.10}$$

and

$$H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) = H^i(\mathbb{P}^d_K, \mathcal{F}) \ \text{ for } i > d-j.$$

Put $G_0 = \mathbf{G}(O_K)$. For any positive integer $n \in \mathbb{N}$, we consider the reduction map

$$p_n : G_0 \to \mathbf{G}(O_K/(\pi^n)). \tag{1.11}$$

Put

$$P^n_{(j+1,d-j)} := p_n^{-1}\big(\mathbf{P_{(j+1,d-j)}}(O_K/(\pi^n))\big).$$

This is a compact open subgroup of G_0 which stabilizes $\mathbb{P}^j_K(\epsilon_n)$. Again, as in the algebraic setting, we have an exact $P^n_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-equivariant topological complex

$$\begin{aligned} 0 \ &\to \ H^{d-j-1}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j-1}((\mathbb{P}^d_K)^{rig} \setminus \mathbb{P}^j_K(\epsilon_n), \mathcal{F}) \to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) \\ &\to \ H^{d-j}(\mathbb{P}^d_K, \mathcal{F}) \to 0. \end{aligned}$$

Proposition 1.3.8. *The action* $P^n_{(j+1,d-j)} \times H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$ *is continuous.*

Proof. We follow the proof of Lemma 1.3 in [ST1]. Since $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$ is a K-Fréchet space it is by the same reasoning as there enough to show that for $m > n$, the orbit maps (into K-Banach spaces) $P^m_{(j+1,d-j)} \to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K, \mathcal{F})$ are locally analytic. We may assume that $\mathcal{F} = \mathcal{O}$, cf. Prop. 2.1' in loc.cit. The cohomology group $H^{d-j}_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K, \mathcal{O})$ is a quotient of $H^0(g \cdot (D_+(X_{j+1})_\epsilon \cap \cdots \cap D_+(X_d)_\epsilon), \mathcal{O})$ for all $g \in P^m_{(j+1,d-j)}$. Let $g \in P^m_{(j+1,d-j)}$ and $F \in H^0(D_+(X_{j+1})_\epsilon \cap \cdots \cap D_+(X_d)_\epsilon, \mathcal{O})$. We choose an open neighborhood Q of g in $P^m_{(j+1,d-j)}$ such that $h\cdot(D_+(X_{j+1})_\epsilon\cap\cdots\cap D_+(X_d)_\epsilon) = g\cdot(D_+(X_{j+1})_\epsilon\cap\cdots\cap D_+(X_d)_\epsilon)\,\forall h\in Q$. Then it suffices to see that the induced map $Q\to H^0(p\cdot(D_+(X_{j+1})_\epsilon\cap\cdots\cap D_+(X_d)_\epsilon),\mathcal{O}),\ h\mapsto hF$, is locally analytic. We may assume that $g=1$. Then we apply the same argument as in Prop. 2.1' loc.cit. $\square$

Corollary 1.3.9. *The dual space $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$ is a locally analytic $P^n_{(j+1,d-j)}$-representation.*

Proof. The dual space $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$ is by Lemma 1.3.2 the locally convex inductive limit

$$H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})' = \varinjlim_{\substack{\epsilon\to\epsilon_n\\ \epsilon_n<\epsilon\in|\overline{K^*}|}} H^i_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K,\mathcal{F})'$$

of duals of K-Banach spaces. In the proof of the previous proposition we have seen that the orbit maps $P^m_{(j+1,d-j)}\to H^i_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K,\mathcal{F})$ are locally analytic. Thus the orbit maps on the dual space are locally analytic. The claim follows. $\square$

Set

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) := \ker\big(H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})\to H^{d-j}(\mathbb{P}^d_K,\mathcal{F})\big).$$

This K-Fréchet space has the structure of a $P^n_{(j+1,d-j)}\ltimes U(\mathfrak{g})$-module in which the algebraic cohomology $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$ is a dense subspace.

We apply Lemma 1.2.1 to obtain a $P_{(j+1,d-j)}$-invariant finite-dimensional K-subspace

$$N_j\subset \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$$

which generates $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$ as $U(\mathfrak{g})$-module. Thus $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$ is a quotient of a generalized Verma module [Le]. More precisely, there is an epimorphism

$$\varphi_j: U(\mathfrak{g})\otimes_{U(\mathfrak{p}_{(j+1,d-j)})} N_j\to \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F})$$

of $U(\mathfrak{g})$-modules. Since the universal enveloping algebra splits into a tensor product $U(\mathfrak{g}) = U(\mathfrak{u}^+_{(j+1,d-j)})\otimes_K U(\mathfrak{p}_{(j+1,d-j)})$, we may regard φ_j as an epimorphism

$$\varphi_j: U(\mathfrak{u}^+_{(j+1,d-j)})\otimes_K N_j\to \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{F}). \tag{1.12}$$

Denote by $\mathfrak{d}_j=\ker(\varphi_j)$ the kernel of this map.

Consider the affine algebraic group $\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})}$. The Levi subgroup $\mathbf{L}_{(\mathbf{j+1},\mathbf{d-j})}$ stabilizes $\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})}$ with respect to the action of conjugation. Let

$$\Phi_j=\{\beta_1,\ldots,\beta_r\}$$

be the set of roots of $\mathfrak{u}^+_{(j+1,d-j)}$. The K-algebra $\mathcal{O}(\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})})$ of algebraic functions on $\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})}$ may be viewed as the polynomial K-algebra in the indeterminates $X_{\beta_1},\ldots,X_{\beta_r}$. Consider the $L_{(j+1,d-j)}\cdot U^+_{(j+1,d-j)}$-equivariant pairing

$$\begin{array}{rcl} \mathcal{O}(\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})})\times U(\mathfrak{u}^+_{(j+1,d-j)}) & \rightarrow & K \\ (f,\mathfrak{z}) & \mapsto & \mathfrak{z}\cdot f(1). \end{array} \tag{1.13}$$

This is a non-degenerate pairing and induces therefore a K-linear $L_{(j+1,d-j)}\cdot U^+_{(j+1,d-j)}$-equivariant injection

$$\mathcal{O}(\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})})\hookrightarrow \mathrm{Hom}_K(U(\mathfrak{u}^+_{(j+1,d-j)}),K).$$

More concretely, this map is given by

$$X^{i_1}_{\beta_1}\cdots X^{i_r}_{\beta_r}\mapsto (i_1)!\cdots(i_r)!\cdot(L^{i_1}_{\beta_1}\cdots L^{i_r}_{\beta_r})^*$$

where

$$\big\{(L^{i_1}_{\beta_1}\cdots L^{i_r}_{\beta_r})^*\mid (i_1,\ldots,i_r)\in\mathbb{N}_0^r\big\}$$

is the dual basis of $\{L^{i_1}_{\beta_1}\cdots L^{i_r}_{\beta_r}\mid (i_1,\ldots,i_r)\in\mathbb{N}_0^r\}$.

Put

$$U^{+,n}_{(j+1,d-j)}=\ker\big(\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})}(O_K)\to\mathbf{U}^+_{(\mathbf{j+1},\mathbf{d-j})}(O_K/(\pi^n))\big).$$

Thus we have the identity

$$P^n_{(j+1,d-j)}=\mathbf{P}_{(j+1,d-j)}(O_K)\cdot U^{+,n}_{(j+1,d-j)}.$$

Further, we may interpret $U^{+,n}_{(j+1,d-j)}\subset U^+_{(j+1,d-j)}$ as an open K-affinoid polydisc, since all entries x in $U^{+,n}_{(j+1,d-j)}$ apart from the diagonal have norm $|x|\leq|\pi^n|$. Hence the ring of K-analytic functions $\mathcal{O}(U^{+,n}_{(j+1,d-j)})$ is a K-Banach algebra. The pairing (1.13) extends by continuity to a non-degenerate $\mathbf{L}_{(j+1,d-j)}(O_K)\cdot U^{+,n}_{(j+1,d-j)}$-equivariant pairing

$$\mathcal{O}(U^{+,n}_{(j+1,d-j)})\times U(\mathfrak{u}^+_{(j+1,d-j)})\to K \tag{1.14}$$

which in turn extends to a $P^n_{(j+1,d-j)}$-equivariant pairing

$$\begin{array}{c} (\ ,\ ):(\mathcal{O}(U^{+,n}_{(j+1,d-j)})\otimes N'_j)\times(U(\mathfrak{u}^+_{(j+1,d-j)})\otimes N_j)\to K. \\ (f\otimes\phi,\mathfrak{z}\otimes n)\mapsto\phi(n)\cdot\mathfrak{z}\cdot f(1) \end{array} \tag{1.15}$$

Here, the subgroup $\mathbf{U}_{(j+1,d-j)}(O_K)\subset P^n_{(j+1,d-j)}$ acts by definition trivially on the K-Banach space $\mathcal{O}(U^+_{(j+1,d-j)})$ respectively on $U(\mathfrak{u}^+_{(j+1,d-j)})$. We put

$$\mathcal{O}(U^{+,n}_{(j+1,d-j)},N'_j)^{\mathfrak{d}_j}:=\Big\{f\in\mathcal{O}(U^{+,n}_{(j+1,d-j)})\otimes N'_j\ \mid (f,\mathfrak{d}_j)=0\Big\}.$$

We obtain an equivariant injection

$$\mathcal{O}(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j} \hookrightarrow \mathrm{Hom}_K(U(\mathfrak{u}^+_{(j+1,d-j)}) \otimes N_j/\mathfrak{d}_j, K) \cong \mathrm{Hom}_K(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}), K).$$

On the other hand, we have an injection of the duals

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})' \hookrightarrow \mathrm{Hom}_K(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}), K),$$

since $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ is dense in $\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$. The following proposition says that for $n \to \infty$, these two topological K-vector spaces coincide in $\mathrm{Hom}_K(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}), K)$. It is based on the same principle as the duality theorem of Morita, cf. [Mo3] Theorem 2.

Proposition 1.3.10. *For $n \in \mathbb{N}$ tending to infinity, we get an isomorphism of (Hausdorff) locally convex K-vector spaces*

$$\varinjlim_{n\in\mathbb{N}} \mathcal{O}(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j} \stackrel{\sim}{\longrightarrow} \varinjlim_{n\in\mathbb{N}} \tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})'$$

compatible with the action of $\varprojlim_n P^n_{(j+1,d-j)} = \mathbf{P}_{(\mathbf{j+1},\mathbf{d-j})}(O_K)$.

Proof. Recall that we can express the K-Fréchet space $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$ by Lemma 1.3.2 as the projective limit of the K-Banach spaces $H^{d-j}_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K, \mathcal{F})$, where $\epsilon \to \epsilon_n$ and $\epsilon_n < \epsilon$. Since $H^{d-j}(\mathbb{P}^d_K, \mathcal{F})$ is finite-dimensional, we see that $\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K, \mathcal{F})$ is closed in $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K, \mathcal{F})$. We deduce the same compatibility for the K-Fréchet space $\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})$, i.e.,

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) = \varprojlim_{\substack{\epsilon \to \epsilon_n \\ \epsilon_n < \epsilon \in |\overline{K^*}|}} \tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon)^-}(\mathbb{P}^d_K, \mathcal{F}).$$

Therefore, we can replace the K-Fréchet spaces in the statement by the K-Banach spaces $\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K, \mathcal{F})$. We set for $\epsilon \in |\overline{K^*}|$,

$$U(\mathfrak{u}^+_{(j+1,d-j)})_\epsilon := \Big\{ \sum_{(i_1,\ldots,i_r)\in\mathbb{N}_0^r} a_{i_1,\ldots,i_r} L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r} \mid a_{i_1,\ldots,i_r} \in K, \quad |(i_1)! \cdots (i_r)! \cdot a_{i_1,\ldots,i_r}| \epsilon^{i_1+\cdots+i_r} \to 0, i_1 + \cdots + i_r \to \infty \Big\}.$$

This is a K-Banach algebra in which the universal enveloping algebra $U(\mathfrak{u}^+_{(j+1,d-j)})$ is a dense subset. We get an epimorphism of K-Banach spaces (use [BGR] Cor. 6 in 1.2),

$$U(\mathfrak{u}^+_{(j+1,d-j)})_{\frac{1}{\epsilon_n}} \otimes N_j \longrightarrow \tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K, \mathcal{F}). \tag{1.16}$$

On the other hand, let

$$\mathcal{O}_b(U^{+,n}_{(j+1,d-j)}) := \Big\{ \sum_{(i_1,\ldots,i_r)\in\mathbb{N}_0^r} a_{i_1,\ldots,i_r} X_{\beta_1}^{i_1}\cdots X_{\beta_r}^{i_r} \mid a_{i_1,\ldots,i_r}\in K, \sup_{(i_1,\ldots,i_r)} |a_{i_1,\ldots,i_r}|\epsilon_n^{i_1+\cdots+i_r} < \infty \Big\}$$

resp.

$$\mathcal{O}_b(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j} := \Big\{ f\in \mathcal{O}_b(U^{+,n}_{(j+1,d-j)})\otimes N'_j \mid (f,\mathfrak{d}_j)=0\Big\}$$

be the K-Banach spaces of bounded functions on $U^{+,n}_{(j+1,d-j)}$. Then

$$\varinjlim_n \mathcal{O}_b(U^{+,n}_{(j+1,d-j)}) = \varinjlim_n \mathcal{O}(U^{+,n}_{(j+1,d-j)})$$

resp.

$$\varinjlim_n \mathcal{O}_b(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j} = \varinjlim_n \mathcal{O}(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j}$$

are identities of locally convex K-vector spaces. But $\mathcal{O}_b(U^{+,n}_{(j+1,d-j)})$ is the topological dual of $U(\mathfrak{u}^{+}_{(j+1,d-j)})_{\frac{1}{\epsilon_n}}$ (cf. Example in [S2] ch. I, §3). We deduce from (1.15) together with (1.16) that $\mathcal{O}_b(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j}$ is the topological dual of $\tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)^-}(\mathbb{P}^d_K,\mathcal{F})$. The claim follows now from [Mo1] Theorem 3.4 respectively [S2] Prop. 16.10 on the duality of projective limits of K-Fréchet spaces and injective limits of K-Banach spaces with compact transition maps. □

Remark 1.3.11. The inductive limit $\varinjlim_{n\in\mathbb{N}} \tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$ identifies by Proposition 1.3.3 with the strong dual of the analytic local cohomology group $\tilde{H}^{d-j}_{(\mathbb{P}^j_K)^{rig}}((\mathbb{P}^d_K)^{rig},\mathcal{F})$. In particular, the action of $\mathbf{P_{(j+1,d-j)}}(O_K)$ on $\varinjlim_{n\in\mathbb{N}} \tilde{H}^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$ extends to one of $P_{(j+1,d-j)}$. On the other hand, the expression $\varinjlim_{n\in\mathbb{N}} \mathcal{O}(U^{+,n}_{(j+1,d-j)}, N'_j)^{\mathfrak{d}_j}$ can been thought as the stalk in the point $\mathbb{P}^j_K$ of the Grassmannian $\mathrm{Gr}_{j+1}(K^{d+1})$ of a certain "sheaf". Here the action extends to one of $P_{(j+1,d-j)}$, as well. It is easily seen that the isomorphism of Proposition 1.3.10 is even $P_{(j+1,d-j)}$-equivariant, where the unipotent radical $U_{(j+1,d-j)}$ acts on the pairings (1.15) via N'_j. Furthermore, it follows from 1.3.9 that the map is even an isomorphism of locally analytic $P_{(j+1,d-j)}$-representations. □

1.4. **Algebraic local cohomology II.** Let $\mathcal{F}$ be a homogeneous vector bundle on $\mathbb{P}^d_K$ which arises by a representation of the Levi subgroup $\mathbf{L_{(1,d)}}$ of $\mathbf{P_{(1,d)}}$, i.e., such that the unipotent radical $\mathbf{U_{(1,d)}}$ acts trivially on the fibre. In this section we compute explicit formulas for the K-vector spaces.

$$\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{F}) = \ker\big(H^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{F}) \to H^i(\mathbb{P}^d_K,\mathcal{F})\big)$$

as representations of $\mathbf{P_{(d-i+1,i)}} \ltimes U(\mathfrak{g})$. First we consider the local cohomology groups with respect to the closed subschemes $V(X_0,\ldots,X_{i-1}) \subset \mathbb{P}^d_K$ defined by the vanishing of the first i coordinate functions. Note that the stabilizer of this subvariety is the upper triangular parabolic subgroup $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}+\mathbf{1}-\mathbf{i})} \subset \mathbf{G}$. Afterwards, we use the conjugacy of $V(X_0, \ldots, X_{i-1})$ and $V(X_{d-i+1}, \ldots, X_d)$ within $\mathbb{P}^d_K$ via the action of $\mathbf{G}$ on $\mathbb{P}^d_K$. The reason is that we follow the notation used by Kempf in [Ke].

Let $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{P}_{(\mathbf{1},\mathbf{d})}$ be the projection map and identify $\mathbf{G}/\mathbf{P}_{(\mathbf{1},\mathbf{d})} \cong \mathbb{P}^d_K$ as described in section one. Let

$$\lambda' = (\lambda_1 \geq \ldots \geq \lambda_d) \in \mathbb{Z}^d$$

be a dominant integral weight of $\mathbf{GL_d}$. This gives rise to a finite-dimensional irreducible algebraic representation $V_{\lambda'}$ of $\mathbf{GL_d}$. Let $\lambda_0 \in \mathbb{Z}$ be arbitrary and put

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^{d+1}.$$

Extend the action of $\mathbf{L}_{(\mathbf{1},\mathbf{d})}$ on $V_{\lambda'}$ to one of $\mathbf{P}_{(\mathbf{1},\mathbf{d})}$ on the same space $V_\lambda := V_{\lambda'}$, such that $\mathbf{U}_{(\mathbf{1},\mathbf{d})}$ acts trivially on it and such that $\mathbb{G}_\mathbf{m}$ acts via multiplication

$$(x, v) \mapsto x^{\lambda_0} \cdot v,$$

$v \in V_\lambda \otimes_K \overline{K},\ x \in \mathbb{G}_\mathbf{m}(\overline{K})$. Denote by $\mathcal{F}_\lambda = \mathcal{F}_{V_\lambda}$ the corresponding homogeneous vector bundle on $\mathbb{P}^d_K$, cf. (1.2). Furthermore, if we add to λ the tuple $r \cdot (1, \ldots, 1)$, $r \in \mathbb{N}$, then the $\mathbf{G}$-linearization on $\mathcal{F}_\lambda$ is twisted by $\det^{\otimes r}$. Finally, we point out that $H^0(D_+(X_0), \mathcal{F}_\lambda)$ is isomorphic to $K[\frac{X_1}{X_0}, \ldots, \frac{X_d}{X_0}] \otimes V_\lambda$ as $\mathbf{P}^+_{(\mathbf{1},\mathbf{d})} \ltimes U(\mathfrak{g})$-module.

Example 1.4.1. The following identifications can be seen by the procedure used in [Ja], part II, 2.16. Note that we work with the contragredient identification of $\mathbf{G}/\mathbf{P}_{(\mathbf{1},\mathbf{d})}$ with $\mathbb{P}^d_K$.

(1) Let $\lambda = (0, \ldots, 0)$. Then $\mathcal{F}_\lambda = \mathcal{O}$ is the structure sheaf on $\mathbb{P}^d_K$.

(2) Let $\lambda = (r, 0, \ldots, 0),\ r \in \mathbb{Z}$. Then $\mathcal{F}_\lambda = \mathcal{O}(r)$ is a twisted sheaf.

(3) Let $\lambda = (-1, 1, 0, \ldots, 0)$. Then $\mathcal{F}_\lambda = \Omega^1$ is the cotangent sheaf on $\mathbb{P}^d_K$.

(4) Let $\lambda = (-d, 1, \ldots, 1)$. Then $\mathcal{F}_\lambda = \Omega^d$ is the canonical bundle on $\mathbb{P}^d_K$. $\square$

Let W be the Weyl group of $\mathbf{G}$. Set

$$w_i := s_i \cdot s_{i-1} \cdot \cdots \cdot s_1 \in W,$$

where $s_i \in W$ is the simple reflection with respect to the simple root $\alpha_i \in \Delta$. We put $w_0 = 1$. Let $W_{\mathbf{L}_{(\mathbf{1},\mathbf{d})}}$ be the Weyl group of $\mathbf{L}_{(\mathbf{1},\mathbf{d})}$. Then the reflections $w_i, i = 0, \ldots, d-1$, are just the representatives of shortest length in W with respect to

$W/W_{\mathbf{L_{(1,d)}}}$. Let $\mathbf{B}^+ \subset \mathbf{G}$ be the Borel subgroup of upper triangular matrices. The Schubert cells in $\mathbb{P}^d_K$ are given by

$$\begin{aligned}
X_{w_0} := \mathbf{B}^+ w_0 \mathbf{P}_{(\mathbf{1},\mathbf{d})}/\mathbf{P}_{(\mathbf{1},\mathbf{d})} &= D_+(X_0) \\
X_{w_1} := \mathbf{B}^+ w_1 \mathbf{P}_{(\mathbf{1},\mathbf{d})}/\mathbf{P}_{(\mathbf{1},\mathbf{d})} &= V(X_0) \cap D_+(X_1) \\
X_{w_2} := \mathbf{B}^+ w_2 \mathbf{P}_{(\mathbf{1},\mathbf{d})}/\mathbf{P}_{(\mathbf{1},\mathbf{d})} &= V(X_0, X_1) \cap D_+(X_2) \\
&\vdots \\
X_{w_d} := \mathbf{B}^+ w_d \mathbf{P}_{(\mathbf{1},\mathbf{d})}/\mathbf{P}_{(\mathbf{1},\mathbf{d})} &= V(X_0, X_1, \ldots, X_{d-1}) \cap D_+(X_d) = \{[0:0:\cdots:1]\}
\end{aligned}$$

The corresponding Schubert varieties $\overline{X}_{w_i}$, i.e., the Zariski closures of the cells X_{w_i} are just the closed subschemes $V(X_0, \ldots, X_{i-1}) \subset \mathbb{P}^d_K$, $0 \le i \le d$.

Denote by $*$ the dot action of W on $X^*(\mathbf{T})_{\mathbb{Q}}$ given by

$$w * \chi = w(\chi + \rho) - \rho,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^-} \alpha$. Note that the set of negative roots Φ^- correspond to the set of positive roots with respect to the Borel subgroup $\mathbf{B}^+$. We get

$$\begin{aligned}
w_0 * \lambda &= \lambda \\
w_1 * \lambda &= (\lambda_1 - 1, \lambda_0 + 1, \lambda_2, \ldots, \lambda_d) \\
w_2 * \lambda &= (\lambda_1 - 1, \lambda_2 - 1, \lambda_0 + 2, \lambda_3, \ldots, \lambda_d) \\
&\vdots \\
w_i * \lambda &= (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_i - 1, \lambda_0 + i, \lambda_{i+1}, \ldots, \lambda_d) \\
&\vdots \\
w_d * \lambda &= (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_d - 1, \lambda_0 + d)\,.
\end{aligned}$$

In particular, there is at most one integer $0 \le i \le d$, such that $w_i * \lambda$ is dominant with respect to $\mathbf{B}^+$. This integer is characterized by the non-vanishing of $H^i(\mathbb{P}^d_K, \mathcal{F}_\lambda)$, cf. [Bo] Theorem IV'. We denote this integer by i_0 if it exists. Otherwise, there is a unique integer $i_0 < d$ with $w_{i_0} * \lambda = w_{i_0+1} * \lambda$. We get

$$w_i * \lambda \succ w_{i+1} * \lambda \tag{1.17}$$

for all $i \geq i_0$ (resp. $i > i_0$ if $w_{i_0} * \lambda = w_{i_0+1} * \lambda$), and

$$w_i * \lambda \prec w_{i+1} * \lambda \tag{1.18}$$

for all $i < i_0$, with respect to the dominance order $\succ$ on $X^*(\mathbf{T})_{\mathbb{Q}}$. We put

$$\mu_{i,\lambda} := \begin{cases} w_{i-1} * \lambda & : i \leq i_0 \\ w_i * \lambda & : i > i_0. \end{cases}$$

This is a $\mathbf{L_{(i,d-i+1)}}$-dominant weight. Let $V_{i,\lambda}$ be the finite dimensional $\mathbf{L_{(i,d-i+1)}}$-module with highest weight $\mu_{i,\lambda}$. By considering the trivial action of $\mathbf{U}^+_{\mathbf{(i,d-i+1)}}$ on it, we may view it as a $\mathbf{P}^+_{\mathbf{(i,d-i+1)}}$-module.

Proposition 1.4.2. *For $1 \leq i \leq d$, the $\mathbf{P}^+_{\mathbf{(i,d-i+1)}} \ltimes U(\mathfrak{g})$-module $\tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$ is a quotient of the $\mathbf{P}^+_{\mathbf{(i,d-i+1)}}$-module*[4] $\bigoplus_{\substack{k_0,\ldots,k_{i-1} \leq 0 \\ k_i,\ldots,k_d \geq 0 \\ k_0+\cdots+k_d=0}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} \otimes V_{i,\lambda}$.

Proof. Set $\mathcal{F} = \mathcal{F}_\lambda$. We consider the Grothendieck-Cousin complex of $\mathcal{F}$ with respect to the covering $(X_{w_i})_{i=0,\ldots,d}$ of $\mathbb{P}^d_K$, i.e., the complex

$$0 \to H^0_{X_{w_0}}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\delta_0} H^1_{X_{w_1}}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{d-1}} H^d_{X_{w_d}}(\mathbb{P}^d_K, \mathcal{F}) \to 0.$$

The i-th cohomology of this complex yields exactly $H^i(\mathbb{P}^d_K, \mathcal{F})$, cf. [Ke] Theorem 8.7. Furthermore, it is compatible with the action of $\mathbf{B}^+$ and $\mathfrak{g}$. We have for each $0 \leq i \leq d-1$, an exact sequence

$$\begin{aligned} 0 \ \to \ & H^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) \to H^i_{X_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) \to H^{i+1}_{\overline{X}_{w_{i+1}}}(\mathbb{P}^d_K, \mathcal{F}) \\ \to \ & H^{i+1}_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) \to 0. \end{aligned}$$

Since $H^{i+1}_{\overline{X}_{w_{i+1}}}(\mathbb{P}^d_K, \mathcal{F}) \to H^{i+1}_{X_{w_{i+1}}}(\mathbb{P}^d_K, \mathcal{F})$ is injective, we see that $H^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F})$ is the kernel of $H^i_{X_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\delta_i} H^{i+1}_{X_{w_{i+1}}}(\mathbb{P}^d_K, \mathcal{F})$. It follows that

$$\tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) = \ker\left(H^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}) \to H^i(\mathbb{P}^d_K, \mathcal{F})\right)$$

is isomorphic to the image of the boundary homomorphism

$$H^{i-1}_{X_{w_{i-1}}}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\delta_{i-1}} H^i_{X_{w_i}}(\mathbb{P}^d_K, \mathcal{F})$$

resp. to the cokernel of

$$H^{i-2}_{X_{w_{i-2}}}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\delta_{i-2}} H^{i-1}_{X_{w_{i-1}}}(\mathbb{P}^d_K, \mathcal{F})$$

[4] As for the $\mathbf{P}^+_{\mathbf{(i,d-i+1)}}$-module structure on $\bigoplus K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$ we refer to 3.1. In the proof we will see that we can realize $V_{i,\lambda}$ as a submodule of $\tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$.

if $H^{i-1}(\mathbb{P}^d_K,\mathcal{F}) \neq 0$. Here we put $H^{-1}_{X_{w_{-1}}}(\mathbb{P}^d_K,\mathcal{F}) = H^0(\mathbb{P}^d_K,\mathcal{F})$. By excision we get the following chain of $\mathbf{T} \ltimes \mathfrak{g}$-isomorphisms

$$\begin{aligned} H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{F}) &\cong H^i_{V(X_0,\ldots,X_{i-1})\cap D_+(X_i)}(D_+(X_i),\mathcal{F}) \\ &\cong H^i_{V(X_0,\ldots,X_{i-1})\cap D_+(X_i)}(D_+(X_i),\mathcal{O}) \otimes \mathcal{F}(w_i), \end{aligned}$$

where $\mathcal{F}(w_i)$ denotes the fibre of $\mathcal{F}$ in $w_i \cdot \mathbf{P_{(1,d)}}$. A simple computation gives

$$H^i_{V(X_0,\ldots,X_{i-1})\cap D_+(X_i)}(D_+(X_i),\mathcal{O}) = \bigoplus_{\substack{k_0,\ldots,k_{i-1}<0 \\ k_{i+1},\ldots,k_d \geq 0 \\ k_i \in \mathbb{Z} \\ k_0+\cdots+k_d=0}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}.$$

We can rewrite this expression as follows. Recall that for a root $\alpha = \alpha_{k,l} \in \Phi$ the symbol $L_{(k,l)}$ denotes the standard generator of the weight space $\mathfrak{g}_\alpha \subset \mathfrak{g}$. Analogously, we put

$$X_{(k,l)} := X_{\alpha_{k,l}} := \frac{X_k}{X_l} \in K[X_0,\ldots,X_d]_{X_0\cdots X_d}.$$

Then we get (compare [Ke] Corollary 11.10)

$$H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{O}) = K[X_{(i+1,i)},\ldots,X_{(d,i)}] \otimes \sum_{(n_0,\ldots,n_{i-1})\in\mathbb{N}^i} L^{n_0}_{(i,0)} \cdot L^{n_1}_{(i,1)} \cdots L^{n_{i-1}}_{(i,i-1)} \cdot \frac{X_i^i}{X_0\cdots X_{i-1}}.$$

Thus we obtain

$$\begin{aligned} H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{F}) &= H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{O}) \otimes \mathcal{F}(w_i) \\ &= K[X_{(i+1,i)},\ldots,X_{(d,i)}] \otimes \sum_{(n_0,\ldots,n_{i-1})} L^{n_0}_{(i,0)} \cdots L^{n_{i-1}}_{(i,i-1)} \cdot \frac{X_i^i}{X_0\cdots X_{i-1}} \otimes \mathcal{F}(w_i). \end{aligned}$$

The weights of $H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{O})$ are given by

$$\Big\{ w_i*(0,\ldots,0) - n_0\cdot\alpha_{0,i} - \cdots - n_{i-1}\cdot\alpha_{i-1,i} - n_i\cdot\alpha_{i,i+1} - \cdots - n_{d-1}\cdot\alpha_{i,d} \mid n_0,\ldots,n_{d-1} \in \mathbb{N} \Big\}.$$

Here the highest weight is $w_i * (0,\ldots,0)$. The highest weight of the fibre $\mathcal{F}(w_i)$ is given by $w_i \cdot \lambda$. We conclude that the highest weight of $H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{F})$ is given by $w_i * \lambda$. A highest weight vector is given by

$$v_{i,\lambda} = \frac{X_i^i}{X_0\cdots X_{i-1}} \otimes w_i(v_\lambda),$$

where $v_\lambda = v_{0,\lambda}$ is a highest weight vector of V_λ.

Reconsider the homomorphism

$$H^{i-1}_{X_{w_{i-1}}}(\mathbb{P}^d_K,\mathcal{F}) \stackrel{\delta_{i-1}}{\to} H^i_{X_{w_i}}(\mathbb{P}^d_K,\mathcal{F}).$$

If $i < i_0$ then by (1.18) the highest weight of the image is $w_{i-1} * \lambda$. If $i \geq i_0$ then by (1.17) the highest weight of the image is $w_i * \lambda$. Thus the highest weight of the image is $\mu_{i,\lambda} \in X^*(\mathbf{T})$. Furthermore, the weight vectors $\delta_{i-1}(v_{i-1,\lambda})$ and $v_{i,\lambda}$ differ by the factor $(\frac{X_i}{X_{i-1}})^{\lambda_0-\lambda_i+i}$. More precisely, we conclude by weight reasons

$$\delta_{i-1}(v_{i-1,\lambda}) = (\frac{X_i}{X_{i-1}})^{\lambda_0-\lambda_i+i} v_{i,\lambda} \quad \text{if} \quad w_{i-1} * \lambda \preceq w_i * \lambda,$$

$$\delta_{i-1}((\frac{X_i}{X_{i-1}})^{\lambda_0-\lambda_i+i} v_{i-1,\lambda}) = v_{i,\lambda} \quad \text{if} \quad w_{i-1} * \lambda \succeq w_i * \lambda. \tag{1.19}$$

Since $\tilde{H}^i_{X_{w_i}}(\mathbb{P}^d_K, \mathcal{F})$ is also a $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$-module, it follows that it contains the irreducible algebraic $\mathbf{L}_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})} = \mathbf{GL_i} \times \mathbf{GL_{d-i+1}}$-representation $V_{i,\lambda}$ corresponding to the highest weight $\mu_{i,\lambda}$. One checks that $\mathrm{im}(\delta_{i-1}) = \tilde{H}^i_{X_{w_i}}(\mathbb{P}^d_K, \mathcal{F})$ is equal to[5]

(1.20)
$$U(L_{(i-1,0)}, \ldots, L_{(i-1,i-2)})(K[\tfrac{X_i}{X_{i-1}}, \ldots, \tfrac{X_d}{X_{i-1}}] \cdot V_{i,\lambda}) = \bigoplus_{\substack{k_0,\ldots,k_{i-1} \leq 0 \\ k_i,\ldots,k_d \geq 0 \\ k_0+\cdots+k_d=0}} K \cdot X_0^{k_0} \cdots X_d^{k_d} \cdot V_{i,\lambda}.$$

Here $U(L_{(i-1,0)}, \ldots, L_{(i-1,i-2)})$ denotes the subalgebra of $U(\mathfrak{g})$ generated by $L_{(i-1,0)}, \ldots, L_{(i-1,i-2)}$. Indeed, the above expression is contained in the image. As for the other inclusion, we note that

$$V_\lambda = U(\mathrm{Lie}(R_u(L_{(1,d)} \cap B)))v_\lambda,$$

where $R_u(L_{(1,d)} \cap B)$ denotes the unipotent radical of $L_{(1,d)} \cap B$, cf. [Ja] p. 204. Thus, if $L_{(k,l)}$ is a root of $\mathfrak{g}$ contained in $\mathrm{Lie}(R_u(L_{(1,d)} \cap B))$ then $k > l, k \geq 2$ and $l \geq 1$. Let $w_{i-1} * \lambda \prec w_i * \lambda$. Then

$$\begin{aligned} \frac{X_{i-1}^{i-1}}{X_0 \cdots X_{i-2}} \cdot w_{i-1}(L_{(k,l)} v_\lambda) &= \frac{X_{i-1}^{i-1}}{X_0 \cdots X_{i-2}} \cdot L_{(w_{i-1}(k), w_{i-1}(l))} w_{i-1}(v_\lambda) \\ = \quad L_{(w_{i-1}(k), w_{i-1}(l))}(v_{i-1,\lambda}) &- \big(L_{(w_{i-1}(k), w_{i-1}(l))} \frac{X_{i-1}^{i-1}}{X_0 \cdots X_{i-2}}\big) \cdot w_{i-1}(v_\lambda). \end{aligned}$$

Since $k > l \geq 1$, we conclude that $w_{i-1}(k) > w_{i-1}(l)$. If $w_{i-1}(l) \notin \{0, \ldots, i-2\}$ or $w_{i-1}(k) \in \{0, \ldots, i-2\}$ we deduce that $(L_{(w_{i-1}(k), w_{i-1}(l))} \frac{X_{i-1}^{i-1}}{X_0 \cdots X_{i-2}}) \cdot w_{i-1}(v_\lambda) = 0$. Otherwise, we get

$$\big(L_{(w_{i-1}(k), w_{i-1}(l))} \frac{X_{i-1}^{i-1}}{X_0 \cdots X_{i-2}}\big) \cdot w_{i-1}(v_\lambda) = -\frac{X_{w_{i-1}(k)}}{X_{w_{i-1}(l)}} \cdot v_{i-1,\lambda}.$$

[5]Note that $U(L_{(i-1,0)}, \ldots, L_{(i-1,i-2)})$ leaves $V_{i,\lambda}$ invariant, since $V_{i,\lambda}$ is a $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$-module.

In any case, since $L_{(w_{i-1}(k),w_{i-1}(l))}(v_{i-1,\lambda})$ is contained in (1.20), we see that $\frac{X_{i-1}^{i-1}}{X_0\cdots X_{i-2}}\cdot w_{i-1}(L_{(k,l)}v_\lambda)$ is contained in (1.20) as well. The case $w_{i-1}*\lambda \succ w_i*\lambda$ is treated similarly by using identity (1.19). $\square$

Example 1.4.3. *Let $\lambda=(0,\ldots,0)$. Then $\mu_{1,\lambda}=w_1*\lambda=(-1,1,0\ldots,0)$ and $V_{1,\lambda}=K\frac{X_1}{X_0}\oplus\cdots\oplus K\frac{X_d}{X_0}$. So $\tilde{H}^1_{X_{w_1}}(\mathbb{P}^d_K,\mathcal{O})$ is a quotient of $K[\frac{X_1}{X_0},\ldots,\frac{X_d}{X_0}]\otimes V_{1,\lambda}$ with non-trivial kernel. On the other hand, if $\lambda=(-1,1,0,\ldots,0)$ then $\mu_{1,\lambda}=w_0*\lambda=\lambda$ and $V_{1,\lambda}=V_\lambda=Kd(\frac{X_1}{X_0})\oplus\cdots\oplus Kd(\frac{X_d}{X_0})$. In this situation the map $K[\frac{X_1}{X_0},\ldots,\frac{X_d}{X_0}]\otimes V_{1,\lambda}\to\tilde{H}^1_{X_{w_1}}(\mathbb{P}^d_K,\Omega^1)$ is an isomorphism.*

We shall determine a $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$-submodule of (1.20) generating it as $U(\mathfrak{g})$-module. We make use of the following statement which can be found in various descriptions in [FH].

Lemma 1.4.4. *Let $\nu\in(\mathbb{Z}^n)_+=\{(\nu_1,\ldots,\nu_n)\mid \nu_1\geq\nu_2\geq\cdots\geq\nu_n\}$ be a dominant weight, $n\in\mathbb{N}$. Let V_ν be the irreducible algebraic representation of GL_n of highest weight ν. For $k\in\mathbb{N}$, we consider the irreducible algebraic representation V_k of highest weight $(k,0,\ldots,0)\in(\mathbb{Z}^n)_+$, i.e., $V_k\cong\mathrm{Sym}^k(K^n)$. Then*[6]

$$V_k\otimes V_\nu=\bigoplus_{\substack{(c_1,\ldots,c_n)\in\mathbb{N}_0^n\\ c_{i+1}\leq\nu_i-\nu_{i+1},\, i=1,\ldots,n-1}} V_{\nu+(c_1,\ldots,c_n)}.$$

Proof. By tensoring V_ν with $\det^{-\nu_n}$, we may suppose that $\nu_n=0$. Set $a_i:=\nu_i-\nu_{i+1},\ i=1,\ldots,n-1$. Then we have $V_\nu=\Gamma_{a_1,\ldots,a_{n-1}}$ and $V_k=\Gamma_{k,0\ldots,0}$ with the terminology in [FH] §15. By loc.cit. Prop. 15.25 a) we deduce the decomposition (as SL_n-modules)

$$V_k\otimes V_\nu=\bigoplus_{b_1,\ldots,b_{n-1}}\Gamma_{b_1,\ldots,b_{n-1}},$$

where the sum is over all non-negative integers $b_1,\ldots,b_{n-1}$ for which there are non-negative integers $c_1,\ldots,c_n$ with $\sum_i c_i=k$, with $c_{i+1}\leq a_i$ and with $b_i=a_i+c_i-c_{i+1}$ for $1\leq i\leq n-1$. The highest weight of $\Gamma_{b_1,\ldots,b_{n-1}}$ is $\nu+(c_1-c_n,c_2-c_n,\ldots,c_{n-1}-c_n,0)$. Since we deal with GL_n-modules of total weight $\nu_1+\cdots+\nu_n+k$, we have to replace $V_{\nu+(c_1-c_n,c_2-c_n,\ldots,c_{n-1}-c_n,0)}$ by $V_{\nu+(c_1,\ldots,c_n)}$. $\square$

We start to investigate the extreme cases $i=1$ and $i=d$. For $k\geq 0$, let $K[\frac{X_1}{X_0},\cdots,\frac{X_d}{X_0}]_k$ be the set of polynomials of degree k in the indetermines $\frac{X_1}{X_0},\ldots,\frac{X_d}{X_0}$. Then we get identifications

$$K[\tfrac{X_1}{X_0},\cdots,\tfrac{X_d}{X_0}]_k\cong\mathrm{Sym}^k(K\tfrac{X_1}{X_0}\oplus\cdots\oplus K\tfrac{X_d}{X_0})\cong V_{(-k,k,0\ldots,0)}.$$

[6]Note that the condition $c_{i+1}\leq\nu_i-\nu_{i+1}$, $i=1,\ldots,n-1$, implies that $\nu+(c_1,\ldots,c_n)\in(\mathbb{Z}^n)_+$

For an integral vector $\nu = (\nu_1, \ldots, \nu_n) \in (\mathbb{Z}^n)_+$, we set

$$|\nu| = \nu_1 - \nu_n.$$

Lemma 1.4.5. *(i) Let $\mu_{1,\lambda} = (\mu_0, \mu')$ with $\mu_0 \in \mathbb{Z}$ and $\mu' \in (\mathbb{Z}^d)_+$. Then $\tilde{H}^1_{\mathcal{X}_{w_1}}(\mathbb{P}^d_K, \mathcal{F}_\lambda) = K[\frac{X_1}{X_0}, \cdots, \frac{X_d}{X_0}] \cdot V_{1,\lambda}$ is generated as $U(\mathfrak{g})$-module by $\oplus_{k \leq |\mu'|} K[\frac{X_1}{X_0}, \cdots, \frac{X_d}{X_0}]_k \cdot V_{1,\lambda}$.*

(ii) The module $\tilde{H}^d_{\mathcal{X}_{w_d}}(\mathbb{P}^d_K, \mathcal{F}_\lambda) = K[\frac{X_d}{X_0}, \cdots, \frac{X_d}{X_{d-1}}] \cdot V_{d,\lambda}$ is generated as $U(\mathfrak{g})$-module by $V_{d,\lambda}$.

Proof. (i) We identify $K[\frac{X_1}{X_0}, \ldots, \frac{X_d}{X_0}] \otimes V_{1,\lambda}$ with $\bigoplus_{k \geq 0} V_{(-k,k,0\ldots,0)} \otimes V_{1,\lambda}$. We may apply the previous lemma with $n = d$ and $\nu = \mu'$. We deduce that the number of irreducible summands in $V_{(-k,k,0\ldots,0)} \otimes V_{1,\lambda}$ is the same for $k \geq a_1 + a_2 + \cdots + a_{n-1}$ with $a_i = \mu'_i - \mu'_{i+1}$. But the latter sum is exactly $\mu'_1 - \mu'_d = |\mu'|$. The Lie algebra $\mathfrak{g}$ maps $V_{(-k,k,0\ldots,0)} \cdot V_{1,\lambda}$ to $V_{(-k-1,k+1,0\ldots,0)} \cdot V_{1,\lambda}$ and its image is again a $\mathbf{P}^+_{(\mathbf{1},\mathbf{d})}$-module. The claim follows since irreducible submodules are mapped to different irreducible submodules by weight reasons.

(ii) By (1.20) it is enough to show that $K[\frac{X_d}{X_{d-1}}] \cdot V_{d,\lambda} \subset U(\mathfrak{g}) \cdot V_{d,\lambda}$. We consider the following two cases.

Case 1: $w_d * \lambda \preccurlyeq w_{d-1} * \lambda$. Then $v_{d,\lambda}$ is a highest weight vector of $V_{d,\lambda}$. We compute

$$\begin{aligned}
\frac{X_d}{X_i} \cdot v_{d,\lambda} &= \frac{X_d}{X_i} \cdot \frac{X_d}{X_0 \cdots X_{d-1}} \cdot w_d(v_\lambda) = -(L_{(d,i)} \frac{X_d}{X_0 \cdots X_{d-1}}) \cdot w_d(v_\lambda) \\
&= -L_{(d,i)}(\frac{X_d}{X_0 \cdots X_{d-1}} \cdot w_d(v_\lambda)) + \frac{X_d}{X_0 \cdots X_{d-1}} \cdot L_{(d,i)} w_d(v_\lambda).
\end{aligned}$$

But $L_{(d,i)} w_d(v_\lambda) = w_d(L_{(0,i+1)} v_\lambda) = 0$ since $L_{(0,i+1)} v_\lambda = 0$. Thus we obtain $\frac{X_d}{X_i} \cdot v_{d,\lambda} = -L_{(d,i)} v_{d,\lambda}$. On the other hand, the module $V_{d,\lambda}$ is equal to $U(\mathrm{Lie}(R_u(L_{(d,1)} \cap B))) v_{d,\lambda}$. If $L_{(k,l)}$ is a root contained in $\mathrm{Lie}(R_u(L_{(d,1)} \cap B))$ we necessarily have $l < k < d$. We deduce that

$$\begin{aligned}
\frac{X_d}{X_{d-1}} \cdot L_{(k,l)} v_{d,\lambda} &= L_{(k,l)}(\frac{X_d}{X_{d-1}} \cdot v_{d,\lambda}) = -L_{(k,l)}(L_{(d,d-1)} v_{d,\lambda}) \\
&= -L_{(d,d-1)}(L_{(k,l)} v_{d,\lambda}) + [L_{(k,l)}, L_{(d,i)}] \cdot v_{d,\lambda} \\
&= -L_{(d,d-1)}(L_{(k,l)} v_{d,\lambda}) + \delta_{i,k} L_{(d,l)} \cdot v_{d,\lambda}
\end{aligned}$$

is contained in $U(\mathfrak{g}) \cdot V_{d,\lambda}$. The case of polynomials of higher degree is treated in the same way.

Case 2: $w_d * \lambda \succ w_{d-1} * \lambda$. Then $\delta_{d-1}(v_{d-1,\lambda})$ is a highest weight vector of $V_{d,\lambda}$. We get $\delta_{d-1}(v_{d-1,\lambda}) = (\frac{X_d}{X_{d-1}})^n \cdot v_{d,\lambda}$ for some $n > 0$, cf. (1.19). We compute

$$\begin{aligned} & \frac{X_d}{X_i} \cdot \delta_{d-1}(v_{d-1,\lambda}) = \frac{X_d}{X_i} \cdot (\frac{X_d}{X_{d-1}})^n \cdot v_{d,\lambda} \\ = \; & (\frac{X_d}{X_{d-1}})^n \cdot \frac{X_d}{X_i} \cdot v_{d,\lambda} = -(\frac{X_d}{X_{d-1}})^n \cdot L_{(d,i)} v_{d,\lambda} \text{ (by case 1)} \\ = \; & -L_{(d,i)}((\frac{X_d}{X_{d-1}})^n \cdot v_{d,\lambda}) + (L_{(d,i)}(\frac{X_d}{X_{d-1}})^n) \cdot v_{d,\lambda} \\ = \; & -L_{(d,i)}(\delta_{d-1}(v_{d-1,\lambda})) + (L_{(d,i)}(\frac{X_d}{X_{d-1}})^n) \cdot v_{d,\lambda}. \end{aligned}$$

But $L_{(d,i)}(\frac{X_d}{X_{d-1}})^n \cdot v_{d,\lambda} = 0$ if $i \neq d-1$. If $i = d-1$ then

$$\begin{aligned} L_{(d,i)}(\frac{X_d}{X_{d-1}})^n \cdot v_{d,\lambda} & = -n(\frac{X_d}{X_{d-1}})^{n+1} \cdot v_{d,\lambda} \\ & = -n\frac{X_d}{X_i} \cdot \delta_{d-1}(v_{d-1,\lambda}), \end{aligned}$$

so $(n+1)\frac{X_d}{X_i} \cdot \delta_{d-1}(v_{d-1,\lambda}) = -L_{(d,i)}\delta_{d-1}(v_{d-1,\lambda})$. If $w \in V_{d,\lambda}$ is an arbitrary vector we argue as in case 1. $\square$

Now we treat the general case which is a mixture between the above extreme cases. Note that $K[X_{(m,n)} \mid m \geq i,\ n \leq i-1] = \bigoplus_{\substack{k_0,\ldots,k_{i-1} \leq 0 \\ k_i,\ldots,k_d \geq 0 \\ k_0+\cdots+k_d=0}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$.

Lemma 1.4.6. *Write $\mu_{i,\lambda} = (\mu', \mu'')$ with $\mu' \in (\mathbb{Z}^i)_+$ and $\mu'' \in (\mathbb{Z}^{d+1-i})_+$. Then the $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})} \ltimes U(\mathfrak{g})$-module $\tilde{H}^i_{\mathcal{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}_\lambda) = K[X_{(m,n)} \mid m \geq i,\ n \leq i-1] \cdot V_{i,\lambda}$ is generated by the $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$-submodule $\oplus_{k \leq |\mu''|} K[X_{(m,n)} \mid m \geq i,\ n \leq i-1]_k \cdot V_{i,\lambda}$.*

Proof. Write $V_{i,\lambda} = V_{\mu'} \boxtimes V_{\mu''}$. We may identify $K[X_{(m,n)} \mid m \geq i,\ n \leq i-1]_k$ with the outer tensor product representation $V_{(0,\ldots,0,-k)} \boxtimes V_{(k,0\ldots,0)}$ of $\mathbf{L}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$. We get

$$K[X_{(m,n)} \mid m \geq i,\ n \leq i-1]_k \otimes V_{i,\lambda} \cong (V_{(0,\ldots,0,-k)} \otimes V_{\mu'}) \boxtimes (V_{(k,0\ldots,0)} \otimes V_{\mu''}).$$

By the proof of Proposition 1.4.2 we saw that $K[X_{(k,l)} \mid k \geq i,\ l \leq i-1] \cdot V_{i,\lambda}$ is already generated as $U(\mathfrak{g})$-module by $K[\frac{X_i}{X_{i-1}}, \ldots, \frac{X_d}{X_{i-1}}] \cdot V_{i,\lambda}$. Now we apply the proof of part (i) of the previous lemma to deduce the claim. $\square$

For $\mu_{i,\lambda} = (\mu', \mu'')$ with $\mu' \in (\mathbb{Z}^i)_+$ and $\mu'' \in (\mathbb{Z}^{d-i+1})_+$, we define

$$\Phi_{i,\lambda} = \bigcup_{k=0}^{|\mu''|} \Big\{ \big(\mu' - (d_i, \ldots, d_1), \mu'' + (c_1, \ldots, c_{d-i+1})\big) \mid \textstyle\sum_j c_j = \sum_j d_j = k, c_1 = 0 \text{ or } d_1 = 0,\ c_{j+1} \leq \mu''_j - \mu''_{j+1},\ j = 1, \ldots, d-i,\ d_{j+1} \leq \mu'_{i-j} - \mu'_{i-j+1},\ j = 1, \ldots, i-1 \Big\}.$$

We let

$$M_{i,\lambda} := \bigoplus_{\mu \in \Phi_{i,\lambda}} V_\mu \quad \subset \quad K[X_{(m,n)} \mid m \geq i,\ n \leq i-1] \otimes V_{i,\lambda}$$

be the sum of the irreducible $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})}$-modules with respect to the weights appearing in $\Phi_{i,\lambda}$. Let

$$p_i : K[X_{(m,n)} \mid m \geq i,\ n \leq i-1] \otimes V_{i,\lambda} \to \tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$$

be the quotient map.

Corollary 1.4.7. *Then for $1 \leq i \leq d$, the $\mathbf{P}^+_{(\mathbf{i},\mathbf{d}-\mathbf{i}+\mathbf{1})} \ltimes U(\mathfrak{g})$-module $\tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$ is generated as $U(\mathfrak{g})$-module by $p_i(M_{i,\lambda})$.*

Proof. We write $V_{(0,\ldots,0,-k)} \otimes V_{\mu'} = (V_{(0,\ldots,0,-k)} \otimes V_{\mu'})^{**} = (V_{(k,\ldots,0,0)} \otimes V_{(\mu')^*})^*$ where $(\mu')^* = (-\mu_i, \ldots, -\mu_1)$ and * indicates the dual representation, cf. [FH] Ex. 15.50. By Lemma 1.4.4 we have a decomposition

$$V_{(k,\ldots,0,0)} \otimes V_{(\mu')^*} = \bigoplus_{\substack{(d_1,\ldots,d_i) \in \mathbb{N}_0^i \\ d_{j+1} \leq (\mu')^*_j - (\mu')^*_{j+1},\, j=2,\ldots,i}} V_{(\mu')^* + (d_1,\ldots,d_i)}.$$

Thus we get

$$V_{(0,\ldots,0,-k)} \otimes V_{\mu'} = \Big(\bigoplus_{\substack{(d_1,\ldots,d_i) \in \mathbb{N}_0^i \\ d_{j+1} \leq (\mu')^*_j - (\mu')^*_{j+1},\, j=2,\ldots,i}} V_{(\mu')^* + (d_1,\ldots,d_i)} \Big)^* = \bigoplus_{\substack{(d_1,\ldots,d_i) \in \mathbb{N}_0^i \\ d_{j+1} \leq (\mu')^*_j - (\mu')^*_{j+1},\, j=2,\ldots,i}} V_{\mu' - (d_i,\ldots,d_1)}$$

But $(\mu')^*_j - (\mu')^*_{j+1} = \mu'_{i-j} - \mu'_{i-j+1}$. Finally, if $c_1 > 0$ and $d_1 > 0$ then

$$p_i(V_{\mu' - (d_i,\ldots,d_1)} \boxtimes V_{\mu'' + (c_1,\ldots,c_{d-i+1})}) = \mathfrak{g} \cdot p_i(V_{\mu' - (d_i,\ldots,d_1 - 1)} \boxtimes V_{\mu'' + (c_i - 1,\ldots,c_{d-1+1})}).$$

The claim follows. $\square$

Remark 1.4.8. *We point out that for some weights λ, some of the irreducible submodules $V_\mu \subset M_{i,\lambda}, \mu \in \Phi_{i,\lambda}$, apart from $\mu_{i,\lambda}$ are mapped to zero under the quotient map p_i. We refer to section 3 for examples.*

Now we translate the above result for the computation of $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K, \mathcal{F})$. Consider the block matrix

$$z_i := \begin{pmatrix} 0 & I_i \\ I_{d+1-i} & 0 \end{pmatrix} \in G,$$

where $I_j \in \mathrm{GL}_j$ denotes the $j \times j$-identity matrix. Then $V(X_0, \dots X_{i-1}) = \overline{X}_{w_i}$ is transformed into $V(X_{d-i+1}, \dots, X_d) = \mathbb{P}^{d-i}_K$ under the action of z_i. We have

$$z_i \cdot \mathbf{P_{(d-i+1,i)}} \cdot z_i^{-1} = \mathbf{P}^+_{\mathbf{(i,d+1-i)}}$$

and on the Levi subgroups the conjugacy map is given by

$$\mathbf{L_{(d-i+1,i)}} \ni \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \in \mathbf{L_{(i,d-i+1)}}.$$

Thus we get an isomorphism

$$\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K, \mathcal{F}) \xrightarrow{\sim} \tilde{H}^i_{\overline{X}_{w_i}}(\mathbb{P}^d_K, \mathcal{F})$$

compatible with the action of the parabolic subgroups. Hence in order to determine the $\mathbf{P_{(d-i+1,i)}} \ltimes U(\mathfrak{g})$-representation of $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K, \mathcal{F})$ in terms of highest weight vectors, we have to apply z_i - regarded as an element in the Weyl group W - to them. Clearly the dominant weights are respected by this transformation. We set

$$\begin{aligned}
\Psi_{i,\lambda} &= z_i^{-1} \cdot \Phi_{i,\lambda} \\
&= \bigcup_{k=0}^{|\mu''|} \Big\{ (\mu'' + (c_1, \dots, c_{d-i+1}), \mu' - (d_i, \dots, d_1)) \mid \textstyle\sum_j c_j = \sum_j d_j = k, c_1 = 0 \\
&\quad \text{ or } d_1 = 0,\ c_{j+1} \leq \mu''_i - \mu''_{j+1},\ j = 1, \dots, d-i,\ d_{j+1} \leq \mu'_{i-j} - \mu'_{i-j+1}, \\
&\quad j = 1, \dots, i-1 \Big\}
\end{aligned}$$

and

$$N_{i,\lambda} = \bigoplus_{\mu \in \Psi_{i,\lambda}} V_\mu \quad \subset \quad K[X_{(m,n)} \mid m \leq d-i,\ n \geq d-i+1] \otimes V_{z_i^{-1}\mu_{i,\lambda}}.$$

Let

$$q_i : K[X_{(m,n)} \mid m \leq d-i,\ n \geq d-i+1] \otimes V_{z_i^{-1}\mu_{i,\lambda}} \to \tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$$

be the quotient map. We obtain:

Corollary 1.4.9. *For $1 \leq i \leq d$, the $\mathbf{P_{(d-i+1,i)}} \ltimes U(\mathfrak{g})$-module $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$ is generated by $q_i(N_{i,\lambda})$.*

2. The G-representation $H^0(\mathcal{X},\mathcal{F})$

2.1. **The fundamental complex.** In this section we recall the construction [O3] of an acyclic resolution of the constant sheaf $\mathbb{Z}$ on the boundary of $\mathcal{X}$ considered as an object in the category of pseudo-adic spaces [H].

In order to determine the structure of $\mathcal{F}(\mathcal{X})' = H^0(\mathcal{X},\mathcal{F})'$ as a locally analytic G-representation, we proceed as follows. Let

$$\mathcal{Y} = (\mathbb{P}^d_K)^{rig} \setminus \mathcal{X}$$

be the set-theoretical complement of $\mathcal{X}$. Consider the topological exact sequence of locally convex K-vector spaces with continuous G-action

$$0 \to H^0(\mathbb{P}^d_K,\mathcal{F}) \to H^0(\mathcal{X},\mathcal{F}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F}) \to H^1(\mathbb{P}^d_K,\mathcal{F}) \to 0.$$

Note that the higher cohomology groups $H^i(\mathcal{X},\mathcal{F}),\ i>0$, vanish since $\mathcal{X}$ is a Stein space [K2]. The G-representations $H^0(\mathbb{P}^d_K,\mathcal{F}),\ H^1(\mathbb{P}^d_K,\mathcal{F})$ are finite-dimensional algebraic. A more delicate problem is to understand the structure of the G-representation $H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F})$ which is a K-Fréchet space. More precisely, it is by Proposition 1.3.3 and Proposition 1.1.3 a projective limit of K-Banach spaces

$$H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F}) = \varprojlim_n H^1_{\mathcal{Y}_n^-}(\mathbb{P}^d_K,\mathcal{F}).$$

In [O3] we constructed acyclic resolutions of overconvergent étale sheaves on the boundary of period domains. We want to apply this construction to our situation. The construction makes use of Huber's adic spaces [H]. In the appendix we give an alternative approach avoiding these spaces. In the following, the symbol X^{ad} indicates the adic space attached to a scheme X or to a rigid analytic variety X defined over K.

We take the complement of $\mathcal{X}$ in the category of adic spaces, i.e., we set

$$\mathcal{Y}^{ad} := (\mathbb{P}^d_K)^{ad} \setminus \mathcal{X}^{ad}.$$

This is a closed pseudo-adic subspace of $\mathbb{P}^d_K$. Let $\{e_0,\ldots,e_d\}$ be the standard basis of $V = K^{d+1}$. For any $\alpha_i \in \Delta$, put

$$V_i = \bigoplus_{j=0}^{i} K\cdot e_j \ \text{ and } \ Y_i = \mathbb{P}(V_i)$$

For any subset $I \subset \Delta$ with $\Delta\backslash I = \{\alpha_{i_1} < \ldots < \alpha_{i_r}\}$, let Y_I be the closed K-subvariety of $\mathbb{P}^d_K$ defined by

$$Y_I = \mathbb{P}(V_{i_1}).$$

Furthermore, let P_I be the lower parabolic subgroup of G, such that I coincides with the simple roots appearing in the Levi factor of P_I. Hence the group P_I stabilizes Y_I. We obtain

$$\mathcal{Y}^{ad} = \bigcup_{I \subset \Delta} \bigcup_{g \in G/P_I} g \cdot Y_I^{ad} = \bigcup_{g \in G} g \cdot Y^{ad}_{\Delta \setminus \{\alpha_{d-1}\}}. \tag{2.1}$$

For any compact open subset $W \subset G/P_I$, put

$$Z_I^W := \bigcup_{g \in W} g Y_I^{ad}.$$

We proved in [O3], Lemma 3.2, that Z_I^W is a closed pseudo-adic subspace of $(\mathbb{P}^d_K)^{ad}$. By (2.1) it follows that

$$\mathcal{Y}^{ad} = \bigcup_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} Z_I^{G/P_I} = Z_{\Delta\setminus\{\alpha_{d-1}\}}^{G/P_{\Delta\setminus\{\alpha_{d-1}\}}}.$$

Starting from the constant étale sheaf $\mathbb{Z}$ on $\mathcal{Y}^{ad}$ we constructed a sheaf of locally constant sections on the same space. We recall the definition. Consider the natural closed embeddings of pseudo-adic spaces

$$\Phi_{g,I} : gY_I^{ad} \longrightarrow \mathcal{Y}^{ad}$$

resp.

$$\tilde{\Phi}_{g,I,W} : gY_I^{ad} \longrightarrow Z_I^W$$

resp.

$$\Psi_{I,W} : Z_I^W \longrightarrow \mathcal{Y}^{ad}.$$

Put

$$\mathbb{Z}_{g,I} := (\Phi_{g,I})_*(\Phi^*_{g,I}\, \mathbb{Z})$$

resp.

$$\mathbb{Z}_{Z_I^W} := (\Psi_{I,W})_*(\Psi^*_{I,W}\, \mathbb{Z})$$

and let

$$\tilde{\Phi}^{\#}_{g,I,W} : \mathbb{Z}_{Z_I^W} \longrightarrow \mathbb{Z}_{g,I}$$

be the natural homomorphism given by restriction. Let $\mathcal{C}_I$ be the category of compact open disjoint coverings of G/P_I where the morphisms are given by the refinement-order. For a covering $c = (W_j)_j \in \mathcal{C}_I$, we denote by $\mathbb{Z}_c$ the sheaf on $\mathcal{Y}^{ad}$ defined by

$$\mathbb{Z}_c(U) \ := \ \Big\{(s_g)_g \in \prod_{g\in G/P_I} \mathbb{Z}_{g,I}(U) \mid \ \text{there are sections } \ s_j \ \in \mathbb{Z}_{Z_I^{W_j}}(U), \ \text{such}$$

$$\text{that } \tilde{\Phi}^{\#}_{g,I,W_j}(s_j) = s_g \text{ for all } g \in W_j\Big\}.$$

Note that $\mathbb{Z}_c$ is just the image of the natural morphism of sheaves

$$\bigoplus_{j\in A} \mathbb{Z}_{Z_I^{W_j}} \hookrightarrow \prod_{g\in G/P_I} \mathbb{Z}_{g,I}.$$

We put

$$\prod_{g\in G/P_I}{}' \ \mathbb{Z}_{g,I} = \varinjlim_{c\in \mathcal{C}_I} \mathbb{Z}_c, \tag{2.2}$$

We obtain the following complex of sheaves on $\mathcal{Y}^{ad}$,

$$\begin{aligned} 0 \to \mathbb{Z} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=1}} \prod_{g\in G/P_I}{}' \ \mathbb{Z}_{g,I} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=2}} \prod_{g\in G/P_I}{}' \ \mathbb{Z}_{g,I} \to \cdots \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=i}} \prod_{g\in G/P_I}{}' \ \mathbb{Z}_{g,I} \to \cdots \\ \cdots \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=d-1}} \prod_{g\in G/P_I}{}' \ \mathbb{Z}_{g,I} \to \prod_{g\in G/P_\emptyset}{}' \ \mathbb{Z}_{g,\emptyset} \to 0. \end{aligned} \tag{2.3}$$

Theorem 2.1.1. *The complex (2.3) is acyclic.*

Proof. This is Theorem 3.3 in [O3]. Strictly speaking, we treated in loc.cit. the case of the constant étale sheaf $\mathbb{Z}/n\mathbb{Z}$. But the proof is the same. $\square$

2.2. **Evaluation of the spectral sequence.** In this section we evaluate the spectral sequence which is induced by the complex (2.3) applied to $\mathrm{Ext}^*(i_*(-),\ \mathcal{F})$. Here $i : \mathcal{Y}^{ad} \hookrightarrow (\mathbb{P}^d_K)^{ad}$ denotes the closed embedding.

By [SGA2] Proposition 2.3 bis., we conclude that

$$\mathrm{Ext}^*(i_*(\mathbb{Z}_{\mathcal{Y}^{ad}}), \mathcal{F}) = H^*_{\mathcal{Y}^{ad}}(\mathbb{P}^d_K, \mathcal{F}),$$

Further, we have $H^*_{\mathcal{Y}^{ad}}(\mathbb{P}^d_K, \mathcal{F}) = H^*_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$ since the topoi of $\mathcal{X}$ and $\mathcal{X}^{ad}$ are equivalent, cf. [H], Prop. 2.1.4. Recall that $G_0 = \mathbf{G}(O_K)$ denotes the compact p-adic group of O_K-valued points of $\mathbf{G}$.

Proposition 2.2.1. *For all $I \subset \Delta$, there is an isomorphism*

$$\mathrm{Ext}^*(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\mathcal{F}) = \varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^*_{gY_I(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}).$$

Proof. Consider the family

$$\left\{gP_I^n \mid g\in G_0, n\in\mathbb{N}\right\}$$

of compact open subsets in G/P_I which yields cofinal coverings in $\mathcal{C}_I$. We obtain by (2.2) the identity

$$\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I} = \varinjlim_{c\in\mathcal{C}_I} \mathbb{Z}_c = \varinjlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} \mathbb{Z}_{Z_I^{gP_I^n}}.$$

Choose an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$. We get

$$\begin{aligned}
\mathrm{Ext}^i(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\mathcal{F}) &= H^i(\mathrm{Hom}(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\mathcal{I}^\bullet))\\
= H^i(\mathrm{Hom}(\varinjlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} i_*(\mathbb{Z}_{Z_I^{gP_I^n}}),\mathcal{I}^\bullet)) &= H^i(\varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} \mathrm{Hom}(i_*(\mathbb{Z}_{Z_I^{gP_I^n}}),\mathcal{I}^\bullet))\\
= H^i(\varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^0_{Z_I^{gP_I^n}}(\mathbb{P}^d_K,\mathcal{I}^\bullet)) & .
\end{aligned}$$

We make use of the following lemma. Here $\varprojlim^{(r)}_{n\in\mathbb{N}}$ is the r-th right derived functor of $\varprojlim_{n\in\mathbb{N}}$.

Lemma 2.2.2. *Let $\mathcal{I}$ be an injective sheaf on $(\mathbb{P}^d_K)^{ad}$. Then*

$$\varprojlim^{(r)}_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^0_{Z_I^{gP_I^n}}(\mathbb{P}^d_K,\mathcal{I}) = 0 \textit{ for } r\geq 1.$$

Proof. It suffices to show that the projective systems

$$\Big(\bigoplus_{g\in G_0/P_I^n} H^0(\mathbb{P}^d_K,\mathcal{I})\Big)_{n\in\mathbb{N}}$$

and

$$\Big(\bigoplus_{g\in G_0/P_I^n} H^0((\mathbb{P}^d_K)^{ad}\setminus Z_I^{gP_I^n},\mathcal{I})\Big)_{n\in\mathbb{N}}$$

are $\varprojlim_{n\in\mathbb{N}}$-acyclic. Clearly the maps $G_0/P_I^n \to G_0/P_I^m$ are surjective for $n\geq m$. It follows that the first projective system is $\varprojlim_{n\in\mathbb{N}}$-acyclic. Since $\mathcal{I}$ is injective, we have surjections

$$H^0((\mathbb{P}^d_K)^{ad}\setminus Z_I^{hP_I^n},\mathcal{I}) \to H^0((\mathbb{P}^d_K)^{ad}\setminus Z_I^{gP_I^m},\mathcal{I})$$

for $n \geq m$ and $hP_I^n \subset gP_I^m$. Thus we see that the transition maps

$$\bigoplus_{h \in G_0/P_I^n} H^0((\mathbb{P}^d_K)^{ad} \setminus Z_I^{hP_I^n}, \mathcal{I}) \to \bigoplus_{g \in G_0/P_I^m} H^0((\mathbb{P}^d_K)^{ad} \setminus Z_I^{gP_I^m}, \mathcal{I})$$

are surjective, as well. The claim follows. □

Thus we get by applying a spectral sequence argument (note that $\varprojlim^{(r)} = 0$ for $r \geq 2$ [Je]) short exact sequences, $i \in \mathbb{N}$,

$$0 \to \varprojlim_n{}^{(1)} \bigoplus_{g \in G_0/P_I^n} H^{i-1}_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F}) \to \mathrm{Ext}^i(i_*(\prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I}), \mathcal{F}) \to \varprojlim_n \bigoplus_{g \in G_0/P_I^n} H^i_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F}) \to 0.$$

Lemma 2.2.3. *The projective system* $\left(\bigoplus_{g \in G_0/P_I^n} H^{i-1}_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F})\right)_{n \in \mathbb{N}}$ *consists of K-Fréchet spaces and satisfies the (topological) Mittag-Leffler property for all $i \geq 1$ (cf.* [EGAIII] *13.2.4).*

Proof. By the same methods as those used in Propositions 1.1.2 and 1.1.3, we can choose a decreasing sequence of admissible open subsets

$$\cdots \supset D_{m-1} \supset D_m \supset D_{m+1} \supset \cdots$$

in $(\mathbb{P}^d_K)^{rig}$ with

$$\bigcap_m (D_m)^{ad} = Z_I^{P_I^n},$$

and such that the complements $(\mathbb{P}^d_K)^{rig} \setminus D_m$ are admissible open. In fact, we can choose these subsets to be coverings of the shape

$$D_m = \bigcup_{h \in R_m} h \cdot Y_I^-(\epsilon_m),$$

where $R_m \subset P_I^n$ are finite subsets. Here we may assume that $R_m \subset R_{m+1}$ and $1 \in R_m$ for all m. By translation with $g \in G_0$ we obtain admissible open subsets $g \cdot D_m$ of $(\mathbb{P}^d_K)^{rig}$ with

$$\bigcap_m g \cdot (D_m)^{ad} = Z_I^{gP_I^n}. \tag{2.4}$$

We shall see that the cohomology groups $H^*((\mathbb{P}^d_K)^{rig} \setminus D_m, \mathcal{F})$ are K-Fréchet spaces and that the transition maps

$$H^*((\mathbb{P}^d_K)^{rig} \setminus D_{m+1}, \mathcal{F}) \to H^*((\mathbb{P}^d_K)^{rig} \setminus D_m, \mathcal{F})$$

have dense image. Let $\Delta \setminus I = \{\alpha_{i_1} < \cdots < \alpha_{i_r}\}$. If $\Delta \setminus I = \{\alpha_{d-1}\}$, i.e., $Y_I \subset \mathbb{P}^d_K$ is a hyperplane, then the covering $((\mathbb{P}^d_K)^{rig} \setminus D_m)_m$ is of the type considered in Proposition

1.1.3 and the statement is a priori clear since it holds for Stein spaces. In general, we may write

$$Y_I^-(\epsilon_m) = \bigcap_{j>i_1} H_j^-(\epsilon_m)$$

where H_j is the hyperplane $V(X_j) \subset \mathbb{P}_K^d$. Then

$$(\mathbb{P}_K^d)^{rig} \setminus D_m = \bigcap_{h\in R_m} h \cdot \Big(\bigcup_{j>i_1} (\mathbb{P}_K^d)^{rig} \setminus H_j^-(\epsilon_m) \Big). \tag{2.5}$$

For a hyperplane $H \subset \mathbb{P}_K^d$, let $\ell_H \in S$ be a unimodular linear polynomial with $V(\ell_H) = H$. Thus a point $z \in (\mathbb{P}_K^d)^{rig}$ is contained in (2.5) if for all $h \in R_m$, there is an index $j > i_1$ with $|\ell_{h \cdot H_j}(z)| \geq \epsilon_m$. For each $h \in R_m$, let $j_h > i_1$ be some integer. Set

$$U_{(j_h)_h} = \{z \in (\mathbb{P}_K^d)^{rig} \mid |\ell_{h\cdot H_{j_h}}(z)| \geq \epsilon_m \ \ \forall h \in R_m\}.$$

Then for varying $(j_h)_h$, the sets $U_{(j_h)_h}$ form an open covering of $(\mathbb{P}_K^d)^{rig} \setminus D_m$ consisting of K-affinoid subsets. In fact, let

$$\mathcal{H}_{(j_h)_h} = \{h \cdot H_{j_h} \mid h \in R_m\}.$$

By the same reasoning as in [SS] Prop. 4, we see that the K-algebra $\mathcal{O}(U_{(j_h)_h})$ of analytic functions on $U_{(j_h)_h}$ is isomorphic to the K-affinoid algebra

$$K\langle T_{H_i,H}, T_{H,H'} \mid 0 \leq i \leq d,\ H, H' \in \mathcal{H}_{(j_h)_h}\}\rangle / I_m,$$

where I_m is the closed ideal generated by the elements

$$\begin{aligned}
&T_{H,H} - \pi^m,\ H \in \mathcal{H}_{(j_h)_h} \\
&T_{H,H'} \cdot T_{H',H''} - \pi^m \cdot T_{H,H''},\ H', H'' \in \mathcal{H}_{(j_h)_h}, H \in \mathcal{H}_{(j_h)_h} \cup \{H_i \mid 0 \leq i \leq d\} \\
&T_{H,H_{j_1}} - \sum_{i=0}^{d} \lambda_i \cdot T_{H_i,H_{j_1}} \text{ if } \ell_H(z) = \sum_{i=0}^{d} \lambda_i z_i,\ H \in \mathcal{H}_{(j_h)_h}.
\end{aligned}$$

The isomorphism is given by $T_{H,H'} \mapsto \pi^m \ell_H / \ell_{H'} \in \mathcal{O}(U_{(j_h)_h})$.

From now on, it suffices to treat the case $\mathcal{F} = \mathcal{O}$. We consider the Čech complex with respect to the covering $U_{(j_h)_h}$ for varying $(j_h)_h$. The analytic functions on the intersections of the various sets $U_{(j_h)_h}$ are described in the same manner. It is checked that the boundary maps are closed. In particular, the cohomology groups $H^*((\mathbb{P}_K^d)^{rig} \setminus D_m, \mathcal{O})$ are K-Fréchet spaces. Furthermore the transition maps are dense. Thus by Proposition 1.3.4 we get[7]

$$H^*((\mathbb{P}_K^d)^{ad} \setminus Z_I^{P_I^n}, \mathcal{F}) = \varprojlim_m H^*((\mathbb{P}_K^d)^{ad} \setminus D_m^{ad}, \mathcal{F}). \tag{2.6}$$

[7]Note that $H^*((\mathbb{P}_K^d)^{ad} \setminus D_{m+1}^{ad}, \mathcal{F}) = H^*((\mathbb{P}_K^d)^{rig} \setminus D_{m+1}, \mathcal{F})$ resp. $H^*((\mathbb{P}_K^d)^{ad} \setminus Z_I^{P_I^n}, \mathcal{F}) = H^*((\mathbb{P}_K^d)^{rig} \setminus P_I^n \cdot Y_I^{rig}, \mathcal{F})$ since the corresponding topoi are equivalent cf. [H] Prop. 2.1.4.

Furthermore, the transition maps

$$H^*((\mathbb{P}^d_K)^{ad} \setminus Z_I^{P_I^{n+1}}, \mathcal{F}) \to H^*((\mathbb{P}^d_K)^{ad} \setminus Z_I^{P_I^n}, \mathcal{F})$$

are dense. Thus, taking (2.4) into account, our projective system consists of K-Fréchet spaces and satisfies the topological Mittag-Leffler property. $\square$

We deduce from [EGAIII] 13.2.4 that

$$\varprojlim\nolimits^{(1)}_{n\in\mathbb{N}} \Big(\bigoplus_{g\in G_0/P_I^n} H^{i-1}_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F}) \Big)_{n\in\mathbb{N}} = 0.$$

We obtain the identity

$$\mathrm{Ext}^i(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathcal{F}) \cong \varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^i_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F}).$$

On the other hand, we have $\bigcap_{n\in\mathbb{N}} Z_I^{P_I^n} = \bigcap_{n\in\mathbb{N}} Y_I(\epsilon_n)^{ad} = Y_I^{ad}$. Again, by applying Proposition 1.3.3, we deduce the identity $\varprojlim_n H^*_{Z_I^{P_I^n}}(\mathbb{P}^d_K, \mathcal{F}) = \varprojlim_n H^*_{Y_I(\epsilon_n)^{ad}}(\mathbb{P}^d_K, \mathcal{F})$. We get

$$\begin{aligned}
\varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^*_{Z_I^{gP_I^n}}(\mathbb{P}^d_K, \mathcal{F}) &= \varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^*_{gY_I(\epsilon_n)^{ad}}(\mathbb{P}^d_K, \mathcal{F}) \\
&= \varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^*_{gY_I(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}).
\end{aligned}$$

Thus the statement of our proposition is proved. $\square$

Consider the spectral sequence

$$(2.7)\quad E_1^{-p,q} = \mathrm{Ext}^q(\bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=p+1}} i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathcal{F}) \Rightarrow \mathrm{Ext}^{-p+q}(i_*(\mathbb{Z}_{\mathcal{Y}^{ad}}), \mathcal{F}) = H^{-p+q}_{\mathcal{Y}^{ad}}(\mathbb{P}^d_K, \mathcal{F})$$

induced by the acyclic complex (2.3), cf. Theorem 2.1.1. By applying the previous proposition to it we compute for the rows $E_1^{\bullet,q}$, $q \in \mathbb{N}$, the following complexes of K-Fréchet spaces. Note that we have

$$H^*_{\mathbb{P}_K^{d-j}(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) = H^j_{\mathbb{P}_K^{d-j}(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}) \oplus \bigoplus_{k=j+1}^{d} H^k(\mathbb{P}^d_K, \mathcal{F})$$

by (1.10).

$$E_1^{\bullet,d}: \varprojlim_n \bigoplus_{g\in G_0/P^n_\emptyset} H^d_{g\mathbb{P}^0_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=1}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^d_{g,I} \to \bigoplus_{\substack{I\subset\Delta\\ \#I=2}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^d_{g,I}$$
$$\to \ldots \to \varprojlim_n \bigoplus_{g\in G_0/P^n_{(1,d)}} M^d_{g,I},$$

where

$$M^d_{g,I} = \begin{cases} H^d_{g\mathbb{P}^0_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_0\notin I \\ \\ H^d(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_0\in I \end{cases},$$

$$E_1^{\bullet,d-1}: \varprojlim_n \bigoplus_{g\in G_0/P^n_{(2,1,\ldots,1)}} H^{d-1}_{g\mathbb{P}^1_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=2\\ \alpha_0\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^{d-1}_{g,I} \to \bigoplus_{\substack{I\subset\Delta\\ \#I=3\\ \alpha_0\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^{d-1}_{g,I}$$
$$\to \ldots \to \varprojlim_n \bigoplus_{g\in G_0/P^n_{(2,d-1)}} M^{d-1}_{g,I},$$

where

$$M^{d-1}_{g,I} = \begin{cases} H^{d-1}_{g\mathbb{P}^1_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_1\notin I \\ \\ H^{d-1}(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_1\in I \end{cases},$$

$$\vdots$$

$$E_1^{\bullet,j}: \varprojlim_n \bigoplus_{g\in G_0/P^n_{(d+1-j,1,\ldots,1)}} H^j_{g\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^j_{g,I} \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+2\\ \alpha_0\cdots\alpha_{d-j-1}\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} M^j_{g,I}$$
$$\to \ldots \to \varprojlim_n \bigoplus_{g\in G_0/P^n_{(d+1-j,j)}} M^j_{g,I},$$

where

$$M^j_{g,I} = \begin{cases} H^j_{g\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_{d-j}\notin I \\ \\ H^j(\mathbb{P}^d_K,\mathcal{F}) & ; \quad \alpha_{d-j}\in I \end{cases},$$

$$\vdots$$

$$E_1^{0,1}: \quad \varprojlim_n \bigoplus_{g\in G_0/P^n_{(d,1)}} H^1_{g\mathbb{P}_K^{d-1}(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}).$$

Here, the very left term in each row $E_1^{\bullet,j}$ sits in degree $-j+1$. We can rewrite these complexes in terms of induced representations. Here we abbreviate

$$(d+1-j,1^j) := (d+1-j,1,\ldots,1)$$

for any decomposition $(d+1-j,1,\ldots,1)$ of $d+1$.

$$\begin{aligned} E_1^{\bullet,d}: \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_\emptyset} H^d_{\mathbb{P}^0_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) &\to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=1}} \mathrm{Ind}^{G_0}_{P^n_I} M^d_{g,I} \to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=2}} \mathrm{Ind}^{G_0}_{P^n_I} M^d_{g,I} \\ &\to \ldots \to \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(1,d)}} M^d_{g,I}, \end{aligned}$$

$$\begin{aligned} E_1^{\bullet,d-1}: \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(2,1^{d-1})}} H^{d-1}_{\mathbb{P}^1_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) &\to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=2\\ \alpha_0\in I}} \mathrm{Ind}^{G_0}_{P^n_I} M^{d-1}_{g,I} \to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=3\\ \alpha_0\in I}} \mathrm{Ind}^{G_0}_{P^n_I} M^{d-1}_{g,I} \\ &\to \ldots \to \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(2,d-1)}} M^{d-1}_{g,I} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} E_1^{\bullet,j}: \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} H^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) &\to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I} M^j_{g,I} \to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+2\\ \alpha_0\cdots\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I} M^j_{g,I} \\ &\to \ldots \to \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}} M^j_{g,I} \end{aligned}$$

$$\vdots$$

$$E_1^{0,1}: \varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(d,1)}} H^1_{\mathbb{P}^{d-1}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}).$$

Proposition 2.2.4. *Each of the complexes $E_1^{\bullet,j}$, $j=1,\ldots,d$, is acyclic apart from the very left and right position.*

Proof. We can write each of the complexes in the shape $E_1^{\bullet,j} = \varprojlim_n K^\bullet_{j,n}$, where $K^\bullet_{j,n}$ is a complex of K-Fréchet spaces which appears in a short exact sequence of complexes of K-Fréchet spaces

$$0 \to K^{\bullet,'}_{j,n} \to K^\bullet_{j,n} \to K^{\bullet,''}_{j,n} \to 0. \tag{2.8}$$

Here, $K^{\bullet,'}_{j,n}$ is the complex

$$\mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,,\alpha_{d-j-1}\in I,\alpha_{d-j}\notin I}} \mathrm{Ind}^{G_0}_{P^n_I}\tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})$$

$$\to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+2\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I,\alpha_{d-j}\notin I}} \mathrm{Ind}^{G_0}_{P^n_I}\tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}) \to \ldots \to \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F}).$$

Furthermore, the complex $K^{\bullet,''}_{j,n}$ is given by

$$\mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} H^j(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}H^j(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+2\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}H^j(\mathbb{P}^d_K,\mathcal{F}) \to$$

$$\ldots \to \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta\setminus I)=1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}H^j(\mathbb{P}^d_K,\mathcal{F}).$$

Since $H^j(\mathbb{P}^d_K,\mathcal{F})$ is a G-module, this complex is isomorphic to

$$\Big(\mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} K \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}K \to \ldots \to \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta\setminus I)=1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}K\Big) \otimes H^j(\mathbb{P}^d_K,\mathcal{F}).$$

It suffices to prove that the complexes $\varprojlim_n K^{\bullet,'}_{j,n}$ and $\varprojlim_n K^{\bullet,''}_{j,n}$ are acyclic apart from the very left and right position. We deduce this property for $K^{\bullet,''}_{j,n}$ by applying the first part of the following lemma to its dual.

Lemma 2.2.5. *(i) For each integer $0\leq j\leq d-1$, the following complex is acyclic apart from the very left and right position:*

$$\Big(\mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} K \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}K \to \ldots \to \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta\setminus I)=1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}K\Big) \otimes H^j(\mathbb{P}^d_K,\mathcal{F}).$$

(ii) Let W be any $\mathbf{P_{(d+1-j,j)}}(O_K/(\pi^n))$-module. Consider W as a $P^n_{(d+1-j,j)}$-module via the inflation map (1.11). Then for each integer $0 \leq j \leq d-1$, the following complex is acyclic apart from the very left and right position:

$$\mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}} W \to \ldots \to \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+2 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I, \alpha_{d-j} \notin I}} \mathrm{Ind}^{G_0}_{P^n_I} W \to \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I, \alpha_{d-j} \notin I}} \mathrm{Ind}^{G_0}_{P^n_I} W \to \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} W$$

Proof. For a subset $I \subset \Delta$, there is a natural isomorphism

$$G_0/P^n_I \xrightarrow{\sim} \mathbf{G}(O_K/(\pi^n))/\mathbf{P_I}(O_K/(\pi^n)).$$

Via this identification the representation $\mathrm{Ind}^{G_0}_{P^n_I} W$ coincides with $\mathrm{Ind}^{\mathbf{G}(O_K/(\pi^n))}_{\mathbf{P_I}(O_K/(\pi^n))} W$. Then statement (i) follows from Theorem 2.5, ch. III in [OR]. In fact, loc.cit. treats the dual complex in the case $n=1$, but the proof for $n>1$ is the same. The proof of part (ii) works by the same reasoning as in loc.cit. In particular, it does not depend on the coefficient system. $\square$

Since the complex $K^{\bullet,''}_{j,n}$ consists of finite-dimensional K-vector spaces, we conclude by the Mittag-Leffler condition that $\varprojlim_n K^{\bullet,''}_{j,n}$ is acyclic apart from the very left and right position. The dual complex $(\varprojlim_n K^{\bullet,''}_{j,n})'$ of $\varprojlim_n K^{\bullet,''}_{j,n}$ is given by

$$\begin{aligned}
\varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} H^j(\mathbb{P}^d_K,\mathcal{F})' \;&\leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_I} H^j(\mathbb{P}^d_K,\mathcal{F})' \leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+2 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_I} H^j(\mathbb{P}^d_K,\mathcal{F})' \\
\leftarrow \ldots \;&\leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#(\Delta \setminus I) = 1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_I} H^j(\mathbb{P}^d_K,\mathcal{F})' \\
= \mathrm{Ind}^{\infty,G}_{P_{(d+1-j,1^j)}} H^j(\mathbb{P}^d_K,\mathcal{F})' \;&\leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \mathrm{Ind}^{\infty,G}_{P_I} H^j(\mathbb{P}^d_K,\mathcal{F})' \leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+2 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \mathrm{Ind}^{\infty,G}_{P_I} H^j(\mathbb{P}^d_K,\mathcal{F})' \\
\leftarrow \ldots \;&\leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#(\Delta \setminus I) = 1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \mathrm{Ind}^{\infty,G}_{P_I} H^j(\mathbb{P}^d_K,\mathcal{F})'.
\end{aligned}$$

Here $\mathrm{Ind}^{\infty,G}_P$ denotes the (unnormalized) smooth induction functor for a parabolic subgroup $P \subset G$, cf. [Ca]. Again, since $H^j(\mathbb{P}^d_K,\mathcal{F})$ is even a G-module, this complex coincides with

$$\Big(\mathrm{Ind}^{\infty,G}_{P_{(d+1-j,1^j)}} K \leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \mathrm{Ind}^{\infty,G}_{P_I} K \leftarrow \ldots \leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#(\Delta \setminus I) = 1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I}} \mathrm{Ind}^{\infty,G}_{P_I} K\Big) \otimes H^j(\mathbb{P}^d_K,\mathcal{F})'.$$

Let

$$v^G_{P_{(d+1-j,1^j)}}(K) := \mathrm{Ind}^{\infty,G}_{P_{(d+1-j,1^j)}} K / \sum_{Q \supsetneq P_{(d+1-j,1^j)}} \mathrm{Ind}^{\infty,G}_{Q} K$$

be the smooth generalized Steinberg representation with respect to the parabolic subgroup $P_{(d+1-j,1^j)}$. It is known that this is an irreducible smooth G-representation, cf. [BW] ch. X. Put

$$v^G_{P_{(d+1-j,1^j)}}(H^j(\mathbb{P}^d_K,\mathcal{F})') := v^G_{P_{(d+1-j,1^j)}}(K) \otimes H^j(\mathbb{P}^d_K,\mathcal{F})'$$

The only non-vanishing cohomology groups of the dual complex $(\varprojlim_n K^{\bullet,''}_{j,n})'$ are therefore given by

$$H^*((\varprojlim_n K^{\bullet,''}_{j,n})') = v^G_{P_{(d+1-j,1^j)}}(H^j(\mathbb{P}^d_K,\mathcal{F})') \oplus H^j(\mathbb{P}^d_K,\mathcal{F})' \text{ for } j \geq 2$$

resp.

$$H^*((\varprojlim_n K^{\bullet,''}_{j,n})') = \mathrm{Ind}^{\infty,G}_{P_{(d,1)}} H^1(\mathbb{P}^d_K,\mathcal{F})' \text{ for } j = 1.$$

Now, we turn to the complexes $\varprojlim_n K^{\bullet,'}_{j,n}$. Each entry in $\varprojlim_n K^{\bullet,'}_{j,n}$ is a compact projective limit of K-Fréchet spaces, hence nuclear, cf. [S2] Proposition 19.9 (compare also the example at the end of chapter 16). By loc.cit. Corollary 19.3 these objects are reflexive. The duality functor is exact on the category of K-Fréchet spaces, cf. [Ba] ch I, Cor. 1.4. So, it suffices to show that the dual complexes (see [S2] Prop. 16.10)

$$\varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})' \leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+1 \\ \alpha_0,\ldots,\alpha_{d-j-1} \in I, \alpha_{d-j} \notin I}} \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_I} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$$

$$\leftarrow \bigoplus_{\substack{I \subset \Delta \\ \#I = d-j+2 \\ \alpha_0 \cdots \alpha_{j-1} \in I, \alpha_{d-j} \notin I}} \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_I} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})' \leftarrow \ldots \leftarrow \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}} \tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'$$

consisting of compact inductive limits of locally convex K-vector spaces are exact apart from the very left and right position. But this follows by the exactness of $\varinjlim$ from Lemma 2.2.5, as well. Thus Proposition 2.2.4 is proved. $\square$

Remark 2.2.6. Alternatively, on could prove the acyclicity of $\varprojlim_n K^{\bullet,'}_{j,n}$ as follows. By the same reasoning as in the proof of Lemma 2.2.5 one shows that even the complex $K^{\bullet,'}_{j,n}$ is exact apart from the very left and right position. Then we apply the topological Mittag-Leffler condition to the projective limit to deduce the claim, cf. Lemma 1.3.7. $\square$

By Proposition 2.2.4 we deduce that the only non-vanishing entries in $E_2^{p,q}$ are given by the indices $(p,q)=(-j+1,j),\ j=1,\ldots,d$, and $(p,q)=(0,j),\ j\geq 2$. For the latter indices, we get

$$E_2^{0,q}=H^q(\mathbb{P}^d,\mathcal{F}).$$

For the other indices, we obtain

$$E_2^{-j+1,j}=\ker(E_1^{-j+1,j}\to E_1^{-j+2,j})=$$
$$\ker\Big(\varprojlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} H^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})\to \varprojlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I} M^j_{g,I}\Big).$$

Thus our spectral sequence has apart from stretching the y-axis the same structure as in the case of constant coefficients, cf. p.70 [SS]. Further, the composed maps $E_2^{0,s}\to H^s_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F})\to H^s(\mathbb{P}^d_k,\mathcal{F})$ where the first map is the edge homomorphism are isomorphisms for $s>1$ and surjective for $s=1$. By the same reasoning as in loc.cit. we conclude that our spectral sequence degenerates at E_2. By duality, i.e., by taking the strong dual of these K-Fréchet spaces, we get locally analytic (cf. [ST3] Cor. 3.3) G_0-representations

$$(E_2^{-j+1,j})'=\mathrm{coker}\Big(\varinjlim_n \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}(M^j_{g,I})'\to \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,1^j)}} H^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'\Big),$$

which are by (2.8) extensions of locally analytic G_0-representations

(2.9)
$$0\to v^G_{P_{(d+1-j,1^j)}}(H^j(\mathbb{P}^d_K,\mathcal{F})')\to (E_2^{-j+1,j})'\to \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}}(\tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'\otimes \mathrm{St}_j)\to 0.$$

Here, $\mathrm{St}_j=v^{GL_j}_{B\cap GL_j}(K)$ denotes the smooth Steinberg representation of the p-adic Lie group GL_j viewed as one of the factors of the Levi subgroup $L_{(d-j+1,j)}$. Now, we plug in the result of Proposition 1.3.10. We obtain isomorphisms of locally analytic G_0-representations:

$$\begin{aligned}\varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}}(\tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K,\mathcal{F})'\otimes \mathrm{St}_j) &\cong \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d-j+1,j)}}(\mathcal{O}(U^{+,n}_{(d-j+1,j)},N'_{d-j})^{\mathfrak{d}_{d-j}}\otimes \mathrm{St}_j)\\ &= \varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d-j+1,j)}}(\mathcal{O}(U^{+,n}_{(d-j+1,j)},N'_{d-j})\otimes \mathrm{St}_j)^{\mathfrak{d}_{d-j}}.\end{aligned}$$

The latter equality holds since St_j is smooth. On the other hand,

$$\varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d-j+1,j)}}(\mathcal{O}(U^{+,n}_{(d-j+1,j)},N'_{d-j})\otimes \mathrm{St}_j)\cong C^{an}(G,P_{(d-j+1,j)};N'_{d-j}\otimes \mathrm{St}_j).$$

Here $C^{an}(G, P_{(d-j+1,j)}; N'_{d-j} \otimes \mathrm{St}_j)$ denotes the locally analytic induced G-representation [F] with values in $N'_{d-j} \otimes \mathrm{St}_j$:

$$C^{an}(G, P_{(d-j+1,j)}; N'_{d-j} \otimes \mathrm{St}_j) = \Big\{ \text{ locally analytic maps } f : G \to N'_{d-j} \otimes \mathrm{St}_j \mid f(g \cdot p) = p^{-1} f(g)\ \forall g \in G, p \in P_{(d-j+1,j)} \Big\}.$$

In the above formula, we have made use of the canonical identity

$$C^{an}(G_0, \mathbf{P_{(d-j+1,j)}}(O_K); N'_{d-j} \otimes \mathrm{St}_j) = C^{an}(G, P_{(d-j+1,j)}; N'_{d-j} \otimes \mathrm{St}_j)$$

as locally analytic G_0-representations. This identity follows from the Iwasawa decomposition $G = G_0 \cdot P_{(d-j+1,j)}$ and by [F] 4.1.4. Hence, the above space possesses even the structure of a locally analytic G-representation. We claim that the above isomorphisms and the resulting extensions (2.9) are even G-equivariant. In fact, this follows essentially from the Iwasawa decomposition and Remark 1.3.11. Altogether, we get an isomorphism

$$\varinjlim_n \mathrm{Ind}^{G_0}_{P^n_{(d+1-j,j)}} (\tilde{H}^j_{\mathbb{P}^{d-j}_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F})' \otimes \mathrm{St}_j) \cong C^{an}(G, P_{(d-j+1,j)}; N'_{d-j} \otimes \mathrm{St}_j)^{\mathfrak{d}_{d-j}}.$$

In the following, we also use for a parabolic subgroup $P \subset G$ and a locally analytic P-representation U, the symbol $\mathrm{Ind}^{an,G}_P(U)$ instead of $C^{an}(G, P; U)$.

Lemma 2.2.7. *Let $V^\bullet = V^{-d+1} \supset V^{-d+2} \supset \cdots \supset V^{-1} \supset V^0 \supset V^1 = (0)$ be the canonical filtration on $V^{-d+1} = H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$ defined by our spectral sequence (2.7). Then the subspaces V^j are closed in the K-Fréchet space $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$.*

Proof. Recall the definition of $V^\bullet$. Let $K^\bullet$ be the simple complex attached to the double complex $E_0^{\bullet,\bullet}$– the 0-th term of our spectral sequence. Let $F^p(K^\bullet) = \sum_{j \geq p} E_0^{j,\bullet}$ be the sum of the columns $E_0^{\bullet,j}$ with $j \geq p$. Then V^{-d+i} is by definition the kernel of the natural homomorphisms $H^1(K^\bullet) \to H^1(K^\bullet / F^{-d+i}(K^\bullet))$, or equivalently, it is the image of the natural homomorphism $H^1(F^{-d+i}(K^\bullet)) \to H^1(K^\bullet)$, cf. 11.2.2 [EGAIII]. We consider the pull-back of $V^\bullet$ to $H^0(\mathcal{X}, \mathcal{F})$ under the boundary map $H^0(\mathcal{X}, \mathcal{F}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$. On $\mathcal{X}$ the vector bundle $\mathcal{F}$ splits, cf. (1.2). Since the canonical filtrations with respect to $\mathcal{O}^{\mathrm{rk}\mathcal{F}}$ and $\mathcal{F}$ differ topologically at most by finite-dimensional Hausdorff K-vector spaces, it is enough to treat the case of $\mathcal{F} = \mathcal{O}^{\mathrm{rk}\mathcal{F}}$ and hence of $\mathcal{F} = \mathcal{O}$. In this case one sees by the definition of our spectral sequence that the algebraic part of V^{-d+i} consists of functions having at most $d - i + 1$ poles, cf. also Proposition 3.1.1. Thus, our filtration coincides up to the numbering with Pohlkamps (3.3) which consists of closed subspaces. $\square$

Summarising the computation of this chapter we obtain the following theorem.

Theorem 2.2.8. *Let*

$$V^{-d+1} \supset V^{-d+2} \supset \cdots \supset V^{-1} \supset V^0 \supset V^1 = (0)$$

be the canonical G-equivariant filtration by closed K-Fréchet spaces on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$ defined by the spectral sequence (2.7). For $j = 1, \ldots, d-1$, there are extensions of locally analytic G-representations

$$0 \to v^G_{P_{(d-j,1^{j+1})}}(H^{j+1}(\mathbb{P}^d_K, \mathcal{F})') \to (V^{-j}/V^{-j+1})' \to \mathrm{Ind}^{an,G}_{P_{(d-j,j+1)}}(N'_{d-j-1} \otimes \mathrm{St}_{j+1})^{\mathfrak{d}_{d-j-1}} \to 0.$$

In the case $j = 0$, there is an extension

$$0 \to \mathrm{Ind}^{\infty,G}_{P_{(d,1)}}(H^1(\mathbb{P}^d_K, \mathcal{F})') \to (V^0)' \to \mathrm{Ind}^{an,G}_{P_{(d,1)}}(N'_{d-1} \otimes \mathrm{St}_1)^{\mathfrak{d}_{d-1}} \to 0.$$

Proof. This follows from the above computation $\square$

Consider the topological exact G-equivariant sequence of K-Fréchet spaces

$$0 \to H^0(\mathbb{P}^d_K, \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{F}) \xrightarrow{p} H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F}) \to H^1(\mathbb{P}^d_K, \mathcal{F}) \to 0.$$

For $i = 0, \ldots, -d$, we set

$$W^i := p^{-1}(V^{i+1}).$$

Thus we get a G-equivariant filtration by closed K-Fréchet spaces

$$W^{-d} \supset W^{-d+1} \supset \cdots \supset W^{-1} \supset W^0$$

on $W^{-d} = H^0(\mathcal{X}, \mathcal{F})$.

Corollary 2.2.9. *For $j = 1, \ldots, d$, there are extensions of locally analytic G-representations*

$$0 \to v^G_{P_{(d-j+1,1^j)}}(H^j(\mathbb{P}^d_K, \mathcal{F})') \to (W^{-j}/W^{-j+1})' \to \mathrm{Ind}^{an,G}_{P_{(d+1-j,j)}}(N'_{d-j} \otimes \mathrm{St}_j)^{\mathfrak{d}_{d-j}} \to 0.$$

In the case $j = 0$, we get

$$W^0 = H^0(\mathbb{P}^d, \mathcal{F}).$$

$\square$

For vector bundles, where the unipotent radical $\mathbf{U_{(1,d)}}$ acts trivially on the fibre, we can make our result more precise using the main result Corollary 1.4.9.

Theorem 2.2.10. *Let $\mathcal{F} = \mathcal{F}_\lambda$ be a homogeneous vector bundle on $\mathbb{P}^d_K$ corresponding to the $L_{(1,d)}$-dominant weight $\lambda \in \mathbb{Z}^{d+1}$. Let $i_0 \in \mathbb{N}$ be the unique integer with $w_i * \lambda \succeq w_{i+1} * \lambda$ for all $i \geq i_0$, and $w_i * \lambda \prec w_{i+1} * \lambda$ for all $i < i_0$. For $1 \leq j \leq d$, let*

$$\mu_{j,\lambda} := \begin{cases} w_{j-1} * \lambda & : j \leq i_0 \\ w_j * \lambda & : j > i_0 \end{cases} .$$

Write $\mu_{j,\lambda} = (\mu', \mu'')$ with $\mu' \in \mathbb{Z}^j$ and $\mu'' \in \mathbb{Z}^{d-j+1}$. Further, set

$$\Psi_{j,\lambda} = \bigcup_{k=0}^{|\mu''|} \Big\{ (\mu'' + (c_1, \ldots, c_{d-j+1}), \mu' - (d_j, \ldots, d_1)) \mid \textstyle\sum_i c_i = \sum_i d_i = k, c_1 = 0$$
$$\text{or } d_1 = 0,\ c_{i+1} \leq \mu''_i - \mu''_{i+1},\ i = 1, \ldots, d-j,\ d_{i+1} \leq \mu'_{j-i} - \mu'_{j-i+1},$$
$$i = 1, \ldots, j-1 \Big\}$$

and let

$$N_{j,\lambda} = \bigoplus_{\mu \in \Psi_{j,\lambda}} V_\mu \quad \subset \quad K[X_{(m,n)} \mid m \leq d-j,\ n \geq d-j+1] \otimes V_{(\mu'',\mu')}$$

be the sum of the irreducible algebraic $P_{(d+1-j,j)}$-representations V_μ attached to μ. Let

$$q_j : K[X_{(m,n)} \mid m \leq d-j,\ n \geq d-j+1] \otimes V_{(\mu'',\mu')} \to \tilde{H}^j_{\mathbb{P}^{d-j}_K}(\mathbb{P}^d_K, \mathcal{F}_\lambda)$$

be the quotient map of Corollary 1.4.9. Then we can choose N_{d-j} to be $q_j(N_{j,\lambda})$.

Proof. This follows from Corollary 1.4.9 . □

Remark 2.2.11. By renumbering the filtration $W^\bullet$ on $H^0(\mathcal{X}, \mathcal{F})$, i.e., if we set

$$\mathcal{F}(\mathcal{X})^i := W^{-d+i}$$

for $i = 0, \ldots d$, we get the filtration on $H^0(\mathcal{X}, \mathcal{F})$ mentioned in the introduction. □

Conclusion: Although we have generalized the cases of $\mathcal{F} = \Omega^d$ [ST1] respectively $\mathcal{F} = \mathcal{O}$ [P] to arbitrary homogeneous vector bundles on $\mathbb{P}^d_K$, our result has the lack that it does not yield explicit isomorphisms (partial boundary value maps) as in loc. cit. We hope to determine these isomorphisms in a future paper.

3. Examples

3.1. $\mathcal{F} = \mathcal{O}_{\mathbb{P}^d_K}$**.** In this chapter we compare our result in the case $\mathcal{F} = \mathcal{O} = \mathcal{O}_{\mathbb{P}^d_K}$ to that of [P]. For this purpose, we have to recall some more notation used there.

In loc.cit. there is defined a filtration by closed K-Fréchet spaces on $\mathcal{O}(\mathcal{X})$ as follows. For a character $\mu = \sum_{l=0}^{d} m_l \epsilon_l \in X_*(\overline{T})$, we set

$$J_-(\mu) = \{l \mid 0 \leq l \leq d \text{ and } m_l < 0\}.$$

For a subset $J \subset \{0, \ldots, d\}$, we put

$$\mathfrak{a}_J := \sum_{\{\mu \mid J_-(\mu) \subset J\}} K \cdot \mu.$$

This is a $U(\mathfrak{g})$-submodule of

$$\mathcal{O}_{\mathrm{inf}}(\mathcal{X}) := \sum_{\mu \in X^*(\overline{T})} K \cdot \Xi_\mu \subset \mathcal{O}(\mathcal{X})$$

and we have

$$J' \subset J \iff \mathfrak{a}_{J'} \subset \mathfrak{a}_J.$$

The extreme cases are

$$\mathfrak{a}_\emptyset = K \cdot \Xi_0 \text{ resp. } \mathfrak{a}_{\{0,\ldots,d\}} = \mathcal{O}_{\mathrm{inf}}(\mathcal{X}).$$

We set $\mathfrak{a}_J^< := \sum_{J' \subsetneq J} \mathfrak{a}_{J'}$ for $J \neq \emptyset$ and $\mathfrak{a}_\emptyset^< = 0$. We obtain K-vector space isomorphisms

$$\mathfrak{a}_J/\mathfrak{a}_J^> \xrightarrow{\sim} \sum_{\{\mu \mid J = J_-(\mu)\}} K \cdot \Xi_\mu.$$

Put

$$\begin{aligned} A(J) &:= \{\mu \in X^*(\overline{T}) \mid m_j = -1 \text{ for } j \in J_-(\mu) \text{ and } J_-(\mu) = J\} \\ M_J &:= \sum_{\mu \in A(J)} K \cdot \Xi_\mu + \mathfrak{a}_J^</\mathfrak{a}_J^< \\ \mathfrak{p}_J &= \{(g_{i,j}) \mid g_{i,j} = 0, \text{ if } i \notin J \text{ and } j \in J\}. \end{aligned}$$

Then $M_J \subset \mathfrak{a}_J/\mathfrak{a}_J^<$ is a $\mathfrak{p}_J$-submodule of $\mathcal{O}_{\mathrm{inf}}(\mathcal{X})$. For $0 \leq j \leq d$, we set

$$\mathfrak{a}_j = \sum_{\{\mu \mid \# J_-(\mu) \leq j\}} K \cdot \Xi_\mu.$$

We get a filtration by $\mathfrak{g}$-submodules

$$K = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_{d-1} \subset \mathfrak{a}_d = \mathcal{O}_{\mathrm{inf}}(\mathcal{X}) \tag{3.1}$$

on $\mathcal{O}_{\mathrm{inf}}(\mathcal{X})$. For $0 \leq j \leq d$, we put

$$\overline{j} := \{d - j + 1, \ldots, d\}.$$

Let $P_{\overline{j}} \subset G$ be the parabolic subgroup with Lie algebra $\mathfrak{p}_{\overline{j}}$. Let $U_{\overline{j}} \subset P_{\overline{j}}$ be its unipotent radical and let $L_{\overline{j}} \subset P_{\overline{j}}$ be its standard Levi subgroup. The group $L_{\overline{j}}$ splits into a product

$$L_{\overline{j}} = L(\overline{j}) \times L'(\overline{j})$$

with $L'(\overline{j}) \cong \mathrm{GL}_j$ and $L(\overline{j}) \cong \mathrm{GL}_{d-j+1}$. We get

$$\mathfrak{l}_{\overline{j}} = \mathfrak{l}(\overline{j}) \times \mathfrak{l}'(\overline{j}).$$

In [P], chapter 1, it is shown that $\mathfrak{l}(\overline{j})$ acts on $M_{\overline{j}}$ via the j-th symmetric power on the K-vector space K^{d+1-j}. Further $\mathfrak{l}'(\overline{j})$ acts on $M_{\overline{j}}$ by the trace character, i.e., by

$$\mathfrak{t} \cdot \Xi_\mu = (-\sum\nolimits_{i \in \overline{j}} \mathfrak{t}_i) \cdot \Xi_\mu,\ \ \mathfrak{t} \in \mathfrak{l}'(\overline{j}).$$

Denote by $\mathfrak{d}_{\overline{j}} := \ker\, \varphi_{\overline{j}}$ the kernel of the epimorphism

$$\varphi_{\overline{j}} : U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\overline{j}})} M_{\overline{j}} \to \mathfrak{a}_{\overline{j}}/\mathfrak{a}_{\overline{j}}^{\leq}.$$

Let $\mathcal{O}_{\mathrm{alg}}(\mathcal{X}) \subset \mathcal{O}(\mathcal{X})$ be the subspace of algebraic functions. More concretely,

$$\begin{aligned}\mathcal{O}_{\mathrm{alg}}(\mathcal{X}) \ \ = \ \ & \Big\{F = \frac{P}{Q} \mid Q = \prod_{j=1}^{r}(\sum_{i=0}^{d} c_{ij}X_i)^{l_j} \mid c_{ij} \in K, \\ & \qquad\qquad P \text{ is a homogeneous polynomial of degree } \sum_{j=1}^{r} l_j\Big\}.\end{aligned}$$

This is a dense subset of the K-Fréchet algebra $\mathcal{O}(\mathcal{X})$, comp. [ST1] 3.3. We denote by

$$\imath : \mathcal{O}_{\mathrm{alg}}(\mathcal{X}) \to \mathbb{N}$$

the index function. This is a G-invariant map taking values in the interval $[0, d]$. For a function $F = \frac{P}{Q} \in \mathcal{O}_{\mathrm{alg}}(\mathcal{X})$, with $(P, Q) = 1$ and pairwise different $(c_{0,j}, \ldots, c_{d,j}) \in K^{d+1}$, $j = 1, \ldots, r$, it is defined by $\imath(\frac{P}{Q}) = \imath_o(\frac{P}{Q}) = r$. In general, it is given by

$$\imath(F) = \min\max\Big\{\imath_o(\frac{P_k}{Q_k}) \ \mid\ k\Big\},$$

where the minimum is taken over all representations $F = \sum_k \frac{P_k}{Q_k}$ with $\frac{P_k}{Q_k} \in \mathcal{O}_{\mathrm{alg}}(\mathcal{X})$. Put

$$\mathcal{O}_{\mathrm{alg}}(\mathcal{X})_j := \Big\{F \in \mathcal{O}_{\mathrm{alg}}(\mathcal{X}) \mid \imath(F) \leq j\Big\}. \tag{3.2}$$

We obtain a filtration

$$K = \mathcal{O}_{\mathrm{alg}}(\mathcal{X})_0 \subset \mathcal{O}_{\mathrm{alg}}(\mathcal{X})_1 \subset \cdots \subset \mathcal{O}_{\mathrm{alg}}(\mathcal{X})_{d-1} \subset \mathcal{O}_{\mathrm{alg}}(\mathcal{X})_d = \mathcal{O}_{\mathrm{alg}}(\mathcal{X})$$

on $\mathcal{O}_{\mathrm{alg}}(\mathcal{X})$. The relation between the filtration (3.1) on $\mathcal{O}_{\mathrm{inf}}(\mathcal{X})$ and this one is

$$\mathcal{O}_{\mathrm{alg}}(\mathcal{X})_j = \sum_{g\in G} g\cdot \mathfrak{a}_j,\ j=0,\ldots,d.$$

This follows from some of the results in [GV] and is explained in [P] resp. [ST1]. Finally, for $j=1,\ldots,d$, let

$$\mathcal{O}(\mathcal{X})_j := \overline{\mathcal{O}_{\mathrm{alg}}(\mathcal{X})_j} \subset \mathcal{O}(\mathcal{X})$$

be the topological closure of $\mathcal{O}_{\mathrm{alg}}(\mathcal{X})_j$ in $\mathcal{O}(\mathcal{X})$. We get a G-equivariant filtration by closed K-Fréchet spaces

$$K=\mathcal{O}(\mathcal{X})_0 \subset \mathcal{O}(\mathcal{X})_1 \subset \cdots \subset \mathcal{O}(\mathcal{X})_d = \mathcal{O}(\mathcal{X}) \tag{3.3}$$

on $\mathcal{O}(\mathcal{X})$. Each subquotient is a reflexive K-Fréchet space with a continuous G-action (see [ST1] Prop. 6) and its dual is a locally analytic G-representation. Similarly to [ST1], Pohlkamp constructs for $1\leq j\leq d$, isomorphisms

$$(\mathcal{O}(\mathcal{X})_j/\mathcal{O}(\mathcal{X})_{j-1})' \xrightarrow{\sim} C^{an}(G,P_{\overline{j}}\,;M'_{\overline{j}}\otimes \mathrm{St}_j)^{\mathfrak{d}_{\overline{j}}=0}$$

of locally analytic G-representations. Here, the unipotent radical of $P_{\overline{j}}$ acts trivially on $M'_{\overline{j}}\otimes \mathrm{St}_j$. The group $L(\overline{j})$ acts in the obvious way. The action of $L'(\overline{j})$ is given by the inverse of the determinant character and on St_j by the Steinberg representation.

We return to our computation. First of all, the cohomology of the structure sheaf $\mathcal{O}$ is given by

$$H^*(\mathbb{P}^d_K,\mathcal{O}) = H^0(\mathbb{P}^d_K,\mathcal{O}) = K.$$

So, in this case all the contributions $v^G_{P_{(d+j+1,1,\ldots,1)}}(H^{-j}(\mathbb{P}^d_K,\mathcal{O})),\ j=-1,\ldots,-d$, in Theorem 2.2.10 vanish. Moreover, we have $\mathcal{O}(\mathcal{X})_0 = K = W^0$. It remains to compute the locally analytic part of our formula. The structure sheaf corresponds to the weight $\lambda=(0,\ldots,0)\in\mathbb{Z}^{d+1}$. We get[8]

$$w_j * \lambda = (\underbrace{-1,\ldots,-1}_{j} \mid j,0,\ldots,0)$$

and

$$\mu_{j,\lambda} = w_j * \lambda \ \text{ for all } j=1,\ldots,d.$$

Further we compute

$$\Phi_{j,\lambda} = \bigcup_{k=0}^{j} \{(\underbrace{-1,\ldots,-1}_{j-1}, -1-k \mid j,k,0\ldots,0)\}$$

[8]Instead of a further comma, we use the symbol $\mid$ for a better distinction of the individual vectors.

resp.

$$\Psi_{j,\lambda} = \bigcup_{k=0}^{j} \{(j,k,0\ldots,0 \mid \underbrace{-1,\ldots,-1}_{j-1}, -1-k)\}.$$

By Corollary 1.4.9 we deduce that the $U(\mathfrak{g})$-module $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{O}) = H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{O})$ is generated by a quotient of the $\mathbf{P_{(j+1,d-j)}}$-representation $N_{d-j,\lambda} = \bigoplus_{\mu\in\Psi_{d-j,\lambda}} V_\mu$, where V_μ is the irreducible algebraic $\mathbf{L_{(j+1,d-j)}}$-representations with highest weight μ. Actually, by the following proposition and the $U(\mathfrak{g})$-structure with respect to $\mathcal{O}$, cf. (1.5), it turns out that the representation to the one with highest weight

$$z_{d-j}^{-1}\cdot\mu_{d-j,\lambda} = (d-j,0,\ldots,0 \mid \underbrace{-1,\ldots,-1)}_{d-j}$$

generates $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{O})$ as $U(\mathfrak{g})$-module.

Proposition 3.1.1. *For $0\leq j\leq d-1$, we have*

$$H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K,\mathcal{O}) = \bigoplus_{\substack{k_0,\ldots,k_j\geq 0\\ k_{j+1},\ldots,k_d<0\\ k_0+\ldots+k_d=0}} K\cdot X_0^{k_0}X_1^{k_1}\cdots X_d^{k_d}$$

Proof. The proof follows easily from the formula (1.7). □

Hence we can choose N_j to be the $\mathbf{P_{(j+1,d-j)}}$-module isomorphic to the outer tensor product

$$\mathrm{Sym}^{d-j}(K^{j+1})\boxtimes \det{}^{-1}.$$

By the proposition above, we may identify N_j with the K-vector space generated by the elements

$$\frac{P}{X_{j+1}\cdots X_d} \in \bigoplus_{\substack{k_0,\ldots,k_j\geq 0\\ k_{j+1},\ldots,k_d<0\\ k_0+\ldots+k_d=0}} K\cdot X_0^{k_0}X_1^{k_1}\cdots X_d^{k_d},$$

where P is homogeneous polynomial in $X_0,\ldots,X_j$ of degree $d-j$. These are precisely the elements

$$\Xi_\mu\in\mathcal{O}(\mathcal{X}) \text{ where } \mu\in A(\overline{d-j}).$$

In particular, we have

$$N_j = M_{\overline{d-j}}$$

as subspaces of $\mathcal{O}_{\inf}(\mathcal{X})$. Further, the set of differential equations $\mathfrak{d}_j=\mathfrak{d}_{\overline{d-j}}$ coincide. By Theorem 2.2.10 we have a filtration $W^{-d}\supset W^{-d+1}\supset\cdots\supset W^{-1}\supset W^0$ on $H^0(\mathcal{X},\mathcal{O})$ with

$$(W^j/W^{j+1})'\cong C^{an}(G,P_{(d+j+1,j)};N'_{d+j}\otimes \mathrm{St}_{-j})^{\mathfrak{d}_{d+j}},\ j=-1,\ldots,-d.$$

Hence we have shown that the graded pieces of our filtration coincides with that in [P]. Moreover, by definition of the canonical filtration on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{O})$ induced by the spectral sequence (2.7) (cf. Lemma 2.2.7), we see that both filtrations are the same. In fact, this follows from Proposition 3.1.1 and (3.1).

3.2. $\mathcal{F} = \Omega^d_{\mathbb{P}^d_K}$. In this chapter we compare our result in the case $\mathcal{F} = \Omega^d = \Omega^d_{\mathbb{P}^d_K}$ to that of Schneider and Teitelbaum. In [ST1] the authors define a G-equivariant decreasing filtration by closed K-Fréchet spaces

$$\Omega^d(\mathcal{X})^0 \supset \Omega^d(\mathcal{X})^1 \supset \cdots \supset \Omega^d(\mathcal{X})^{d-1} \supset \Omega^d(\mathcal{X})^d \supset \Omega^d(\mathcal{X})^{d+1} = 0$$

on $\Omega^d(\mathcal{X})^0 = H^0(\mathcal{X}, \Omega^d)$. As in [P] this construction involves an index function $\imath$ on the algebraic differential forms $\Omega^d_{\mathrm{alg}}(\mathcal{X})$, which counts the negative prime divisors without multiplicities of a given differential form. For any integer $j \in \mathbb{N}$, the filtration step $\Omega^d(\mathcal{X})^j$ is defined by the topological closure in $H^0(\mathcal{X}, \Omega^d)$ of its algebraic differential forms

$$\Omega^d_{\mathrm{alg}}(\mathcal{X})^j := \Big\{\eta \in \Omega^d_{\mathrm{alg}}(\mathcal{X}) \mid \imath(\eta) \le d+1-j\Big\}.$$

Furthermore, they construct explicit isomorphisms (*boundary value maps*)

$$(\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1})' \xrightarrow{\sim} C^{an}(G, P_{\underline{j}}; M'_{\underline{j}} \otimes \mathrm{St}_{d+1-j})^{\mathfrak{d}_{\underline{j}}=0}$$

of locally analytic G-representations. Here, $P_{\underline{j}}{=}P_{(j,d+1-j)} \subset G$ is the (lower) standard-parabolic subgroup to the decomposition $(j, d+1-j)$ of $d+1$ and

$$\underline{j} := \{0, \ldots, j-1\}.$$

The K-vector space $M_{\underline{j}} \subset U(\mathfrak{g})$ is given by the sum

$$M_{\underline{j}} = \sum_{\mu \in B(\underline{j})} K \cdot L_\mu,$$

where

$$B(\underline{j}) = \Big\{\mu = \sum\nolimits_k m_k \epsilon_k \in X^*(\overline{T}) \ \mid \ m_k = 1 \text{ for } k \in \underline{j},\ m_k \le 0 \text{ for } k \notin \underline{j}\Big\}.$$

Further, the symbol L_μ denotes a (sorted) element of $U(\mathfrak{g})$, cf. [ST1] p. 31. It is of weight μ and satisfies $L_\mu \cdot \xi = -\Xi_\mu \cdot \xi$, where

$$\xi = \frac{X_0^d}{X_1 \cdots X_d} \cdot d(\frac{X_1}{X_0}) \wedge d(\frac{X_2}{X_0}) \wedge \cdots \wedge d(\frac{X_d}{X_0}).$$

The unipotent radical of $P_{\underline{j}}$ acts trivially on the tensor product $M'_{\underline{j}} \otimes \mathrm{St}_{d+1-j}$. The second factor of the Levi subgroup $L_{(j,d+1-j)} = \mathrm{GL}_j \times \mathrm{GL}_{d+1-j}$ acts on $M'_{\underline{j}}$ via the symmetric power $\mathrm{Sym}^j(K^{d+1-j})$. On St_{d+1-j} it acts via the Steinberg representation.

The action of GL_j is given by the inverse of the determinant character. In particular the case $j=0$ yields the Steinberg representation St_{d+1}.

We turn to our computation. We have

$$H^*(\mathbb{P}^d_K, \Omega^d) = H^d(\mathbb{P}^d_K, \Omega^d) = K \cdot \xi.$$

Consequently, all the contributions $v^G_{P_{(d+j+1,1,\ldots,1)}}(H^{-j}(\mathbb{P}^d_K, \Omega^d))$, $j = -1, \ldots, -d$, of Theorem 2.2.10 vanish except for $j = -d$. In the latter case we obtain the Steinberg representation $v^G_{P_{(1,1,\ldots,1)}}(K) = \mathrm{St}_{d+1}$. The canonical bundle corresponds to the homogeneous vector bundle $\mathcal{F}_\lambda$ of weight $\lambda = (-d, 1, \ldots, 1) \in \mathbb{Z}^{d+1}$, cf. 1.4.1. We have

$$w_j * \lambda = (0, \ldots, 0, -d+j \mid \underbrace{1, \ldots, 1}_{d-j}) \quad \text{for } j = 1 \ldots, d.$$

It follows that

$$\mu_{j,\lambda} = w_{j-1} * \lambda \quad \text{for } j = 1, \ldots, d$$

and

$$\Phi_{j,\lambda} = \{\mu_{j,\lambda}\}.$$

We deduce that $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^d)$ is generated as $U(\mathfrak{g})$-module by the $\mathbf{P_{(j+1,d-j)}}$-representation $N_j = N_{d-j,\lambda}$ corresponding to the irreducible $\mathbf{L_{(j+1,d-j)}}$-representation with highest weight

$$z_{d-j}^{-1} \cdot \mu_{d-j,\lambda} = (\underbrace{1, \ldots, 1}_{j+1} \mid 0, \ldots, 0, -j-1).$$

It follows that the $\mathbf{P_{(j+1,d-j)}}$-module N_j is isomorphic to

$$\det \ \boxtimes \ \mathrm{Sym}^{j+1}(K^{d-j})'.$$

Again, we want to realize the corresponding representation concretely.

Proposition 3.2.1. *For $j \geq 0$, we have*

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{O}(-d-1)) = \bigoplus_{\substack{k_0,\ldots,k_j \geq 0 \\ k_{j+1},\ldots,k_d < 0 \\ k_0+\ldots+k_d = -d-1}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}.$$

Proof. The proof is similar to that of Proposition 3.1.1 □

By identifying Ω^d with the twisted sheaf $\mathcal{O}(-d-1)$, the element ξ corresponds to the fraction $\frac{1}{X_0 \cdots X_d}$. It follows that N_j is the K-vector space generated by the elements

$$\frac{X_0 \cdots X_j}{X_{j+1}^{m_{j+1}} \cdots X_d^{m_d}} \cdot \xi,$$

with $m_{j+1}+\cdots+m_d=j+1$. The fractions $\frac{X_0\cdots X_j}{X_{j+1}^{m_{j+1}}\cdots X_d^{m_d}}$ are exactly the elements Ξ_μ, with $\mu\in B(\underline{j+1})$, cf. [ST1], p. 65. We get for $j=0,\ldots,d-1$, isomorphisms

$$M_{\underline{j+1}} \xrightarrow{\sim} N_j$$
$$L_\mu \mapsto L_\mu\cdot\xi = -\Xi_\mu\cdot\xi.$$

Moreover, the set of differential equations $\mathfrak{d}_j$ and $\mathfrak{d}_{\underline{j+1}}$ are the same. By Theorem 2.2.10 we have a filtration $W^{-d}\supset W^{-d+1}\supset\cdots\supset W^{-1}\supset W^0$ on $H^0(\mathcal{X},\Omega^d)$ where $(W^{-d}/W^{-d+1})'$ is an extension

(3.4) $0\to v^G_{P_{(1,1,\ldots,1)}}(K)\to (W^{-d}/W^{-d+1})'\to C^{an}(G,P_{(1,d)};N_0'\otimes \mathrm{St}_d)^{\mathfrak{d}_0}\to 0$

and

$$(W^j/W^{j+1})'\cong C^{an}(G,P_{(d+1+j,-j)};N'_{d+j}\otimes\mathrm{St}_{-j})^{\mathfrak{d}_{d+j}}.$$

Thus we see that the graded pieces of our filtration coincide with that of [ST1]. Moreover, by looking at the pole order of sections in $\Omega^d(\mathcal{X})$ as in the case of the structure sheaf we see that the filtrations are the same apart from the first filtration step. The difference is just given by the extension above. In other words, we have an extension

$$0\to\Omega^d(\mathcal{X})^1/\Omega^d(\mathcal{X})^2\to W^{-d}/W^{-d+1}\to\Omega^d(\mathcal{X})^0/\Omega^d(\mathcal{X})^1\to 0,$$

such that its dual coincides with (3.4).

3.3. $\mathcal{F}=\Omega^1_{\mathbb{P}^d_K}$. This chapter provides another example for our computation. It treats the cotangent bundle $\mathcal{F}=\Omega^1=\Omega^1_{\mathbb{P}^d_K}$ on $\mathbb{P}^d_K$.

We have

$$H^*(\mathbb{P}^d_K,\Omega^1)=H^1(\mathbb{P}^d_K,\Omega^1)=K.$$

Therefore, all the contributions $v^G_{P_{(d+j+1,1-j)}}(H^{-j}(\mathbb{P}^d_K,\Omega^d))$ in Theorem 2.2.10 vanish except for $j=-1$. In the latter case we obtain the generalized Steinberg representation $v^G_{P_{(d,1)}}(K)$. The cotangent bundle corresponds to the homogeneous vector bundle $\mathcal{F}_\lambda$ given by the weight $\lambda=(-1,1,0,\ldots,0)\in\mathbb{Z}^{d+1}$. By Theorem 2.2.10 we have a filtration $W^{-d}\supset W^{-d+1}\supset\cdots\supset W^{-1}\supset W^0$ on $H^0(\mathcal{X},\Omega^1)$ where $(W^{-1}/W^0)'$ is an extension

$$0\to v^G_{P_{(d,1)}}(K)\to(W^{-1}/W^0)'\to C^{an}(G,P_{(d,1)};N'_{d-1}\otimes\mathrm{St}_1)^{\mathfrak{d}_{d-1}}$$

and

$$(W^j/W^{j+1})'\cong C^{an}(G,P_{(d+1+j,-j)};N'_{d+j}\otimes\mathrm{St}_{-j})^{\mathfrak{d}_{d+j}}$$

for $j \neq -1$. A computation shows that

$$\mu_{j,\lambda} := \begin{cases} \lambda & : j = 1 \\ (0, \underbrace{-1, \ldots, -1}_{j-1} \mid j-1, 0 \ldots, 0) & : j > 1. \end{cases}$$

Further, for $j > 1$

$$\Phi_{j,\lambda} = \{\mu_{j,\lambda}\} \cup \bigcup_{k=1}^{j-1} \{(l, -1, \ldots, -1, -1-l-k \mid j-1, k, 0 \ldots, 0) \mid l = 0, -1\}$$

resp.

$$\Psi_{j,\lambda} = \{z_i^{-1} \cdot \mu_{j,\lambda}\} \cup \bigcup_{k=1}^{j-1} \{(j-1, k, 0 \ldots, 0 \mid l, -1, \ldots, -1, -1-l-k) \mid l = 0, -1\}.$$

In the case $j = 1$, we compute

$$\Phi_{1,\lambda} = \{\mu_{1,\lambda}, (-2 \mid 1, 1, 0 \ldots, 0)\}$$

resp.

$$\Psi_{1,\lambda} = \{z_i^{-1} \cdot \mu_{1,\lambda}, (1, 1, 0 \ldots, 0 \mid -2)\}$$

We will see that for $n > 1$ the weights with $k \leq 1$ yield a generating system of $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^1)$.

We deduce that $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^1)$ contains the irreducible algebraic $\mathbf{L}_{(\mathbf{j+1},\mathbf{d-j})}$-representation V_μ with highest weight

$$\mu = z_{d-j}^{-1} \cdot \mu_{d-j,\lambda} := \begin{cases} (1, 0, \ldots, 0 \mid -1) & : j = d-1 \\ (d-j-1, 0 \ldots, 0 \mid 0, \underbrace{-1, \ldots, -1}_{d-j-1}) & : j < d-1. \end{cases}$$

It follows that $V_{z_{d-j}^{-1} \cdot \mu_{d-j,\lambda}}$ is isomorphic to

$$\mathrm{Sym}^{d-j-1}(K^{j+1}) \boxtimes (K^{d-j} \otimes \det{}^{-1}) \text{ for } j < d-1.$$

For $j = d-1$, we get $V_{z_1^{-1} \cdot \mu_{1,\lambda}} \cong K^d \boxtimes \det{}^{-1}$. We shall give an explicit realization of $V_{z_{d-j}^{-1} \cdot \mu_{d-j,\lambda}}$.

Let V_j be the finite-dimensional K-vector space generated by the elements

$$\frac{P \cdot X_k^2}{X_{j+1} \cdots X_d} \cdot d(\frac{X_l}{X_k}),\ k \in \{0, \ldots, j\}, l \in \{j+1, \ldots, d\},$$

where P is a homogeneous polynomial of degree $d-j-2$ in the indeterminates $X_0,\ldots,X_j$. In the case $j=d-1$, let V_{d-1} be generated by the elements $d(\frac{X_k}{X_d})$, $k\in\{0,\ldots,d-1\}$. Consider the K-linear map

$$\begin{aligned}\mathrm{Sym}^{d-j-2}(K^{j+1})\otimes K^{j+1}\otimes K^{d-j} &\longrightarrow V_j\\ P\otimes e_k\otimes e_l &\mapsto \frac{P\cdot X_k^2}{X_{j+1}\cdots X_d}\cdot d(\frac{X_l}{X_k}).\end{aligned}$$

This map is clearly a surjective linear map of K-vector spaces. Consider the following action of $\mathbf{L_{(j+1,d-j)}}$ on the LHS. On K^{j+1} the action of $\mathbf{L(j+1)}$ is the standard representation. On $\mathrm{Sym}^{d-j+2}(K^{j+1})$ it acts via the $(d-j+2)$-th symmetric power. On K^{d-j} the operation of $\mathbf{L(j+1)}$ is the trivial one. The action of $\mathbf{L(d-j)}$ on K^{d-j} is the standard representation. On the other factor it operates via the inverse of the determinant character. From the identity

$$\frac{P\cdot X_k^2}{X_{j+1}\cdots X_d}\cdot d(\frac{X_l}{X_k})=-\frac{P\cdot X_l^2}{X_{j+1}\cdots X_d}\cdot d(\frac{X_k}{X_l}),\ k\in\{0,\ldots,j\},l\in\{j+1,\ldots,d\},$$

it follows that V_j is a finite-dimensional $\mathbf{L_{(j+1,d-j)}}$ -module and that the map above is a surjection of $\mathbf{L_{(j+1,d-j)}}$-representations. Inside the representation $\mathrm{Sym}^{d-j-2}(K^{j+1})\otimes K^{j+1}$ we have the irreducible $\mathbf{L(j+1)}$-subrepresentation $\mathrm{Sym}^{d-j-1}(K^{j+1})$, which corresponds to the highest weight $(d-j-1,0,\ldots,0)$ of $\mathbf{L(j+1)}$. A computation shows that the above map restricts to an isomorphism

$$\mathrm{Sym}^{d-j-1}(K^{j+1})\boxtimes(K^{d-j}\otimes\det{}^{-1})\xrightarrow{\sim}V_j.$$

For $j<d-2$, the representation $V_j=V_{z_{d-j}^{-1}\cdot\mu_{d-j,\lambda}}$ is a quotient of some representation containing the representation $\mathrm{Sym}^{d-j-2}(K^{j+1})$. We have realized the latter one as a subrepresentation of $K[X_0,\ldots,X_j]$. Thus $K[X_0,\ldots,X_j]\cdot(\mathrm{Sym}^{d-j-2}(K^{j+1})\otimes K^{j+1}\otimes K^{d-j})\subset K[X_0,\ldots,X_j]\cdot(K^{j+1}\otimes K^{d-j})$. By the discussion in 1.4 it suffices to consider in Lemma 1.4.6 the representations $K^{j+1}\otimes K^{d-j}$ instead of $\mathrm{Sym}^{d-j-2}(K^{j+1})\otimes K^{j+1}\otimes K^{d-j}$. Since the highest weight of $K^{j+1}\otimes K^{d-j}$ is $(1,0\ldots,0\mid 0,-1,\ldots,-1)$ we deduce that the weights in $z_{d-j}^{-1}\Phi_{d-j,\lambda}$ with $k\leq 1$ yield a generating system.

As for the other irreducible subrepresentations, we note that in the case $j=d-1$ the irreducible $\mathbf{P}_{(d,1)}$-representation corresponding to the weight $(1,1,0,\ldots,0\mid -2)$ is generated by the expressions

$$\frac{X_i}{X_d}d(\frac{X_j}{X_d})-\frac{X_j}{X_d}d(\frac{X_i}{X_d}).$$

For $j < d-1$, we write down a highest weight vector for the remaining weights. In the case of $(d-j-1, 1, 0\ldots, 0 \mid -1, -1, \ldots, -1, -1)$ it is given by

$$\frac{X_0^{d-j-1}}{X_{j+2}\cdots X_d} d(\frac{X_1}{X_{j+1}}) - \frac{X_0^{d-j-2}X_1}{X_{j+2}\cdots X_d} d(\frac{X_0}{X_{j+1}}).$$

In the case of $(d-j-1, 1, 0\ldots, 0 \mid 0, -1, \ldots, -1, -2)$ it is given by

$$\frac{X_0^{d-j-2}X_{j+1}^2}{X_{j+1}\cdots X_d}\Big(\frac{X_0}{X_d} d(\frac{X_1}{X_{j+1}}) - \frac{X_1}{X_d} d(\frac{X_0}{X_{j+1}})\Big).$$

4. Appendix

In this final part of our paper we present another approach for the main computation. We replace the acyclic complex of Theorem 2.1.1 by a similar complex avoiding adic spaces. It is based purely on rigid analytic varieties. The construction is similar to that of [SS] in the case of constant coefficients. In that case, the main difference is essentially that in loc.cit. the authors regard intersections of hyperplanes, whereas we deal with arbitrary subspaces directly. Our approach follows the construction of [O3]. For this purpose, we have further to investigate the neighborhoods of the closed subvarieties $Y_U = \mathbb{P}(U) \subset \mathbb{P}^d_K$, where $U \subset K^{d+1}$ is a linear subspace.

Let

$$\Lambda = \bigoplus_{i=0}^{d} O_K \cdot e_i$$

be the O_K-lattice of V generated by our fixed basis $\{e_0, \ldots, e_d\}$. Let O be the ring of integers in $\mathbb{C}_p$. For $n \in \mathbb{N}$, we put

$$O_K^{(n)} := O_K/\pi^n O_K,\ O^{(n)} := O/\pi^n O,\ \Lambda^{(n)} := \Lambda/\pi^n\Lambda.$$

Consider for $n \in \mathbb{N}$, the mod-n reduction map

$$\mathrm{red}_n : \mathbb{P}^d_K(\mathbb{C}_p) \longrightarrow \mathbb{P}(\Lambda)(O^{(n)}).$$

If $L_x \subset \mathbb{C}_p^{d+1}$ is a line representing a point $x \in \mathbb{P}^d_K(\mathbb{C}_p)$, then

$$\mathrm{red}_n(x) = (L_x \cap (\Lambda \otimes_{O_K} O)) \otimes O^{(n)}.$$

Let $I_U \subset K[T_0, \ldots, T_d]$ be the vanishing ideal of the linear subvariety Y_U. The schematic closure of Y_U in $\mathbb{P}^d_{O_K}$ is defined by the the ideal

$$\tilde{I}_U = I_U \cap O_K[T_0, \ldots, T_d].$$

Consider the ϵ_n-neighborhood $Y_U(\epsilon_n)$ for a given non-trivial K-subspace $U \subsetneq V$. Let $f \in \tilde{I}_U$ be a polynomial, such that at least one coefficient is a unit. Then we have for $x \in \mathbb{P}^d_K(\mathbb{C}_p)$ (take any unimodular representative of x),

$$\begin{aligned} x \in Y_U(\epsilon_n)(\mathbb{C}_p) &\Leftrightarrow |f(x)| \leq \epsilon_n \\ &\Leftrightarrow \pi^n \mid f(x) \\ &\Leftrightarrow f(x) = 0 \ (\mathrm{mod}\ \pi^n). \end{aligned}$$

So $Y_U(\epsilon_n)$ is nothing else but

$$Y_U(\epsilon_n) = \Big\{ x \in (\mathbb{P}^d_K)^{rig} \mid \mathrm{red}_n(x) \in \tilde{Y}_U(O^{(n)}) \Big\},$$

where $\tilde{Y}_U$ is the closed subscheme of $\mathbb{P}^d_{O_K}$ defined by $\tilde{I}_U$. Moreover, we see that

$$Y_U(\epsilon_n) = Y_{U'}(\epsilon_n) \tag{4.1}$$

if $U \cap \Lambda \equiv U' \cap \Lambda \bmod \pi^n$, compare also Lemma 2, ch. 1 in [SS].

Let

$$U_\bullet = \big((0) \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_s \subsetneq \Lambda^{(n)}\big)$$

be a filtration of free $O_K^{(n)}$-modules. Choose a lift of $U_\bullet$ to a K-filtration

$$\tilde{U}_\bullet = \big((0) \subsetneq \tilde{U}_1 \subsetneq \tilde{U}_2 \subsetneq \cdots \subsetneq \tilde{U}_s \subsetneq V\big)$$

of K-subspaces of V. We set

$$Y_{U_\bullet} := Y_{\tilde{U}_1}(\epsilon_n).$$

This definition depends by (4.1) only on the $O_K^{(n)}$-module U_1. Hence, the set $Y_{U_\bullet}$ is a well-defined rigid analytic open subvariety of $(\mathbb{P}^d_K)^{rig}$. Thus we have associated to each filtration $U_\bullet$ of $\Lambda^{(n)}$ by free $O_K^{(n)}$-submodules a quasi-compact rigid analytic subset $Y_{U_\bullet} \subset (\mathbb{P}^d_K)^{rig}$. For $0 \leq i \leq d$, let

$$\Lambda_i^{(n)} = \bigoplus_{j=0}^{i} O_K^{(n)} \cdot e_j \subset \Lambda^{(n)}$$

be the free $O_K^{(n)}$-submodule generated by our first $i+1$ basis vectors. For any subset $I = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Delta$, we put

$$\Lambda_I^{(n)} = \big((0) \subset \Lambda_{i_1}^{(n)} \subset \Lambda_{i_2}^{(n)} \subset \cdots \subset \Lambda_{i_r}^{(n)} \subset \Lambda^{(n)}\big).$$

Then we get $Y_{\Lambda_I^{(n)}} = Y_I(\epsilon_n)$. Thus, analogously to (2.3), we can construct for every étale (resp. Zariski) sheaf $\mathcal{G}$ on $\mathcal{Y}_n$ the following complex of étale (resp. Zariski) sheaves on $\mathcal{Y}_n$:

$$\begin{aligned} 0 \to \mathcal{G} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=1}} \bigoplus_{g\in G_0/P_I^n} (\phi^n_{g,I})_*(\phi^n_{g,I})^*\mathcal{G} \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=2}} \bigoplus_{g\in G_0/P_I^n} (\phi^n_{g,I})_*(\phi^n_{g,I})^*\mathcal{G} \to \\ \cdots \to \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=d-1}} \bigoplus_{g\in G_0/P_I^n} (\phi^n_{g,I})_*(\phi^n_{g,I})^*\mathcal{G} \to \bigoplus_{g\in G_0/P_\emptyset^n} (\phi^n_{g,\emptyset})_*(\phi^n_{g,\emptyset})^*\mathcal{G} \to 0, \end{aligned} \tag{4.2}$$

where $\phi^n_{g,I}$ denotes the open embedding $Y_{g\Lambda_I^{(n)}} \hookrightarrow \mathcal{Y}_n$ of rigid analytic varieties. In contrast to (2.3), this complex is not acyclic as the following example shows. This was pointed out to me by P. Schneider some years ago.

Example 4.1.1. Let $d = 2$. Then the above complex is nothing else but

$$0 \to \mathcal{G} \to \bigoplus_{g \in G_0/P^n_{\{\alpha_1\}}} (\phi^n_{g,\{\alpha_1\}})_*(\phi^n_{g,\{\alpha_1\}})^*\mathcal{G} \ \oplus \bigoplus_{g \in G_0/P^n_{\{\alpha_2\}}} (\phi^n_{g,\{\alpha_2\}})_*(\phi^n_{g,\{\alpha_2\}})^*\mathcal{G}$$

$$\to \bigoplus_{g \in G_0/P^n_{\emptyset}} (\phi^n_{g,\emptyset})_*(\phi^n_{g,\emptyset})^*\mathcal{G} \to 0.$$

We have

$$\begin{aligned} Y_{\{\alpha_1\}}(\epsilon_n)(\mathbb{C}_p) &= \Big\{x \in \mathbb{P}^d_K(\mathbb{C}_p) \mid \mathrm{red}_n(x) = \Lambda^{(n)}_1 \otimes_{O^{(n)}_K} O^{(n)}\Big\} \\ Y_{\{\alpha_2\}}(\epsilon_n)(\mathbb{C}_p) &= \Big\{x \in \mathbb{P}^d_K(\mathbb{C}_p) \mid \mathrm{red}_n(x) \subset \Lambda^{(n)}_2 \otimes_{O^{(n)}_K} O^{(n)}\Big\} \\ Y_{\emptyset}(\epsilon_n)(\mathbb{C}_p) &= Y_{\{\alpha_1\}}(\epsilon_n)(\mathbb{C}_p). \end{aligned}$$

Let $x \in \mathbb{P}^2_K(\mathbb{C}_p)$ be a point such that the corresponding line $L_x \subset \mathbb{C}^3_p$ has the shape $L_x = \mathbb{C}_p \cdot (\pi^{n-1}e_0 + ae_2),\ a \in O^\times$, with $[a] \in (O^{(n)})^\times \setminus (O^{(n)}_K)^\times$. Consider the planes

$$E = O^{(n)}_K \cdot (e_0 + \pi e_1) \oplus O^{(n)}_K \cdot e_2 \text{ and } E' = O^{(n)}_K \cdot e_0 \oplus O^{(n)}_K \cdot e_2$$

in $\Lambda^{(n)}$. Then

$$\mathrm{red}_n(x) \in E \otimes_{O^{(n)}_K} O^{(n)} \ \cap\ E' \otimes_{O^{(n)}_K} O^{(n)},$$

but $\mathrm{red}_n(x) \neq L \otimes_{O^{(n)}_K} O^{(n)}$ for all $L \in \mathbb{P}(\Lambda)(O^{(n)}_K)$. So, localizing the above complex in x yields a sequence

$$0 \to \mathcal{G}_x \to \mathcal{G}^r_x \to 0$$

with $r = \#\{E \subset \Lambda^{(n)} \mid E \text{ is free with } \mathrm{rk}(E) = 2, \mathrm{red}_n(x) \subset E \otimes_{O^{(n)}_K} O^{(n)}\} \geq 2$. Hence, the complex (4.2) cannot be acyclic in general. $\square$

The above example suggests to fill the gaps in (4.2). Let U be a $O^{(n)}_K$-submodule of $\Lambda^{(n)}$, not necessarily free. We define

$$Y_U := \Big\{x \in (\mathbb{P}^d_K)^{rig} \mid \mathrm{red}_n(x) \in U \otimes_{O^{(n)}_K} O^{(n)}\Big\}.$$

If U is a free $O^{(n)}_K$-module then this definition coincides with the previous one. Put

$$\mathrm{rk}(U) = \dim_{O_K/\pi O_K}(U + \pi\Lambda^{(n)}/\pi\Lambda^{(n)}).$$

This is just the rank of the torsion free submodule of U. We also have the ordinary rank

$$\mathrm{rk}'(U) := \min\{n \in \mathbb{N} \mid \text{ there are } m_1, \ldots, m_n \in U \text{ which generate } U\}.$$

We have $\mathrm{rk}(U) \leq \mathrm{rk}'(U)$ where the equality holds if and only if U is free ($\Leftrightarrow$ torsion free).

Proposition 4.1.2. *The set Y_U is a quasi-compact rigid analytic open subset of $(\mathbb{P}^d_K)^{rig}$.*

Proof. Compare also Lemma 5 in [SS]. Without loss of generality, we may suppose that U is generated by

$$e_0, e_1, \ldots, e_i, \pi^{n_{i+1}} e_{i+1}, \ldots, \pi^{n_j} e_j$$

for certain integers $0 < n_k < n$, $k = i+1, \ldots, j$. Then Y_U is just the quasi-compact subset

$$\Big\{x = [x_0 : \cdots : x_d] \in (\mathbb{P}^d_K)^{rig} \mid |x_k| \leq |\pi^n|, k = 1, \ldots i,\ |x_k| \leq |\pi^{n-n_k}|,\ k = i{+}1, \ldots, j\Big\}$$

of $(\mathbb{P}^d_K)^{rig}$. $\square$

Let $U_\bullet = \big((0) \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_s \subsetneq \Lambda^{(n)}\big)$ be a filtration of $\Lambda^{(n)}$ by $O_K^{(n)}$-submodules. As in the case of filtrations consisting of free $O_K^{(n)}$-submodules, we put

$$Y_{U_\bullet} = Y_{U_1}.$$

Note that $Y_{U_\bullet} = \emptyset$ is the empty set unless $\mathrm{rk}(U) \geq 1$. With these newly-created rigid analytic subsets of $(\mathbb{P}^d_K)^{rig}$ we modify the complex (4.2) as follows:

$$\begin{aligned} 0 \ \to\ \mathcal{G} \to \bigoplus_{\substack{U_\bullet = (0) \subsetneq U \subsetneq \Lambda^{(n)} \\ \mathrm{rk}'(U) \leq d \\ \mathrm{rk}(U) \geq 1}} (\phi^n_{U_\bullet})_* (\phi^n_{U_\bullet})^* \mathcal{G} \to \bigoplus_{\substack{U_\bullet = (0) \subsetneq U_1 \subsetneq U_2 \subsetneq \Lambda^{(n)} \\ \mathrm{rk}'(U_2) \leq d \\ \mathrm{rk}(U_1) \geq 1}} (\phi^n_{U_\bullet})_* (\phi^n_{U_\bullet})^* \mathcal{G} \to \\ \ldots \ \to \bigoplus_{\substack{U_\bullet = (0) \subsetneq U_1 \subsetneq \ldots \subsetneq U_p \subsetneq \Lambda^{(n)} \\ \mathrm{rk}'(U_p) \leq d \\ \mathrm{rk}(U_1) \geq 1}} (\phi^n_{U_\bullet})_* (\phi^n_{U_\bullet})^* \mathcal{G} \to \ldots \to \bigoplus_{\substack{U_\bullet = (0) \subsetneq U_1 \subsetneq \ldots \subsetneq U_{n\cdot(d-1)} \subsetneq \Lambda^{(n)} \\ \mathrm{rk}'(U_{n\cdot(d-1)}) \leq d \\ \mathrm{rk}(U_1) \geq 1}} (\phi^n_{U_\bullet})_* (\phi^n_{U_\bullet})^* \mathcal{G} \to 0, \end{aligned} \tag{4.3}$$

where $\phi^n_{U_\bullet}$ denotes the open embedding $Y_{U_\bullet} \hookrightarrow \mathcal{Y}_n$. Why we impose on U the condition $\mathrm{rk}'(U) \leq d$ will become clear later on, cf. Prop. 4.1.5. It simply means that U is contained in a proper free submodule of $\Lambda^{(n)}$.

As for the following theorem, we refer to [JP] for the notion of an overconvergent sheaf on a rigid analytic variety. The crucial property of such a sheaf is that it vanishes if and only if all its stalks vanish.

Theorem 4.1.3. *Let $\mathcal{G}$ be an overconvergent étale sheaf on $\mathcal{Y}_n$. Then the complex (4.3) is acyclic.*

The proof of this theorem is similar to Satz 5.3 in [O1]. The main difference is that we are now dealing with modules instead of vector spaces. For proving Theorem 4.1.3, we have to introduce some more notation.

Let $X = (X, \prec)$ be a partially ordered set (poset). We associate to X a simplicial complex

$$X^\bullet = \bigcup_{n \in \mathbb{N}} X^n,$$

where a n-simplex $\tau \in X^n$ is given by a $n+1$-tuple

$$\tau = (x_0 \prec x_1 \prec \cdots \prec x_n)$$

with elements $x_i \in X,\ i = 0, \ldots, n$. In particular, for the 0-simplices X^0 of $X^\bullet$ we have $X^0 = X$.

A morphism $f : X \longrightarrow Y$ of posets is a map which preserves the order. Such a morphism induces a simplicial map

$$f^\bullet : X^\bullet \longrightarrow Y^\bullet$$

of simplicial complexes. Thus we obtain a functor from the category of posets to the category of simplicial complexes.

Example 4.1.4. *a)* Let $R = O_K^{(n)}, O_K, K$ resp. $M = \Lambda^{(n)}, \Lambda, V$. Put

$$T_R(M) := \Big\{ R\text{-submodules of } M \mid \ \mathrm{rk}(U) \geq 1 \text{ and } \mathrm{rk}'(U) \leq d \Big\}.$$

We supply this set with the structure of a poset by considering the canonical order given by inclusion. Thus a n-simplex $\tau \in T_R(M)^n$ is a flag

$$\tau = \big((0) \subsetneq U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n \subsetneq M \big)$$

of submodules with $\mathrm{rk}(U_0) \geq 1$ and $\mathrm{rk}'(U_n) \leq d$.

b) In the situation above, let $T_R^f(M)$ be the subposet consisting of all non-trivial R-modules such that M/U is free. We get a morphism of posets $T_R^f(M) \hookrightarrow T_R(M)$.

c) Let $R = K$. Then $T_K(V)$ is nothing else but the Tits complex of $\mathrm{GL}(V)$, compare also [Q]. □

Proposition 4.1.5. *The morphism of posets*

$$\begin{aligned} \psi : T_K(V) &\longrightarrow T_{O_K}(\Lambda) \\ W &\longmapsto W \cap \Lambda \end{aligned}$$

induces a homotopy equivalence

$$T_{O_K}(\Lambda)^\bullet \simeq T_K(V)^\bullet$$

Proof. Using Proposition 1.6 of [Q] it is enough to show that for every proper submodule $U \in T_{O_K}(\Lambda)$ the subposet $\{W \in T_K(V) \mid U \subset W \cap \Lambda\}$ is contractible. Note that these subsets are non-empty since $\mathrm{rk}'(U) \leq d$. The contractibility follows easily from the next proposition. □

Proposition 4.1.6. *(*[Q]*, 1.5) Let X be a poset and let $x_0 \in X$ be a fixed element. Further, let $f : X \longrightarrow X$ be an endomorphism of posets with*

$$x \succeq f(x) \preceq x_0 \ \text{, for all } x \in X.$$

Then $X^\bullet$ is contractible.

Remark 4.1.7. It is easily seen that we may identify $T_K(V)$ with $T^f_{O_K}(\Lambda)$. Under this identification the morphism ψ corresponds to the inclusion $T^f_{O_K}(V) \hookrightarrow T_{O_K}(\Lambda)$. □

We continue with the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3: Since $\mathcal{G}$ is overconvergent and all the morphisms $\phi^n_{U_\bullet}$ are quasi-compact, we conclude (cf. [JP] 3.5) that all the appearing étale sheaves in the complex are overconvergent, as well. Thus it is enough to show the acyclicity of the localized complex with respect to any étale point of $\mathcal{Y}_n$, cf. loc.cit. 3.4. So let e be an étale point of $\mathcal{Y}_n$. By definition this is just a separable closure H_e of some valued field F_a depending on an analytic point a lying below e. Let F_e be the completion of H_e. The étale point e corresponds to a general morphism

$$\mathrm{Spm}(F_e) \longrightarrow \mathcal{Y}_n$$

of rigid analytic varieties, that is, to a morphism

$$\mathrm{Spm}(F_e) \longrightarrow \mathcal{Y}_n \hat{\otimes}_K F_e$$

of rigid analytic varieties, hence to a line $L_e \in \mathbb{P}^d_K(F_e)$. Localizing the complex (4.3) in e yields a chain complex with values in F_e. The pull back of the complex (4.3) to $\mathbb{P}^d_K \times_K \mathrm{Spm}(F_e)$ is just the complex, where all the appearing objects of (4.3) are defined with respect to the base field F_e. Localizing of this complex in e would give the same chain complex. Hence we can assume without loss of generality that $F_e = \mathbb{C}_p$. The chain complex is induced by a subcomplex $C^\bullet$ of $T^\bullet_{O_K^{(n)}}(\Lambda^{(n)})$, which is generated by its 0-dimensional simplices C^0, a subposet of $T_{O_K^{(n)}}(\Lambda^{(n)})$. Its simplices are given by

$$C^\bullet = \Big\{ U_\bullet \in T^\bullet_{O_K^{(n)}}(\Lambda^{(n)}) \mid \mathrm{red}_n(L_e) \in Y_{U_\bullet} \Big\}.$$

By the next lemma, we will see that $C^\bullet$ is contractible. Theorem 4.1.3 follows, since the étale point was arbitrary. $\square$

Lemma 4.1.8. *The simplicial complex $C^\bullet$ is contractible*

Proof. Put $\overline{T_{O_K^{(n)}}(\Lambda^{(n)})} = T_{O_K^{(n)}}(\Lambda^{(n)}) \cup \{0, \Lambda^{(n)}\}$ and supply this set with the canonical order. Thus we have realized $T_{O_K^{(n)}}(\Lambda^{(n)})$ as a subposet of $\overline{T_{O_K^{(n)}}(\Lambda^{(n)})}$. Let $U_0 \in T_{O_K^{(n)}}(\Lambda^{(n)})$ be a fixed element. Consider the map

$$\begin{array}{rcl} f : C^0 & \longrightarrow & \overline{T_{O_K^{(n)}}(\Lambda^{(n)})} \\ U & \longmapsto & U_0 \cap U \end{array}.$$

This map is a morphism of posets with

$$U \succeq f(U) \preceq U_0 \quad \text{for all } U \in C^0.$$

By Proposition 4.1.6, it suffices to show that the image of f is contained in C^0. But this follows from the inclusion

$$Y_U \cap Y_{U_0} \subset Y_{U \cap U_0},$$

which itsself follows from the flatness of $O_K^{(n)} \hookrightarrow O^{(n)}$. $\square$

Let $\mathcal{F}$ be our fixed homogeneous vector bundle on $\mathbb{P}^d_K$. Consider the projective limit of posets $\varprojlim_n T_{O_K^{(n)}}(\Lambda^{(n)})$ which identifies with $T_{O_K}(\Lambda)$. Let

$$(U^n_\bullet)_{n \in \mathbb{N}} \in \varprojlim_n T_{O_K^{(n)}}(\Lambda^{(n)})$$

be any element which corresponds to $U^\bullet \in T_{O_K}(\Lambda)$. By (the proof in) Proposition 4.1.2, we have inclusions $Y_{U^{n+1}_\bullet} \subset Y_{U^n_\bullet}$ and therefore homomorphisms

$$H^*_{Y_{U^{n+1}_\bullet}}(\mathbb{P}^d_K, \mathcal{F}) \to H^*_{Y_{U^n_\bullet}}(\mathbb{P}^d_K, \mathcal{F})$$

for all $n \in \mathbb{N}$. In particular, we get a projective system $(H^*_{Y_{U^n_\bullet}}(\mathbb{P}^d_K, \mathcal{F}))_{n \in \mathbb{N}}$ of K-vector spaces. As in Lemma 1.3.2 one verifies that this system consists of K-Fréchet spaces and the transition maps are dense. Let $U^f_\bullet$ the largest subobject in $U^\bullet$ such that $U^f_\bullet \in T^f_{O_K}(\Lambda) = T_K(V)$.

Proposition 4.1.9. *There is a topological isomorphism of K-Fréchet spaces*

$$\varprojlim_{n\in\mathbb{N}} H^*_{Y_{U^n_\bullet}}(\mathbb{P}^d_K,\mathcal{F}) \cong H^*_{Y_{U^f_\bullet\otimes K}}(\mathbb{P}^d_K,\mathcal{F}).$$

Proof. This follows from the description of the analytic varieties $Y_{U_\bullet}$ given in the proof of Proposition 4.1.2 together with Proposition 1.3.3 □

We consider the acyclic complex of Theorem 4.1.3 in the case $\mathcal{G}=\mathbb{Z}$. Then the resulting complex induces for each $n\in\mathbb{N}$, a spectral sequence

$$E_1^{-p,q,n} = \bigoplus_{U_\bullet\in T(\Lambda^{(n)})^p} H^q_{Y_{U_\bullet}}(\mathbb{P}^d_K,\mathcal{F}) \Longrightarrow H^{-p+q}_{\mathcal{Y}_n}(\mathbb{P}^d_K,\mathcal{F}).$$

This spectral sequence has a similar shape as the one in section 2.2. Its rows are given by

$$E_1^{\bullet,j,n}: \bigoplus_{\substack{U_\bullet\in T(\Lambda^{(n)})^{n\cdot j}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U_\bullet}}(\mathbb{P}^d_K,\mathcal{F}) \to \bigoplus_{\substack{U_\bullet\in T(\Lambda^{(n)})^{n\cdot j-1}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U_\bullet}}(\mathbb{P}^d_K,\mathcal{F})$$

$$\to \bigoplus_{\substack{U_\bullet\in T(\Lambda^{(n)})^{n\cdot j-2}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U_\bullet}}(\mathbb{P}^d_K,\mathcal{F}) \to \ldots \to \bigoplus_{\substack{U_\bullet\in T(\Lambda^{(n)})^{1}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U_\bullet}}(\mathbb{P}^d_K,\mathcal{F})$$

Passing to the limit and applying Proposition 4.1.9 resp. Proposition 1.3.3 to it yields a spectral sequence

$$E_1^{-p,q} \Longrightarrow H^{-p+q}_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F}).$$

Its rows are given by

$$E_1^{\bullet,j}: \varprojlim_{n\in\mathbb{N}} \bigoplus_{\substack{U_\bullet\in T(\Lambda)^{n\cdot j}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U^f_\bullet\otimes K}}(\mathbb{P}^d_K,\mathcal{F}) \to \varprojlim_{n\in\mathbb{N}} \bigoplus_{\substack{U_\bullet\in T(\Lambda)^{n\cdot j-1}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U^f_\bullet\otimes K}}(\mathbb{P}^d_K,\mathcal{F})$$

$$\to \varprojlim_{n\in\mathbb{N}} \bigoplus_{\substack{U_\bullet\in T(\Lambda)^{n\cdot j-2}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U^f_\bullet\otimes K}}(\mathbb{P}^d_K,\mathcal{F}) \to \ldots \to \varprojlim_{n\in\mathbb{N}} \bigoplus_{\substack{U_\bullet\in T(\Lambda)^{1}\\ \mathrm{rk}(U_1)=d-j}} H^j_{Y_{U^f_\bullet\otimes K}}(\mathbb{P}^d_K,\mathcal{F}).$$

Now we apply Proposition 4.1.5. We see that the spectral sequence (2.7) of section 2.2 is homotopy equivalent to $E_1^{\bullet,\bullet}$. Thus from now on, we may carry on with the computation there. □

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