THE PRO-ÉTALE COHOMOLOGY OF DRINFELD'S UPPER HALF SPACE

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Abstract. We determine the geometric pro-étale cohomology of Drinfeld's upper half space over a p-adic field K. The strategy is different from [CDN] and is based on the approach developed in [O] describing global sections of equivariant vector bundles.

Introduction

Let K be a finite extension of $\mathbb{Q}_p$ with absolute Galois group $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ and let $\mathbb{C}_p$ be the completion of an algebraic closure of K. We denote by

$$\mathcal{X} = \mathbb{P}^d_K \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$$

(the complement of all K-rational hyperplanes in projective space $\mathbb{P}^d_K$) Drinfeld's upper half space [D] of dimension $d \geq 1$ over K. This is a rigid analytic variety over K which is equipped with a natural action of $G = \mathrm{GL}_{d+1}(K)$. In [CDN] Colmez, Dospinescu and Niziol determined the pro-étale cohomology of $\mathcal{X}_{\mathbb{C}_p}$ as a special case considering more generally Stein spaces X which have an underlying structure of a semistable weak formal scheme over the ring of integers O_K. It turns out that these cohomology groups are strictly exact extensions

$$(1) \qquad 0 \to \Omega^{s-1}(\mathcal{X})/D^{s-1}(\mathcal{X})\hat{\otimes}_K \mathbb{C}_p \to H^s(\mathcal{X}_{\mathbb{C}_p}, \mathbb{Q}_p(s)) \to v^G_{P_{(d-s+1,1,\ldots,1)}}(\mathbb{Q}_p)' \to 0$$

of $G \times \Gamma_K$-modules. Here Ω^{s-1} is the sheaf of differential forms of degree $s-1$ on $\mathbb{P}^d_K$, $D^{s-1} = \ker(d^{s-1})$ where $d^{s-1} : \Omega^{s-1} \to \Omega^s$ is the differential morphism and $v^G_{P_{(d-s+1,1,\ldots,1)}}(\mathbb{Q}_p)'$ is the (strong) topological dual of the generalized smooth Steinberg representation attached to the decomposition $(d-s+1, 1, \ldots, 1)$ of $d+1$. From the above extensions one tells that these invariants are finer than the de Rham cohomology of $\mathcal{X}$ determined by Schneider and Stuhler [SS]. The latter objects are essentially (replace $\mathbb{Q}_p$ by K) given by the representations on the RHS of the above sequences. For the proof of their result, Colmez, Dospinescu and Niziol use syntomic and Hyodo-Kato cohomology, comparison isomorphisms (Fontaine-Messing period morphisms) and moreover as already mentioned above a (p-adic) semi-stable weak formal model of $\mathcal{X}$ over O_K. In the meantime Colmez and Niziol [CN2] generalized their results to a large extent to arbitrary smooth rigid analytic (dagger) varieties.

Nevertheless, our goal in this paper is to give an alternative approach for the determination of the (p-adic) pro-étale cohomology groups of $\mathcal{X}_{\mathbb{C}_p}$. The strategy is based on the machinery developed in [O] for describing global sections of equivariant vector bundles on $\mathcal{X}$. The advantage is that it reduces the computation of the pro-étale cohomology to simpler geometric objects, as open or punctured discs. For the latter aspect we follow the idea of Le Bras [LB] who sketched a general strategy for Stein spaces and carried this out for open and essentially for punctured discs.[1] Moreover, we have to consider as a technical ingredient local cohomology groups $H^*_{\mathbb{P}^j_K(\epsilon)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ with support in certain tubes $\mathbb{P}^j_K(\epsilon)$ of projective subspace $\mathbb{P}^j_K$ and with coefficients in the dagger sheaf attached to Ω^{s-1}/D^{s-1}.

Another feature of our approach is that we are able to make more precise the structure of the G-representation in the spirit of [O, OS]. For any integer $j \geq 0$, let $P_{(j+1,d-j)} \subset G$ be the standard parabolic subgroup of G attached to the decomposition $(j+1, d-j)$ of $d+1$ and $L_{(j+1,d-j)}$ its Levi factor. Our main theorem is:

Theorem: *i) For any integer $s \geq 0$, there is an extension*[2]

$$0 \to H^0(\mathcal{X}, \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K \mathbb{C}_p \to H^s(\mathcal{X}_{\mathbb{C}_p}, \mathbb{Q}_p(s)) \to v^G_{P_{(d-s+1,1,\ldots,1)}}(\mathbb{Q}_p)' \to 0$$

of $G \times \Gamma_K$-modules.

ii) For any integer $s = 1, \ldots, d$, there is a descending filtration $(Z^j)_{j=0,\ldots,d}$ on $Z^0 = H^0(\mathcal{X}, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ by closed subspaces together with isomorphisms of locally analytic G-representations

$$(Z^j/Z^{j+1})' \cong \mathcal{F}^G_{P_{(j+1,d-j)}}(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1}/D^{s-1}), St_{d-j}), j = 0, \ldots, d-1$$

where $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1}/D^{s-1})$ is the algebraic local Zariski cohomology and St_{d-j} is the Steinberg representation of $\mathrm{GL}_{d-j}(K)$ considered as representation of $L_{(j+1,d-j)}$.

(Here we refer to [OS] *for the definition of the functors $\mathcal{F}^G_{P_{(j+1,d-j)}}$).*

The content of this paper is organized as follows. The second part deals with the study of the pro-étale local cohomology $H^*_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ with support in the rigid analytic tubes $\mathbb{P}^j_{\mathbb{C}_p}(\epsilon)$. For doing so, we have to analyze in the first part the analytic local cohomology groups $H^*_{\mathbb{P}^j_K(\epsilon)}(\mathbb{P}^d_K, \mathcal{F})$ where $\mathcal{F}$ is one of the sheaves $D^{\dagger,s-1}, \Omega^{\dagger,s-1}/D^{\dagger,s-1}$ from above. As for the latter objects, we cannot apply the same methods of [O] directly, e.g. the Čech complex, since the sheaves are not coherent. We circumvent this problem by proving a vanishing result for the higher cohomology groups of $\mathcal{F}$ by using the machinery of van der Put [vP] for overconvergent sheaves. The local pro-étale cohomology groups $H^*_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$

[1] This strategy has been also used recently by Guido Bosco [Bo] in the case of Drinfeld's upper half space.

[2] If the proof of Proposition 2.1 allows us to show that the extensions mentioned there are strictly exact, then it follows that the extensions here are strictly exact, as well.

are needed in order to evaluate the spectral sequence attached to some acyclic complex on the closed complement of $\mathcal{X}_{\mathbb{C}_p}$ in $\mathbb{P}^d_{\mathbb{C}_p}$ in the final section. This strategy was already used in [O] for equivariant vector bundles. It works for the pro-étale cohomology, as well since we can pull back this acyclic complex to the pro-étale site of $\mathbb{P}^d_{\mathbb{C}_p}$. Finally we show that $H^0(\mathcal{X}, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) = \Omega^{\dagger,s-1}(\mathcal{X})/D^{\dagger,s-1}(\mathcal{X})$ for all $s \geq 1$. This result is needed for showing the compatibility of our result with formula (1) as our approach gives rather rise to the space of global sections of the sheaves $\Omega^{\dagger,s-1}/D^{\dagger,s-1}$.

Notation: We denote by p a prime, by $K \supset \mathbb{Q}_p$ a finite extension of the field of p-adic integers $\mathbb{Q}_p$, by O_K its ring of integers and by π a uniformizer of K. Let $|\ |: K \to \mathbb{R}$ be the normalized norm, i.e., $|\pi| = \#(O_K/(\pi))^{-1}$. We denote by $\mathbb{C}_p$ the completion of an algebraic closure $\overline{K}$ of K and extend the norm $|\ |$ on it. For a locally convex K-vector space V, we denote by V' its strong dual, i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence.

We denote for a scheme X (or rigid analytic variety) over K by X^{rig} (resp. X^{ad}) the rigid analytic variety attached (resp. adic space) to X. If $Y \subset \mathbb{P}^d_K$ is a closed algebraic K-subvariety and $\mathcal{F}$ is a sheaf on $\mathbb{P}^d_K$ we write $H^*_Y(\mathbb{P}^d_K, \mathcal{F})$ for the corresponding local cohomology. If Y is a rigid analytic subvariety (resp. pseudo-adic subspace) of $(\mathbb{P}^d_K)^{rig}$ (resp. $(\mathbb{P}^d_K)^{ad}$) we also write $H^*_Y(\mathbb{P}^d_K, \mathcal{F})$ instead of $H^*_Y((\mathbb{P}^d_K)^{rig}, \mathcal{F}^{rig})$ (resp. $H^*_Y((\mathbb{P}^d_K)^{ad}, \mathcal{F}^{ad})$) to simplify matters. For a scheme X (or an adic space etc.) over $\mathbb{C}_p$, we denote by $H^i(X, \mathbb{Q}_p)$ the p-adic pro-étale cohomology of it, cf. [S, BS].

We use bold letters $\mathbf{G}, \mathbf{P}, \ldots$ to denote algebraic group schemes over K, whereas we use normal letters $G, P, \ldots$ for their K-valued points of p-adic groups. We use Gothic letters $\mathfrak{g}, \mathfrak{p}, \ldots$ for their Lie algebras. The corresponding enveloping algebras are denoted as usual by $U(\mathfrak{g}), U(\mathfrak{p}), \ldots$. Finally, we set $\mathbf{G} := \mathbf{GL_{d+1}}$. Denote by $\mathbf{B} \subset \mathbf{G}$ the Borel subgroup of lower triangular matrices. Let $\mathbf{T} \subset \mathbf{G}$ be the diagonal torus. Let Δ be the set of simple roots with respect to $\mathbf{T} \subset \mathbf{B}$. For a decomposition $(i_1, \ldots, i_r)$ of $d+1$, let $\mathbf{P_{(i_1,\ldots,i_r)}}$ be the corresponding standard-parabolic subgroup of $\mathbf{G}$ and $\mathbf{L_{(i_1,\ldots,i_r)}}$ its Levi component. We consider the algebraic action $m : \mathbf{G} \times \mathbb{P}^d_K \to \mathbb{P}^d_K$ of $\mathbf{G}$ on $\mathbb{P}^d_K$ given by

$$g \cdot [q_0 : \cdots : q_d] := m(g, [q_0 : \cdots : q_d]) := [q_0 : \cdots : q_d]g^{-1}.$$

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1. Some preparations

We start by recalling some further notations used in [O]. Let L be one of the fields K or $\mathbb{C}_p$. Let $\epsilon \in \bigcup_{n\in\mathbb{N}} \sqrt[n]{|K^\times|} = |\overline{K^\times}|$ be a n-th root of some absolute value in $|K^\times|$. For a closed L-subvariety $Y \subset \mathbb{P}^d_L$, the open[3] ϵ-neighborhood of Y is defined by

$$Y(\epsilon) = \Big\{ z \in (\mathbb{P}^d_L)^{rig} \mid \text{ for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| \leq \epsilon \text{ for all } 1 \leq j \leq r \Big\}.$$

Here $f_1, \ldots, f_r \in O_L[T_0, \ldots, T_d]$ are finitely many homogeneous polynomials generating the vanishing ideal of the Zariski closure of Y in $\mathbb{P}^d_{O_L}$. We suppose that each polynomial has at least one coefficient in $O_L^\times$. It is a quasi-compact open rigid analytic subspace of $(\mathbb{P}^d_L)^{rig}$. On the other hand, the set

$$Y^-(\epsilon) = \Big\{ z \in (\mathbb{P}^d_L)^{rig} \mid \text{ for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| < \epsilon \text{ for all } 1 \leq j \leq r \Big\}$$

is the closed ϵ-neighborhood of Y. Again, it is an admissible open subset of $(\mathbb{P}^d_L)^{rig}$, but which is in general not quasi-compact.

Recall that Drinfeld's upper half space $\mathcal{X} = \mathbb{P}^d_K \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$ is a rigid analytic Stein space over K and its algebra of analytic functions $\mathcal{O}(\mathcal{X})$ is a K-Fréchet space [SS]. From this one deduces that for very vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$ the space of global sections $\mathcal{X}$ has the structure of a K-Fréchet space. Since our p-adic group G stabilizes $\mathcal{X}$ we even get for every $\mathbf{G}$-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$ the structure of a continuous G-representation on $\mathcal{F}(\mathcal{X})$. Moreover its (strong) dual has the structure of a locally analytic G-representation in the sense of Schneider and Teitelbaum [ST2].

We consider the de Rham complex of sheaves

$$\mathcal{O} \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \to \ldots \to \Omega^d$$

on the scheme $\mathbb{P}^d_K$ or rigid analytic variety $(\mathbb{P}^d_K)^{\mathrm{rig}}$. Moreover, we let

$$\mathcal{O}^\dagger \xrightarrow{d_0} \Omega^{\dagger,1} \xrightarrow{d_1} \Omega^{\dagger,2} \to \ldots \to \Omega^{\dagger,d}$$

be the de Rham complex on the dagger space $\mathbb{P}^{d,\dagger}_K$ in the sense of Große-Klönne [GK]. In the following we use the notation $\Omega^{(\dagger),s}$ etc. to handle both kind of sheaves simultaneously. For convenience, we do not distinguish between $(\mathbb{P}^d_K)^{\mathrm{rig}}$ and $\mathbb{P}^{d,\dagger}_K$ and also for

[3] In [O] we used this notation since in the category of adic spaces this object is an open neighborhood. Of course, many people would call this neighborhood a closed one in analogy with the classical case.

other geometric objects since the underlying topological spaces (Grothendieck topology) are the same. Both complexes are equivariant for the action of $\mathbf{G}$. Let $D^{(\dagger),s}$ be the kernel of the sheaf homomorphism $d^s : \Omega^{(\dagger),s} \to \Omega^{(\dagger),s+1}$. If $X \subset \mathbb{P}_K^{d,\dagger}$ is as Stein space, then $\Omega^{\dagger,s}(X) = \Omega^s(X)$. In particular we get $D^{\dagger,s}(X) = D^s(X)$. We obtain $\mathbf{G}$-equivariant sheaves $D^{(\dagger),s}$ and $\Omega^{(\dagger),s}/D^{(\dagger),s}$ on $\mathbb{P}_K^d$, $s = 0, \ldots, d$. If we denote by $\mathcal{F}^{(\dagger)}$ one of them then there is an induced action of G on the K-vector space of rigid analytic sections $\mathcal{F}^{(\dagger)}(\mathcal{X})$. Since $D^{(\dagger),s}(\mathcal{X})$ is closed in $\Omega^{(\dagger),s}(\mathcal{X})$ (by the continuity of d^s) it follows from above that[4] $\mathcal{F}^{(\dagger)}(\mathcal{X})$ is a K-Fréchet space and its dual $\mathcal{F}^{(\dagger)}(\mathcal{X})'$ is a locally analytic G-representations, as well.

Remark 1.1. From Proposition 3.6 in the final section it will follow that the identity $\Omega^{(\dagger),s}(\mathcal{X})/D^{(\dagger),s}(\mathcal{X}) = H^0(\mathcal{X}, \Omega^{\dagger,s}/D^{\dagger,s})$ is satisfied.

Fix an integer $0 \le j \le d-1$. Let

$$\mathbb{P}_K^j = V(T_{j+1}, \ldots, T_d) \subset \mathbb{P}_K^d$$

be the closed $\mathbf{P_{(j+1,d-j)}}$ stable K-subvariety defined by the vanishing of the coordinates $T_{j+1}, \ldots, T_d$. The local cohomology groups $H^*_{\mathbb{P}^j}(\mathbb{P}_K^d, \mathcal{F})$ are then by functorialty $\mathbf{P_{(j+1,d-j)}} \ltimes U(\mathfrak{g})$-modules. Indeed, the case of vector bundles was treated in [O], cf. also [Fa]. Since D^s is a G-equivariant sheaf, the same reasoning as in loc.cit. applies to get homomorphisms $\mathfrak{g} \to \mathrm{End}(D^s(U)))$ for every open subset $U \subset X$. Thus we obtain also a homomorphism $\mathfrak{g} \to \mathrm{End}(D^{\dagger,s})$. Alternatively, one could argue that the morphism $d_s : \Omega^s \to \Omega^{s+1}$ is $\mathfrak{g}$-linear.

For any positive integer $n \in \mathbb{N}$, we consider the reduction map

$$p_n : \mathbf{GL_{d+1}}(O_K) \to \mathbf{GL_{d+1}}(O_K/(\pi^n)). \tag{2}$$

Put $P^n_{(j+1,d-j)} := p_n^{-1}\big(\mathbf{P_{(j+1,d-j)}}(O_K/(\pi^n))\big)$. This is a compact open subgroup of G which stabilizes the ϵ_n-neighborhood $\mathbb{P}_K^j(\epsilon_n)$ where $\epsilon_n := |\pi|^n$. Hence $H^*_{\mathbb{P}_K^j(\epsilon_n)}(\mathbb{P}_K^d, \mathcal{F}^{(\dagger)})$ is a $P^n_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-module. Again as in the algebraic setting, we have a long exact sequence of $P^n_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-modules

$$\begin{aligned} \cdots \ \to\ & H^i(\mathbb{P}_K^d, \mathcal{F}^{(\dagger)}) \to H^i(\mathbb{P}_K^d \setminus \mathbb{P}_K^j(\epsilon_n), \mathcal{F}^{(\dagger)}) \to H^{i+1}_{\mathbb{P}_K^j(\epsilon_n)}(\mathbb{P}_K^d, \mathcal{F}^{(\dagger)}) \\ \to\ & H^{i+1}(\mathbb{P}_K^d, \mathcal{F}^{(\dagger)}) \to \cdots . \end{aligned}$$

In the following we study the analytic local cohomology groups $H^*_{\mathbb{P}_K^j(\epsilon)}(\mathbb{P}_K^d, \mathcal{F}^\dagger)$ for

$$\mathcal{F}^\dagger \in \Theta := \{D^{\dagger,s} \mid s = 1, \ldots, d\} \cup \{\Omega^{\dagger,s}/D^{\dagger,s} \mid s = 1, \ldots, d\}.$$

For any real numbers $0 < \delta < \epsilon$, we consider the rigid analytic varieties

$$B(\epsilon) := \{z \in (\mathbb{A}_K^1)^{rig} \mid |z| \le \epsilon\} \text{ and } C(\delta, \epsilon) := \{z \in (\mathbb{A}_K^1)^{rig} \mid \delta \le |z| \le \epsilon\}$$

[4]Note that $\mathcal{F}(\mathcal{X}) = \mathcal{F}^\dagger(\mathcal{X})$

resp.

$$B^-(\epsilon) := \{z \in (\mathbb{A}^1_K)^{rig} \mid |z| < \epsilon\} \text{ and } C^-(\delta,\epsilon) := \{z \in (\mathbb{A}^1_K)^{rig} \mid \delta < |z| < \epsilon\}.$$

Proposition 1.2. *Let $X = \prod_{i=1}^d X_i \subset (\mathbb{P}^d_K)^{\mathrm{rig}}$ be some open subspace where for $i = 1,\ldots,d$, $X_i \in \bigcup_{\delta>0,\epsilon>0}\{B(\epsilon), C(\delta,\epsilon)\}$, Then $H^n(X, D^{\dagger,s}) = 0$ and $H^n(X, \Omega^{\dagger,s}/D^{\dagger,s}) = 0$ for all $n > 0$.*

Proof. Since X is affinoid and $\Omega^{\dagger,s}$ is coherent it suffices (by considering the long exact cohomology sequence attached to $0 \to D^{\dagger,s} \to \Omega^{\dagger,s} \to \Omega^{\dagger,s}/D^{\dagger,s} \to 0$) to prove the vanishing property for the sheaf $D^{\dagger,s}$, cf. [GK2, Prop.3.1].

We follow here the machinery of van der Put [vP]. The proof is by induction on d. The case of the constant sheaf $D^{\dagger,0}$ is treated in loc. cit. In particular this contributes to the base of induction, i.e., for $d = 1$. The sheaf $D^{\dagger,1}$ coincides in this case with the coherent sheaf $\Omega^{\dagger,1}$ whose higher cohomology vanishes anyway, cf. [GK2, Prop.3.1].

Let $d > 1$ and $X' = \prod_{i=1}^{d-1} X_i$. We consider the projection map $\phi : X \to X'$ forgetting the last entry and the induced Leray spectral sequence $H^j(X', R^i\phi_* D^{\dagger,s}) \Rightarrow H^{i+j}(X, D^{\dagger,s})$. By the very definition the sheaves $D^{\dagger,s}$ are overconvergent (resp. constructible in the language of van der Put). Hence for any closed geometric point z of X' there is by Theorem 2.3 of loc.cit. an isomorphism $H^i(\hat{Z}, i^{-1}D^{\dagger,s}) = (R^i\phi_* D^{\dagger,s})_z$ where $i : \hat{Z} \hookrightarrow \hat{X}$ is the inclusion map of the fiber[5] at z. We shall prove that $(R^i\phi_* D^{\dagger,s})_z = 0$ and that $H^i(X, \phi_* D^{\dagger,s}) = 0$ for all $i > 0$. We start with the latter aspect.

The complex $\phi_*\Omega^{\dagger,\bullet}_X$ is the (dagger) tensor product of the de Rham complex $\Omega^{\dagger,\bullet}_{X'}$ on X' with the constant de Rham complex $\mathcal{O}^\dagger(X_d) \to \mathcal{O}^\dagger(X_d)dT_d$ of global sections on X_d. Hence the complex has the form

$$\begin{array}{rcl}
 & \vdots & \\
\phi_*\Omega^{\dagger,s+1}_X = \Omega^{\dagger,s}_{X'} \otimes^\dagger \Omega^{\dagger,1}_{X_d}(X_d) & \bigoplus & \Omega^{\dagger,s+1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d) \\
\uparrow & \nwarrow & \uparrow \\
\phi_*\Omega^{\dagger,s}_X = \Omega^{\dagger,s-1}_{X'} \otimes^\dagger \Omega^{\dagger,1}_{X_d}(X_d) & \bigoplus & \Omega^{\dagger,s}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d) \\
\uparrow & \nwarrow & \uparrow \\
\phi_*\Omega^{\dagger,s-1}_X = \Omega^{\dagger,s-2}_{X'} \otimes^\dagger \Omega^{\dagger,1}_{X_d}(X_d) & \bigoplus & \Omega^{\dagger,s-1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d) \\
 & \vdots &
\end{array}$$

where the maps are the obviuos ones. Let $D^{\dagger,s-1}_{X'}$ be the kernel of the morphism $d_{s-1} : \Omega^{\dagger,s-1}_{X'} \to \Omega^{\dagger,s}_{X'}$. One verifies that $\phi_* D^{\dagger,s}$ is the sum of the sheaves $D^{\dagger,s-1}_{X'} \otimes^\dagger \Omega^{\dagger,1}(X_d)dT_d$,

[5]Here we adopt the notation of [vP] to denote by $\hat{X}$ the space of closed geometric points of X

$D^{\dagger,s}_{X'} \otimes^\dagger K$, and the image $\mathrm{im}(\phi_*(d_{s-1}))$. Here the first two sheaves form obviuosly a direct sum. On the other hand, the image of the map

$$d_{s-2} \otimes id : \Omega^{\dagger,s-2}_{X'} \otimes^\dagger \Omega^{\dagger,1}(X_d)dT_d \to \Omega^{\dagger,s-1}_{X'} \otimes^\dagger \Omega^{\dagger,1}(X_d)dT_d$$

is contained in $D^{\dagger,s-1}_{X'} \otimes^\dagger \Omega^{\dagger,1}(X_d)dT_d$. Hence we may replace the summand $\mathrm{im}(\phi_*(d_{s-1}))$ by the smaller subsheaf $(\phi_*(d_{s-1}))(\Omega^{\dagger,s-1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d))$ which is isomorphic to the quotient $\Omega^{\dagger,s-1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d)/D^{\dagger,s-1}_{X'} \otimes^\dagger K$. The intersection of $(\phi_*(d_{s-1}))(\Omega^{\dagger,s-1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d))$ with $D^{\dagger,s-1}_{X'} \otimes^\dagger \Omega^{\dagger,1}(X_d)dT_d \bigoplus D^{\dagger,s}_{X'} \otimes^\dagger K$ is just $D^{\dagger,s-1}_{X'} \otimes^\dagger \mathrm{im}(\mathcal{O}^\dagger(X_d) \to \Omega^{\dagger,1}(X_d)dT_d)$. The higher cohomology groups of the first two summands vanish by induction and the flatness of $\mathcal{O}^\dagger(X_d)dT_d$ over K. Concerning the vanishing of $\Omega^{\dagger,s-1}_{X'} \otimes^\dagger \mathcal{O}^\dagger(X_d)/D^{\dagger,s-1}_{X'} \otimes^\dagger K$ this follows from the fact that X' is affinoid, $\Omega^{s-1}_{X'}$ is coherent and again by induction and flatness. In the same way, one checks that this vanishing property holds for the intersection of these sheaves. Hence the claim follows from a Mayer-Vietories sequence with respect to these sheaves.

Now we prove that $H^i(\hat{Z}, i^{-1}D^{\dagger,s}) = 0$ for all $i > 0$. We start with the observation that for any admissible open subset $V \subset \hat{X}_d$ and any open admissible subset $U \subset \hat{X}$ with $\{z\} \times V \subset U$ there is some open neighborhood W of z such that $W \times V \subset U$ [6]. From this topological fact, we deduce by the very definition of the functor i^{-1} that the pull-back $i^{-1}\Omega^\bullet_X$ is the (dagger) tensor product of the de Rham complex $\mathcal{O}^\dagger_{X_d} \to \Omega^{\dagger,1}_{X_d}$ on X_d and the complex $\Omega^{\dagger,\bullet}_{X',z}$ given by the localisation of $\Omega^{\dagger,\bullet}_{X'}$ in z, i.e., it has the form

$$\begin{array}{rcccl} & & \vdots & & \\ i^{-1}\Omega^{s+1}_X = \Omega^{\dagger,s}_{X',z} \otimes^\dagger \Omega^{\dagger,1}_{X_d} & & \bigoplus & & \Omega^{\dagger,s+1}_{X',z} \otimes^\dagger \mathcal{O}^\dagger_{X_d} \\ & \uparrow & \nwarrow & \uparrow & \\ i^{-1}\Omega^{s}_X = \Omega^{\dagger,s-1}_{X',z} \otimes^\dagger \Omega^{\dagger,1}_{X_d} & & \bigoplus & & \Omega^{\dagger,s}_{X',z} \otimes^\dagger \mathcal{O}^\dagger_{X_d} \\ & \uparrow & \nwarrow & \uparrow & \\ i^{-1}\Omega^{s-1}_X = \Omega^{\dagger,s-2}_{X',z} \otimes^\dagger \Omega^{\dagger,1}_{X_d} & & \bigoplus & & \Omega^{\dagger,s-1}_{X',z} \otimes^\dagger \mathcal{O}^\dagger_{X_d} \\ & & \vdots & & \end{array}$$

where the maps are the obviuos ones. Since the functor i^{-1} is exact we deduce as above that $i^{-1}D^{\dagger,s}$ is the sum of the sheaves $(D^{\dagger,s}_{X'})_z \otimes^\dagger D^{\dagger,0}_{X_d}$, $(D^{\dagger,s-1}_{X'})_z \otimes^\dagger \Omega^{\dagger,1}_{X_d}$ and and some sheaf isomorphic to $\Omega^{\dagger,s-1}_{X',z} \otimes^\dagger \mathcal{O}^\dagger_{X_d}/(D^{\dagger,s-1}_{X'})_z \otimes^\dagger D^{\dagger,0}_{X_d}$. From here on the argumentation is the same as above. □

Corollary 1.3. *Let* $X^- = \prod_{i=1}^d X_i^- \subset (\mathbb{P}^d_K)^{rig}$ *be some open subspace where for* $i = 1, \ldots, d$, $X_i^- \in \bigcup_{\delta>0,\epsilon>0}\{B^-(\epsilon), C^-(\delta,\epsilon)\}$. *Then* $H^n(X^-, D^{\dagger,s}) = 0$ *and* $H^n(X^-, \Omega^{\dagger,s}/D^{\dagger,s}) = 0$ *for all* $n > 0$.

[6] This is clear for classical points, i.e. for the rigid varieties X without $\hat{}$. It transfers by the very definition of the topology to the enriched rigid varieties $\hat{X}$.

Proof. As above (by considering the corresponding long exact cohomology sequence) it is enough to prove the statement for the sheaf $D^{\dagger,s}$. We start with the observation that X^- is a Stein space for which an admissible affinoid covering $X^- = \bigcup_{k\in\mathbb{N}} U_k$ with affinoid objects U_k as before exists. By the same reasoning as in §2 of [SS] we have short exact sequences

$$0 \to \varprojlim_k{}^{(1)} H^{i-1}(U_k, D^{\dagger,s}) \to H^i(X^-, D^{\dagger,s}) \to \varprojlim_k H^i(U_k, D^{\dagger,s}) \to 0.$$

Thus we get the claim for $i \geq 2$ by applying the previous proposition. For $i = 1$, we need to show that $\varprojlim_k^{(1)} H^0(U_k, D^{\dagger,s}) = 0$. But $\varprojlim_k^{(1)} H^0(U_k, D^s) = 0$ as the projective system $(H^0(U_k, D^s))_k$ consists of Banach spaces where the transition maps have dense image so that the topological Mittag Leffler property is satisfied. Thus it is enough to show that $\varprojlim_k H^0(U_k, D^s/D^{\dagger,s}) = 0$. But by the definition of the sheaves $D^{\dagger,s}$ the transition maps in the projective system $H^0(U_k, D^s/D^{\dagger,s})$ are all zero. Hence we see that $\varprojlim_k H^0(U_k, D^s/D^{\dagger,s}) = 0$. □

Corollary 1.4. *Let Y be one of the rigid analytic varieties X or X^- considered above. Then $\Omega^{\dagger,s}(Y)/D^{\dagger,s}(Y) = H^0(Y, \Omega^{\dagger,s}/D^{\dagger,s})$.*

Proof. This follows from the corresponding long exact cohomology sequence and the vanishing of the first cohomology $H^1(Y, D^{\dagger,s})$. □

As a first application we may deduce:

Lemma 1.5. *Let $s \geq 1$. Then $H^i(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) = 0$ for $i \geq 0$.*

Proof. We consider the standard covering $\mathbb{P}^d_K = \bigcup_{k=0}^d D_+(T_k)_1$ by balls, i.e, where $D_+(T_k)_1 = \{x \in \mathbb{P}^d_k \mid |x_k| \geq |x_j| \quad \forall j \neq k\}$. By applying Proposition 1.2 to the open subvarieties $D_+(T_k)_1, k = 0, \ldots, d$ and their intersections we see that the corresponding Čech complex with values in $D^{\dagger,s-1}$ computes $H^i(\mathbb{P}^d_K, D^{\dagger,s-1})$. Now the proof is by induction on s. For $s = 1$, we consider the short exact sequence

$$0 \to D^{\dagger,0} \to \mathcal{O}^\dagger \to \mathcal{O}^\dagger/D^{\dagger,0} \to 0.$$

Using the fact that $H^0(X, D^{\dagger,0}) = K$ for any rigid analytic variety X appearing in the Čech complex induced by the above covering we see by looking at its nerve that $H^i(\mathbb{P}^d_K, D^{\dagger,0}) = 0$ for $i > 0$. As for $\mathcal{O}^\dagger$ we have $H^i(\mathbb{P}^d_K, \mathcal{O}^\dagger) = H^i(\mathbb{P}^d_K, \mathcal{O}) = 0$ for any $i > 0$ [Ha, Thm 5.1], [GK2, Thm. 3.2]. For $i = 0$, the map $H^0(\mathbb{P}^d_K, D^{\dagger,0}) \to H^0(\mathbb{P}^d_K, \mathcal{O}^\dagger) = K$ is clearly an isomorphism and the base of induction is shown.

Now let $s > 1$. For any X appearing as geometric object in the Čech complex, there are (using Corollary 1.4 and since X is smooth) short exact sequences

$$0 \to (\Omega^{\dagger,s-1}/D^{\dagger,s-1})(X) \to D^{\dagger,s}(X) \to H^s_{dR}(X) \to 0$$

and

$$0 \to D^{\dagger,s} \to \Omega^{\dagger,s} \to \Omega^{\dagger,s}/D^{\dagger,s} \to 0. \tag{3}$$

By the induction hypothesis and by reconsidering the above covering we get from the first exact sequence isomorphisms $H^i(\mathbb{P}^d_K, D^{\dagger,s}) = H^i(H^s_{dR}(\cdot)), i \geq 0$ where $H^s_{dR}(\cdot)$ is the complex

$$\bigoplus_{0\leq k\leq d} H^s_{dR}(D_+(T_k)_1) \to \bigoplus_{0\leq k_1<k_2\leq d} H^s_{dR}(D_+(T_{k_1})_1 \cap D_+(T_{k_2})_1) \to \cdots$$
$$\cdots \to H^s_{dR}(D_+(T_0)_1 \cap \cdots \cap D_+(T_d)_1). \tag{4}$$

Now again by [Ha, III, Exercise 7.3] $H^i(\mathbb{P}^d_K, \Omega^{\dagger,s}) = H^i(\mathbb{P}^d_K, \Omega^s) \neq 0$ iff $i = s$ which is moreover then a one-dimensional K-vector space. This is exactly induced by $H^i(H^s_{dR}(\cdot))$ and all other groups $H^i(H^s_{dR}(\cdot))$ vanish as the above complexes form the E_1-term of the attached covering spectral sequence computing $H^*_{dR}(\mathbb{P}^d_K)$. Thus we deduce that $H^i(\mathbb{P}^d_K, \Omega^{\dagger,s}/D^{\dagger,s}) = 0$ for all $i \geq 0$. □

As a byproduct we see that $D^{\dagger,s}$ has the same cohomology on $\mathbb{P}^d$ as Ω^s, i.e.

$$H^n(\mathbb{P}^d_K, D^{\dagger,s}) = H^n(\mathbb{P}^d_K, \Omega^s) \tag{5}$$

for all $n \geq 0$

As for the next application, we consider for any integer $0 \leq j \leq d-1$, the complement $\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon)$ of the tube $\mathbb{P}^j_K(\epsilon)$ in projective space. One checks that there is a covering

$$\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon) = \bigcup_{k=j+1}^{d} V(k;\epsilon)$$

where

$$V(j+1;\epsilon) = \Big\{[x_0 : \ldots : x_d] \in \mathbb{P}^d_K \mid |x_{j+1}| > |x_l| \cdot \epsilon \;\; \forall\, l < j+1\Big\}$$

and

$$V(k;\epsilon) = \Big\{[x_0 : \ldots : x_d] \in \mathbb{P}^d_K \mid |x_k| > |x_l| \cdot \epsilon \;\; \forall\, l < j+1, |x_k| > |x_l| \;\; \forall\, j+1 \leq l < k\Big\}$$

for $k > j+1$. These are admissible open subsets of $(\mathbb{P}^d_K)^{rig}$ which are even Stein spaces. The same holds true for arbitrary intersections of them. More concretely, there is the following description.

Lemma 1.6. *Let $I = \{i_1 < \cdots < i_r\} \subset \{j+1, \ldots, d\}$. Then*

$$\bigcap_{k\in I} V(k;\epsilon) \cong B^-(1/\epsilon)^{j+1} \times B^-(1)^{i_1-1-j} \times \prod_{k=2}^{r} \left(C^-(0,1) \times B^-(1)^{i_k-i_{k-1}-1}\right) \times \mathbb{A}_K^{d-i_r}$$

Proof. We consider the map $\bigcap_{k\in I} V(k;\epsilon) \to \mathbb{A}^d$ defined by $[x_0 : \cdots : x_d] \to (y_0,\ldots,y_{d-1})$ where $y_i = x_i/x_{i_1}$ for $i < i_1$, $y_i = x_i/x_{i_2}$ for $i_1 \leq i < i_2$, $y_i = x_i/x_{i_3}$ for $i_2 \leq i < i_3$, $\ldots$, $y_i = x_i/x_{i_r}$ for $i_{r-1} \leq i < i_r$ and $y_i = x_i/x_{i_r}$ for $i > i_r$. The image is contained in the RHS of the stated isomorphisms. We define an inverse morphism by $(y_0,\ldots,y_{d-1}) \mapsto [x_0 : \cdots : x_d]$ with $x_i = y_i, i < i_1$, $x_{i_1} = 1$, $x_i = y_i y_{i_1}^{-1}, i_1 < i < i_2$, $x_{i_2} = y_{i_1}^{-1}$, $x_i = y_i y_{i_1}^{-1} y_{i_2}^{-1}, i_2 < i < i_3$, $x_{i_3} = y_{i_1}^{-1} y_{i_2}^{-1}$, $x_i = y_i y_{i_1}^{-1} y_{i_2}^{-1} y_{i_3}^{-1}, i_3 < i < i_4$, $x_{i_3} = y_{i_1}^{-1} y_{i_2}^{-1} y_{i_3}^{-1}$, $\ldots$, $x_i = y_i y_{i_1}^{-1} y_{i_2}^{-1} \cdots y_{i_r}^{-1}, i > i_r$. □

By Lemma 1.6 and Corollary 1.3 we may compute the cohomology $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \mathcal{F}^\dagger)$ for $\mathcal{F}^\dagger \in \Theta$ via the Čech complex

$$(6)\quad C^\bullet_n \mathcal{F}^\dagger : \bigoplus_{j+1\leq k\leq d} \mathcal{F}^\dagger(V(k;\epsilon_n)) \to \bigoplus_{j+1\leq k_1<k_2\leq d} \mathcal{F}^\dagger(V(k_1;\epsilon_n)\cap V(k_2;\epsilon_n)) \to \cdots$$
$$\cdots \to \mathcal{F}^\dagger(V(j+1;\epsilon_n)\cap\cdots\cap V(d;\epsilon_n)).$$

Remark 1.7. *In* [O] *we proved via this approach that the cohomology groups* $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^s) = H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s})$ *and* $H^*_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^s) = H^*_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s})$ *are* K*-Fréchet spaces with the structure of a continuous* $P^n_{(j+1,d-j)} \ltimes U(\mathfrak{g})$*-module in which the algebraic cohomology* $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \Omega^s)$ *resp.* $H^*_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^s)$ *is a dense subspace. Since the differential maps* $d^s : \Omega^s(U) \to \Omega^{s+1}(U)$ *are continuous for any open subvariety* U *appearing in the complex (6) and* $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), D^{\dagger,s}) \cap H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \Omega^s) = H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, D^s)$ *(cf. Prop. 1.9) one checks now easily that the same is satisfied for the modules* $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \mathcal{F}^\dagger)$ *and* $H^*_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathcal{F}^\dagger)$*. This aspect can be made precise as follows. For* $j < k \leq d$ *and* $\epsilon > \epsilon_n$*, consider the open affinoid subvarietes*

$$\bar{V}(j+1;\epsilon) = \Big\{[x_0 : \ldots : x_d] \in \mathbb{P}^d_K \mid |x_{j+1}| \geq |x_l|\cdot\epsilon \;\; \forall\, l < j+1\Big\}$$

and

$$\bar{V}(k;\epsilon) = \Big\{[x_0 : \ldots : x_d] \in \mathbb{P}^d_K \mid |x_k| \geq |x_l|\cdot\epsilon \;\; \forall\, l < j+1, |x_k| \geq |x_l| \;\; \forall\, j+1 \leq l < k\Big\}$$

and form the attached Čech complex

$$(7)\quad C^\bullet_\epsilon \mathcal{F} : \bigoplus_{j+1\leq k\leq d} \mathcal{F}(\bar{V}(k,\epsilon)) \to \bigoplus_{j+1\leq k_1<k_2\leq d} \mathcal{F}(\bar{V}(k_1,\epsilon)\cap\bar{V}(k_2,\epsilon)) \to \cdots$$
$$\cdots \to \mathcal{F}(\bar{V}(j+1,\epsilon)\cap\cdots\cap\bar{V}(d,\epsilon)).$$

If we denote by U_ϵ *some open affinoid subvariety appearing in this complex (and similarly* $U_{\epsilon'}$ *for* $\epsilon' < \epsilon$*), then the restriction maps* $\Omega^s(U_\epsilon) \to \Omega^s(U_{\epsilon'})$ *of Banach spaces are injective, continuous and have dense image. It follows that the same holds true for the maps* $D^s(U_\epsilon) \to D^s(U_{\epsilon'})$ *of Banach spaces (!). As the the map* d^s *is even strict, we deduce that the the functor* $\varprojlim_{\epsilon\to\epsilon_n}$ *is exact with respect to the complexes* $C^\bullet_\epsilon\mathcal{F}$*. Since we additionally have*

$\varprojlim_{\epsilon\to\epsilon_n}\mathcal{F}(U_\epsilon)=\varprojlim_{\epsilon\to\epsilon_n}\mathcal{F}^\dagger(U_\epsilon)=\mathcal{F}^\dagger(U_{\epsilon_n})$ *we may identify* $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon),\mathcal{F}^\dagger)$ *with the projective limit of K-Banach spaces* $\varprojlim_{\epsilon\to\epsilon_n}H^i(C^\bullet_\epsilon\mathcal{F})$.

Next we observe that

$$H^n_{\mathbb{P}^j_K(\epsilon)}(\mathbb{P}^d_K,D^{\dagger,s})=H^n(\mathbb{P}^d_K,D^{\dagger,s})\quad\forall n>d-j \tag{8}$$

by the length of the Čech complex (6). The following result is known for coherent sheaves (by the smoothness of $\mathbb{P}^j_K$), cf. [O].

Lemma 1.8. *Let $0\leq j\leq d-1$. Then the cohomology groups $H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\Omega^{\dagger,s}/D^{\dagger,s})$ and $H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,D^{\dagger,s})$ vanish for $i<d-j$.*

Proof. The case $j=d-1$ is trivial by Lemma 1.5 and since $K=H^0(\mathbb{P}^d_K,D^{\dagger,0})=H^0(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),D^{\dagger,0})$. So let $j<d-1$.

The proof is similar to Lemma 1.5. By Lemma 1.5 we need to show that $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),\Omega^{\dagger,s}/D^{\dagger,s})=0$ for $i<d-j-1$. We consider the covering of $\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n)=\bigcup_{k>j}V(k;\epsilon_n)$. For $s=0$, we have again $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),D^{\dagger,0})=0$ for all $i>0$ and $H^0(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),D^{\dagger,0})=K$. Since the stated vanishing is true for coherent sheaves we see that $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),\mathcal{O}^\dagger)=H^i(\mathbb{P}^d_K,\mathcal{O}^\dagger)$ for all such i and the result follows.

For $s>0$ we reconsider the exact sequence (3). By induction hypothesis the statement is true for the sheaf $\Omega^{\dagger,s-1}/D^{\dagger,s-1}$. Hence $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),D^{\dagger,s})=H^i(H^s_{dR}(\cdot))$ for all $i<d-j-1$ where $H^s_{dR}(\cdot)$ is defined similar as before with respect to the above covering of $\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n)$. The latter term coincides with $H^i(\mathbb{P}^d_K,\Omega^{\dagger,s})$ which is thus $H^i(\mathbb{P}^d_K\setminus\mathbb{P}^j_K(\epsilon_n),\Omega^{\dagger,s})$ since $i<d-j-1$. As before the result follows. □

Proposition 1.9. *There are for all $s\geq1$, strict exact sequences*

$$0\to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,D^{\dagger,s-1})\to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\Omega^{\dagger,s-1})\to H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\Omega^{\dagger,s-1}/D^{\dagger,s-1})\to0.$$

Proof. By Lemma 1.8 the above sequence is exact on the left. As for the right exactness we claim that the map $H^{d-j+1}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,D^{\dagger,s-1})\to H^{d-j+1}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,\Omega^{\dagger,s-1})$ is an isomorphism. Indeed by (8) we have $H^{d-j+1}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K,D^{\dagger,s-1})=H^{d-j+1}(\mathbb{P}^d_K,D^{\dagger,s-1})$. With the same reasoning as in (8) we also have $H^{d-j+1}(\mathbb{P}^d_K,\Omega^{\dagger,s-1})=H^{d-j+1}_{\mathbb{P}^j(\epsilon_n)}(\mathbb{P}^d_K,\Omega^{\dagger,s-1})$. But $H^{d-j+1}(\mathbb{P}^d_K,D^{\dagger,s-1})=H^{d-j+1}(\mathbb{P}^d_K,\Omega^{\dagger,s-1})$ by identity (5). The claim follows.

As for the topological statement this follows easily from the fact that the differential map d^{s-1} is continuous and therefore $D^{\dagger,s-1}(X)$ is closed in $\Omega^{\dagger,s-1}(X)$ for every rigid analytic variety X appearing in the Čech complex. □

Remark 1.10. The sequence

$$0 \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, D^{\dagger,s-1}) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) \to 0$$

consisting of analytic local cohomology groups is exact, as well. This follows by taking the projective limit $\varprojlim_n$ and applying the topological Mittag-Leffler criterion. By density this fact is also true for the corresponding sequence of algebraic local cohomology groups of schemes

$$0 \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, D^{s-1}) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1}) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1}/D^{s-1}) \to 0.$$

In fact this sequence is the same as the pull back of the above sequence to $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1})$.

Remark 1.11. Since the (strong) topological dual $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{s-1})'$ is a locally analytic $P^n_{(j+1,d-j)}$-representation by [O, Cor. 1.3.9], the same is true for $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{s-1}/D^{s-1})'$ as a closed $P^n_{(j+1,d-j)}$-stable subspace.

2. Some local pro-étale cohomology groups

In the sequel we denote for a rigid analytic variety X over K by $X_{\mathbb{C}_p}$ its base change to $\mathbb{C}_p$. We shall determine in this section the local pro-étale cohomology groups $H^*_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$.

As usual there is a long exact cohomology sequence

$$(9) \qquad \begin{aligned} \cdots \;\to\; & H^{i-1}(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p) \to H^i_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to H^i(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \\ \to\; & H^i(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p) \to \cdots . \end{aligned}$$

As for the computation of $H^i(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p)$, we consider the spectral sequence

$$E_1^{p,q} \Rightarrow E^{p+q} = H^{p+q}(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p)$$

with respect to the covering of Stein spaces

$$\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n) = \bigcup_{k=j+1}^{d} V(k;\epsilon_n)_{\mathbb{C}_p}.$$

The line $E^{\bullet,s}$ is given by the complex

$$(10) \qquad \bigoplus_{j+1\leq k\leq d} H^s((V(k;\epsilon_n)_{\mathbb{C}_p}, \mathbb{Q}_p) \to \bigoplus_{j+1\leq k_1<k_2\leq d} H^s(V(k_1;\epsilon_n)_{\mathbb{C}_p} \cap V(k_2;\epsilon_n)_{\mathbb{C}_p}, \mathbb{Q}_p) \to \cdots \to H^s(V(j+1;\epsilon_n)_{\mathbb{C}_p} \cap \cdots \cap V(d;\epsilon_n)_{\mathbb{C}_p}, \mathbb{Q}_p).$$

Let $U = V(k_1;\epsilon_n) \cap V(k_2;\epsilon_n) \cap \ldots \cap V(k_r;\epsilon_n)$ be some intersection of these Stein spaces which appear in the above complex. Then this is a Stein space, as well, and the geometric pro-étale cohomology has the following description.

Proposition 2.1. *For $s \geq 0$, there is an extension*

$$0 \to \Omega^{s-1}(U)/D^{s-1}(U)\hat{\otimes}_K \mathbb{C}_p(-s) \to H^s(U_{\mathbb{C}_p}, \mathbb{Q}_p) \to H^s_{dR}(U, \mathbb{Q}_p)(-s) \to 0$$

where $H^s_{dR}(U, \mathbb{Q}_p) = \wedge^s(\mathbb{Q}_p^{r-1})$ is a $\mathbb{Q}_p$-vector space[7] *with $H^s_{dR}(U, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = H^s_{dR}(U/K)$.*

Proof. The proof is essentially contained in [LB] (which was in the meantime further revisited in [Bo]). We therefore give only a sketch of it. At first we consider the short exact sequence of pro-étale sheaves

$$0 \to \mathbb{Q}_p \to \mathbb{B}[1/t]^{\varphi} \to \mathbb{B}_{dR}/\mathbb{B}^+_{dR} \to 0 \tag{11}$$

on $U_{\mathbb{C}_p}$ and determine the cohomology of the period sheaves. As for latter sheaf, we consider for any integer $k \in \mathbb{N}$ the spectral sequence

$$E_1^{p,q} = H^{p+q}(U_{\mathbb{C}_p}, gr^p(\mathbb{B}^+_{dR}/t^k)) \Rightarrow H^{p+q}(\mathbb{B}^+_{dR}/t^k)$$

which is induced by the filtration $(t^p\mathbb{B}^+_{dR}/t^k)_{p=0,\ldots k-1}$ on $\mathbb{B}^+_{dR}/t^k$. Here we assume that $k > d$. In loc.cit. it is explained that there is an identification $E_1^{p,q} = \Omega^{p+q}(U_{\mathbb{C}_p})(-q)$, $\forall p, q$, since $U_{\mathbb{C}_p}$ is a Stein space. Note that $E_1^{p,q} = 0$ for $p < 0$ or $p \geq k$. Hence we get

$$E_2^{p,q} = H^{p+q}_{dR}(U_{\mathbb{C}_p})(-q) = H^{p+q}_{dR}(U_K)\hat{\otimes}_K \mathbb{C}_p(-q)$$

for $0 < p < k-1$ and $E_2^{0,q} = D^q(U_{\mathbb{C}_p})(-q)$ resp. $E_2^{k-1,q} = \mathrm{coker}(d_{p+q}(U_{\mathbb{C}_p}))(-q)$ for all q. By weight reasons we have $E_2 = E_\infty$. Hence if we set $i = p + q$, then there is a filtration $F^0 \supset F^1 \supset \cdots F^{k-1} \supset F^k = (0)$ on $H^i(\mathbb{B}^+_{dR}/t^k)$ such that $F^{k-1} = \mathrm{coker}(d_i)(k-i-1)$, $F^p/F^{p+1} = H^i_{dR}(U_{\mathbb{C}_p})(p-i)$, $p = 1, \ldots, k-2$ and $F^0/F^1 = \ker(d_i)(-i)$.

Now we write $\ker(d_i)(-i)$ as a (split) extension

$$0 \to \Theta_i \to \ker(d_i)(-i) \to H^i_{dR}(U_{\mathbb{C}_p}, \mathbb{C}_p)(-i) \to 0$$

where

$$\Theta_i := \Omega^{i-1}(U_{\mathbb{C}_p})/D^{i-1}(U_{\mathbb{C}_p})(-i).$$

Passing to the limit as $k \to \infty$ we get an extension

$$0 \to H^i_{dR}(U)(-i) \otimes_K B^+_{dR} \to H^i(U_{\mathbb{C}_p}, \mathbb{B}^+_{dR}) \to \Theta_i \to 0.$$

As for the sheaf $\mathbb{B}_{dR}$ we follow the reasoning in [LB, Prop. 2.3.19] to conclude that

$$H^i(U_{\mathbb{C}_p}, \mathbb{B}_{dR}) = H^i_{dR}(U)(-i) \otimes_K B_{dR}.$$

Hence we see via the long exact cohomology sequence attached to

$$0 \to \mathbb{B}^+_{dR} \to \mathbb{B}_{dR} \to \mathbb{B}_{dR}/\mathbb{B}^+_{dR} \to 0 \tag{12}$$

on $U_{\mathbb{C}_p}$ that $H^i_{dR}(U_{\mathbb{C}_p}, \mathbb{B}_{dR}/\mathbb{B}^+_{dR})$ is an extension

$$0 \to H^i_{dR}(U)(-i) \otimes_K B_{dR}/B^+_{dR} \to H^i(U_{\mathbb{C}_p}, \mathbb{B}_{dR}/\mathbb{B}^+_{dR}) \to \Theta_{i+1} \to 0.$$

[7] which can be expressed via Hyodo-Kato cohomology

Concerning the sheaf $\mathbb{B}[1/t]^{\varphi}$ we use the geometric description of U in Lemma 1.6 in order to apply [LB, Cor. 2.3.31][8] together with the projective limit argument below [LB, Cor. 2.3.31] to deduce that

$$H^i(U_{\mathbb{C}_p}, \mathbb{B}[1/t]) = H^i_{dR}(U_{\mathbb{C}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B[1/t].$$

Finally we follow the argument below [LB, Cor, 2.3.31] concerning φ using the freeness of the latter object over $\mathbb{B}[1/t]$ to see that

$$H^i(U_{\mathbb{C}_p}, \mathbb{B}[1/t]^{\varphi}) = H^i_{dR}(U_{\mathbb{C}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B[1/t]^{\varphi}.$$

Write $H^i_{dR}(U)(-i) \otimes_K B_{dR}/B^+_{dR} = H^i_{dR}(U, \mathbb{Q}_p)(-i) \otimes_{\mathbb{Q}_p} B_{dR}/B^+_{dR}$. Considering the long exact cohomology sequence attached to the short exact sequence (11) we get the claim. □

Remark 2.2. *It seems that this result is also covered by* [CN2, Theorem 1.1 and Theorem 1.3]. *For open balls this was done before by Colmez and Niziol* [CN1, Theorem 3] *resp. Le Bras* [LB, Theorem 2.3.2].

Hence we may write $E_1^{\bullet,\bullet}$ as an extension

$$0 \to F_1^{r,s} \to E_1^{r,s} \to G_1^{r,s} \to 0$$

of double complexes, as well.

The cohomology of the double complex $G_1^{r,s}$ gives rise to a $\mathbb{Q}_p$-form $H^*_{dR}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \mathbb{Q}_p)$ of the de Rham cohomology $H^*_{dR}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n))$ which is $\bigoplus_{i=0}^{d-j-1} \mathbb{Q}_p[-2i]$. On the other hand, by Corollary 1.4 and by the lines before Remark 1.7 the $F_1^{\bullet,s}$-term just computes the cohomology group $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K \mathbb{C}_p(-s)$. More precisely,

$$F_2^{r,s} = H^r(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K \mathbb{C}_p(-s)$$

for all $r, s \geq 0$. The contributions $H^r(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon), \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ vanish (by considering the Čech complex) for $r \geq d-j$. Further, we have the long exact cohomology sequence

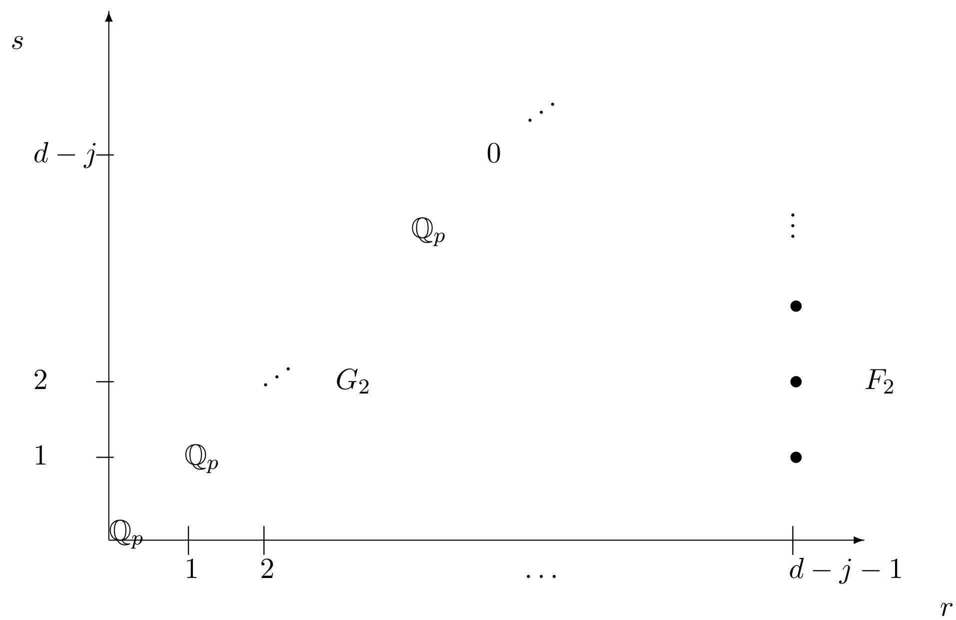
$$\begin{aligned} \ldots \to \;& H^i_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) \;\to H^i(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) \to H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s-1}/D^{\dagger,s-1}) \\ \to \;& H^{i+1}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) \;\to \ldots \end{aligned}$$

The expressions $H^i(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ vanish by Lemma 1.5 for all $i \geq 0$. Hence we get

$$H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s-1}/D^{\dagger,s-1}) = H^{i+1}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$$

for all $i \geq 0$. Since $H^i_{\mathbb{P}^j_K(\epsilon)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1}) = 0$ for $i < d-j$ by Lemma 1.8 we deduce that $F_2^{r,s} = 0$ for $r \neq d-j-1$. Hence the E_2-term consists of two lines.

[8]See also [Bo, Cor. 3.3.16]



Lemma 2.3. *Considered as $U(\mathfrak{g})$-module the representations $F_2^{d-j-1,s}, s = 0, \ldots d$, do not include the trivial representation as composition factor.*

Proof. We consider the weights of $F_2^{d-j-1,s}$ with respect to the Cartan algebra $\mathfrak{t} \subset \mathfrak{g}$. By Prop. 1.9 the representation $F_2^{d-j-1,s} = H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ is a homomorphic image of $H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}) = H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{s-1})$ which in turn is a quotient of a (Fréchet)-completion of some representation of the shape

$$\bigoplus_{\substack{k_0,\ldots,k_j \geq 0 \\ k_{j+1},\ldots,k_d \leq 0 \\ k_0+\cdots+k_d=0}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} \otimes V_{d-j,z_{d-j}\lambda} \tag{13}$$

cf. [O, Prop. 1.4.2, Cor. 1.4.9] for some irreducible algebraic representation $V = V_{d-j,z_{d-j}\lambda}$ of the Levi subgroup $L_{(j+1,d-j)}$ of $P_{(j+1,d-j)}$. Here λ is the weight defining Ω^{s-1} in the sense of loc.cit. It is given by the tuple $(-s+1, 1, \cdots, 1, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ via the identification $X^*(T) = \mathbb{Z}^{d+1}$.[9] Therefore it suffices to check that the latter representation does not contain the trivial representation as composition factor. Going to the definition of V in loc.cit., it turns out that that the weights of this representation are given by all concatenations of all permutations of the individual arrays $(0, \ldots, 0, -s+d-j)$ (of length $j+1$) and $(1, \ldots, 1, 0 \ldots, 0)$ (of length $d-j$) for $d-j \leq s-1$ resp. $(0, \ldots, 0, -1, \ldots, -1)$ (of length $j+1$) and $(d-j+1-s, 0, \ldots, 0)$ (of length $d-j$) for $d-j > s-1$. Since the weight of a polynomial $f = X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$ is given by $(k_0, \ldots, k_d)$ we see that the trivial weight $(0, \ldots, 0)$ is not realizable in the representation (13). □

[9] Moreover $i_0 = s-1$ in the notation of loc.cit. as $H^k(\mathbb{P}^d_K, \Omega^k) = K$ for all $k \geq 0$.

In particular, we see that $E_2 = E_\infty$ and that $H^i(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p)$ is an extension

$$0 \to A \to H^i(\mathbb{P}^d_{\mathbb{C}_p} \setminus \mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n), \mathbb{Q}_p) \to B \to 0$$

where

$$A = H^{d-j-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K \mathbb{C}_p(-s)$$

with $s = i - (d-j-1) \geq 1$ and

$$B = H^i_{dR}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K(\epsilon_n), \mathbb{Q}_p)(-\frac{i}{2}).$$

In particular the term A vanishes for $i < d-j$. Further the term B vanishes for $i \geq 2(d-j)$.

Thus we have proved by applying the long exact sequence (9) the following statement.

Proposition 2.4. *Let $j \geq 0$.*

a) If $i < d-j$, then $H^i_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) = 0$.

b) If $i \geq d-j$, then $H^i_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ is an extension

$$0 \to C \to H^i_{\mathbb{P}^j_{\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_K, \mathbb{Q}_p) \to D \to 0$$

where

$$C = H^{d-j}_{\mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger, i-(d-j)-1}/D^{\dagger,(i-(d-j)-1})\hat{\otimes}_K \mathbb{C}_p(-(i-(d-j))$$

and

$$D = H^i_{dR, \mathbb{P}^j_K(\epsilon_n)}(\mathbb{P}^d_K, \mathbb{Q}_p)(-\frac{i}{2})$$

denotes the "local de Rham cohomology" of $\mathbb{P}^d_K$ with support in $\mathbb{P}^j_K(\epsilon_n)$.

3. The proof of the main theorem

Let $\mathcal{Y}_{\mathbb{C}_p}$ be the set-theoretical complement of $\mathcal{X}_{\mathbb{C}_p}$ in $\mathbb{P}^d_{\mathbb{C}_p}$. Consider the topological exact sequence of locally convex $\mathbb{Q}_p$-vector spaces with continuous G-action

$$\begin{aligned} \ldots \to \;& H^i_{\mathcal{Y}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to H^i(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to H^i(\mathcal{X}_{\mathbb{C}_p}, \mathbb{Q}_p) \\ \to \;& H^{i+1}_{\mathcal{Y}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to \ldots \end{aligned} \tag{14}$$

The pro-étale cohomology groups of $\mathbb{P}^d_{\mathbb{C}_p}$ look like as in the classical setting by the quasi-compactness of projective space, i.e. $H^*(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) = \bigoplus_{i=0}^d \mathbb{Q}_p[-2i]$. Hence it suffices to understand the objects $H^i_{\mathcal{Y}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ and the maps $H^i_{\mathcal{Y}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to H^i(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$. For this we recall the construction [O] of an acyclic resolution of the constant sheaf $\mathbb{Z}$ on $\mathcal{Y}_{\mathbb{C}_p}$ considered as an object in the category of pseudo-adic spaces [H].

Let L be again one of our fields K or $\mathbb{C}_p$. Set

$$\mathcal{Y}_L^{ad} := (\mathbb{P}_L^d)^{ad} \setminus \mathcal{X}_L^{ad}.$$

This is a closed pseudo-adic subspace of $(\mathbb{P}_L^d)^{ad}$. For any subset $I \subset \Delta$ with $\Delta \setminus I = \{\alpha_{i_1} < \ldots < \alpha_{i_r}\}$, set

$$j(I) := i_1 \text{ and } Y_{I,L} = \mathbb{P}_L^{j(I)}.$$

Furthermore, let $P_I \subset G$ be the standard parabolic subgroup attached to I. Hence the group P_I stabilizes $Y_{I,L}$. We obtain

$$\mathcal{Y}_L^{ad} = \bigcup_{I \subset \Delta} \bigcup_{g \in G/P_I} g \cdot Y_{I.L}^{ad} = \bigcup_{g \in G} g \cdot Y_{\Delta \setminus \{\alpha_{d-1}\},L}^{ad}. \tag{15}$$

For any compact open subset $W \subset G/P_I$, we set

$$Z_{I,L}^W := \bigcup_{g \in W} g Y_{I,L}^{ad}.$$

Thus

$$\mathcal{Y}_L^{ad} = \bigcup_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} Z_I^{G/P_I} = Z_{\Delta \setminus \{\alpha_{d-1}\},L}^{G/P_{\Delta \setminus \{\alpha_{d-1}\}}}.$$

We consider the natural closed embeddings of pseudo-adic spaces

$$\Phi_{g,I} : g Y_{I,L}^{ad} \longrightarrow \mathcal{Y}_L^{ad} \text{ and } \Psi_{I,W} : Z_{I,L}^W \longrightarrow \mathcal{Y}_L^{ad}.$$

Put

$$\mathbb{Z}_{g,I} := (\Phi_{g,I})_* (\Phi_{g,I}^* \mathbb{Z}) \text{ and } \mathbb{Z}_{Z_{I,L}^W} := (\Psi_{I,W})_* (\Psi_{I,W}^* \mathbb{Z})$$

Let $\mathcal{C}_I$ be the category of compact open disjoint coverings of G/P_I where the morphisms are given by the refinement-order. For a covering $c = (W_j)_j \in \mathcal{C}_I$, we denote by $\mathbb{Z}_c$ the sheaf on $\mathcal{Y}_L^{ad}$ defined by the image of the natural morphism of sheaves

$$\bigoplus_{W_j \in \mathcal{C}_I} \mathbb{Z}_{Z_{I,L}^{W_j}} \hookrightarrow \prod_{g \in G/P_I} \mathbb{Z}_{g,I}.$$

We put

$$\prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} = \varinjlim_{c \in \mathcal{C}_I} \mathbb{Z}_c, \tag{16}$$

We obtain the following complex $C_L^\bullet$ of sheaves on $\mathcal{Y}_L^{ad}$,

$$\begin{aligned} 0 \to \mathbb{Z} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 2}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = i}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \cdots \\ \cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = d-1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \prod_{g \in G/P_\emptyset}{}' \mathbb{Z}_{g,\emptyset} \to 0. \end{aligned} \tag{17}$$

Concerning the next statement see [O, Thm 2.1.1].

Theorem 3.1. *The complex $C^\bullet_L$ is acyclic.* □

We consider the morphism of topoi $\nu : \widetilde{(\mathbb{P}^d_{\mathbb{C}_p})^{ad}_{\text{proét}}} \to \widetilde{(\mathbb{P}^d_{\mathbb{C}_p})^{ad}_{\text{ét}}}$. By pulling back the complex $i_*(C^\bullet_{\mathbb{C}_p})$ to $(\mathbb{P}^d_{\mathbb{C}_p})^{ad}_{\text{proét}}$ where $i : \mathcal{Y}^{ad}_{\mathbb{C}_p} \hookrightarrow (\mathbb{P}^d_{\mathbb{C}_p})^{ad}$ is the inclusion, we get a resolution of the pro-étale sheaf $i_*(\mathbb{Z}_{\mathcal{Y}^{ad}_{\mathbb{C}_p}})$ on $(\mathbb{P}^d_{\mathbb{C}_p})^{ad}$ since ν^* is an exact functor. We denote this complex for simplicity by the same symbols. In fact, we could have defined this complex directly on the pro-étale site as the sheaves $\mathbb{Z}$ are constant. In this section we evaluate the spectral sequence which is induced by the complex (17) applied to $\mathrm{Ext}^*_{\widetilde{(\mathbb{P}^d_{\mathbb{C}_p})^{ad}_{\text{proét}}}}(i_*(-), \mathbb{Q}_p)$. In the following we also simply write $\mathrm{Ext}^i(\cdot,\cdot)$ for the i^{th} Ext group in the category $\widetilde{(\mathbb{P}^d_{\mathbb{C}_p})^{ad}_{\text{proét}}}$.

As usual there is the identification

$$\mathrm{Ext}^*(i_*(\mathbb{Z}_{\mathcal{Y}^{ad}_{\mathbb{C}_p}}), \mathbb{Q}_p) = H^*_{\mathcal{Y}^{ad}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p).$$

Further, we have $H^*_{\mathcal{Y}^{ad}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) = H^*_{\mathcal{Y}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ by the very definition of pro-étale cohomology.

Proposition 3.2. *For all subsets $I \subset \Delta$, there is an isomorphism*

$$\mathrm{Ext}^*(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathbb{Q}_p) = \varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^*_{gY_{I,\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p).$$

Proof. Consider the family

$$\left\{ gP_I^n \mid g \in G_0, n \in \mathbb{N} \right\}$$

of compact open subsets in G/P_I which yields cofinal coverings in $\mathcal{C}_I$. We obtain by (16) the identity

$$\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I} = \varinjlim_{c\in\mathcal{C}_I} \mathbb{Z}_c = \varinjlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} \mathbb{Z}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}.$$

Choose an injective resolution $\mathcal{I}^\bullet$ of the sheaf $\mathbb{Q}_p$. We get

$$\begin{aligned} \mathrm{Ext}^i(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathbb{Q}_p) &= H^i(\mathrm{Hom}(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathcal{I}^\bullet)) \\ = H^i(\mathrm{Hom}(\varinjlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} i_*(\mathbb{Z}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}), \mathcal{I}^\bullet)) &= H^i(\varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} \mathrm{Hom}(i_*(\mathbb{Z}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}), \mathcal{I}^\bullet)) \\ = H^i(\varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^0_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathcal{I}^\bullet)) & . \end{aligned}$$

We make use of the following lemma. Here $\varprojlim^{(r)}_{n\in\mathbb{N}}$ is the r-th right derived functor of $\varprojlim_{n\in\mathbb{N}}$.

Lemma 3.3. *Let $\mathcal{I}$ be an injective sheaf on the proétale site of $(\mathbb{P}^d_{\mathbb{C}_p})^{ad}$. Then*

$$\varprojlim^{(r)}_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^0_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathcal{I}) = 0 \text{ for } r\geq 1.$$

Proof. The proof works in the same way as in [O, Lemma 2.2.2]. □

Thus we get by applying a spectral sequence argument (note that $\varprojlim^{(r)} = 0$ for $r\geq 2$ [Je]) short exact sequences, $i\in\mathbb{N}$,

$$0\to \varprojlim_n{}^{(1)} \bigoplus_{g\in G_0/P_I^n} H^{i-1}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p)\to \mathrm{Ext}^i(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\mathbb{Q}_p)\to \varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^{i}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p)\to 0.$$

Lemma 3.4. *The projective system $\left(\bigoplus_{g\in G_0/P_I^n} H^{i-1}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p)\right)_{n\in\mathbb{N}}$ consists of $\mathbb{Q}_p$-Fréchet spaces and satisfies the (topological) Mittag-Leffler property for all $i\geq 1$.*

Proof. The proof works in the same way as in [O, Lemma 2.2.3]. Additionally one replaces the Zariski local cohomology groups in loc.cit by the extensions in Proposition 2.4 and considers the corresponding LHSs and RHSs separately. Whereas the situation of the RHSs is trivial the LHSs are treated in the same as in loc.cit. □

We deduce from [EGAIII] 13.2.4 that

$$\varprojlim^{(1)}_{n\in\mathbb{N}} \Big(\bigoplus_{g\in G_0/P_I^n} H^{i-1}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p)\Big)_{n\in\mathbb{N}} = 0.$$

We obtain the identity

$$\mathrm{Ext}^i(i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\mathbb{Q}_p) \cong \varprojlim_{n\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^n} H^{i}_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p).$$

On the other hand, we have

$$\bigcap_{n\in\mathbb{N}} Z^{P_I^n}_{I,\mathbb{C}_p} = \bigcap_{n\in\mathbb{N}} Y_{I,\mathbb{C}_p}(\epsilon_n)^{ad} = Y^{ad}_{I,\mathbb{C}_p}.$$

Again, by applying [O, Proposition 1.3.3][10], we deduce the identity

$$\varprojlim_n H^*_{Z^{P_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p) \quad = \quad \varprojlim_n H^*_{Y_{I,\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p).$$

We get

$$\varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^*_{Z^{gP_I^n}_{I,\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p) \quad = \quad \varprojlim_n \bigoplus_{g\in G_0/P_I^n} H^*_{gY_{I,\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p},\mathbb{Q}_p)$$

Thus the statement of our proposition is proved. □

[10] The assumption that the sheaf is coherent is not needed here

We analyze now the spectral sequence

$$E_1^{-p,q} = \mathrm{Ext}^q(\bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=p+1}} i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), \mathbb{Q}_p) \Rightarrow \mathrm{Ext}^{-p+q}(i_*(\mathbb{Z}_{\mathcal{Y}^{ad}_{\mathbb{C}_p}}), \mathbb{Q}_p) = H^{-p+q}_{\mathcal{Y}^{ad}_{\mathbb{C}_p}}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \tag{18}$$

induced by the acyclic complex (17) in Theorem 3.1. By applying Proposition 2.4 the term $H^q_{gY_{I,\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ which appears in $E_1^{-p,q}$ as a direct summand is for $q \geq 2(d-j(I))$ a an extension

$$0 \to F \to H^q_{gY_{I,\mathbb{C}_p}(\epsilon_n)}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p) \to G \to 0$$

where

$$F = H^{d-j(I)}_{g\mathbb{P}^{j(I)}_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,q-(d-j(I))-1}/D^{\dagger,(q-(d-j(I))-1})\hat{\otimes}_K\mathbb{C}_p(q-(d-j(I)))$$

and

$$G = H^q_{dR}(\mathbb{P}^d_K, \mathbb{Q}_p)(-\frac{q}{2}).$$

It is equal to

$$H^{d-j(I)}_{g\mathbb{P}^{j(I)}_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,q-(d-j(I))-1}/D^{\dagger,(q-(d-j(I))-1})\hat{\otimes}_K\mathbb{C}_p(q-(d-j(I)))$$

for $2(d-j(I)) > q \geq d-j(I)$. Hence we may write $E_1^{\bullet,\bullet}$ as an extension of double complexes

$$0 \to F_1^{\bullet,\bullet} \to E_1^{\bullet,\bullet} \to G_1^{\bullet,\bullet} \to 0.$$

The F_1-term splits as a direct sum $F_1 = \bigoplus_{s=1}^d F_{1,s}$ where $F_{1,s}$ is the sub double complex with fixed Tate twist $s = q-(d-j(I))$, i.e.,

$$F^{p,q}_{1,s} = \varprojlim_n \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=p+1}} \bigoplus_{g\in G_0/P_I^n} H^{q-s}_{g\mathbb{P}^{d-(q-s)}_K(\epsilon_n)}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K\mathbb{C}_p(s).$$

Up to the tensor factor $\hat{\otimes}_K\mathbb{C}_p(s)$ the object $F_{1,s}$ is the E_1-term of a spectral sequence considered in [O] with respect to the equivariant sheaf $\Omega^{\dagger,s-1}/D^{\dagger,s-1}$. The computation in [O, p. 633] shows that the only non-vanishing entries $F^{p,q}_{2,s}$ are given by the tuples $(p,q) = (-j+1, d-s+j), j = 1,\ldots,d$ and that $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \Omega^{\dagger,s-1}/D^{\dagger,s-1})\hat{\otimes}_K\mathbb{C}_p(s)$ is a successive extension of these non-vanishing objects.

Concerning the double complex $G_1^{\bullet,\bullet}$ there are the following (non-trivial) rows

$$\begin{aligned} G_1^{\bullet,2d} \quad : (\varprojlim_n \bigoplus_{g\in G_0/P^n_\emptyset} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta \\ \#I=1}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta \\ \#I=2}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p \\ \leftarrow \ldots \leftarrow \bigoplus_{\substack{I\subset\Delta \\ \#I=d-1}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p)(-d) \end{aligned}$$

$$G_1^{\bullet,2(d-1)} : \Big(\varprojlim_n \bigoplus_{g\in G_0/P^n_{(2,1,\ldots,1)}} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=2\\ \alpha_0\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=3\\ \alpha_0\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p$$

$$\leftarrow \ldots \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=d-1\\ \alpha_0\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p\Big)(-(d-1))$$

$$\vdots$$

$$G_1^{\bullet,2j} : \Big(\varprojlim_n \bigoplus_{g\in G_0/P^n_{(d+1-j,1,\ldots,1)}} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+2\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p$$

$$\leftarrow \ldots \leftarrow \bigoplus_{\substack{I\subset\Delta\\ \#I=d-1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \varprojlim_n \bigoplus_{g\in G_0/P^n_I} \mathbb{Q}_p\Big)(-j)$$

$$\vdots$$

$$G_1^{0,2} : \varprojlim_n \bigoplus_{g\in G_0/P^n_{(d,1)}} \mathbb{Q}_p(-1).$$

Here, the very left term in each row $G_1^{\bullet,2j}$ sits in degree $-j+1$. We can rewrite these complexes in terms of induced representations. Here we abbreviate

$$(d+1-j,1^j) := (d+1-j,1,\ldots,1)$$

for any decomposition $(d+1-j,1,\ldots,1)$ of $d+1$. Let $\mathrm{Ind}_P^{\infty,G}$ denote the (unnormalized) smooth induction functor for a parabolic subgroup $P\subset G$. The dual of the row $G_1^{\bullet,2j}$ coincides with the complex

$$\mathrm{Ind}^{\infty,G}_{P_{(d+1-j,1^j)}}\mathbb{Q}_p \to \bigoplus_{\substack{I\subset\Delta\\ \#I=d-j+1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{\infty,G}_{P_I}\mathbb{Q}_p \to \ldots \to \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta\setminus I)=1\\ \alpha_0,\ldots,\alpha_{d-j-1}\in I}} \mathrm{Ind}^{\infty,G}_{P_I}\mathbb{Q}_p.$$

Each of the complexes $G_1^{\bullet,2j}$, $j=1,\ldots,d$, is acyclic apart from the very left and right position [DOR, Thm. 7.1.9]. Indeed, let

$$v^G_{P_{(d+1-j,1^j)}}(\mathbb{Q}_p) := \mathrm{Ind}^{\infty,G}_{P_{(d+1-j,1^j)}}\mathbb{Q}_p / \sum_{Q\supsetneq P_{(d+1-j,1^j)}} \mathrm{Ind}^{\infty,G}_Q\mathbb{Q}_p$$

be the smooth generalized Steinberg representation with respect to the parabolic subgroup $P_{(d+1-j,1^j)}$. This is an irreducible smooth G-representation, cf. [BW, ch. X]. We deduce that the only non-vanishing entries in $G_2^{p,q}$ are given by the indices $(p,q)=(-j+1,j),\ j=1,\ldots,d$, and $(p,q)=(0,2j),\ j\geq 2$. Here we get for $j\geq 2$,

$$G_2^{-j,j+1}=v^G_{P_{(d+1-j,1^j)}}(\mathbb{Q}_p)(-j)' \text{ and } G_2^{0,2j}=\mathbb{Q}_p(-j)$$

and

$$G_2^{0,2}=(\mathrm{Ind}^{\infty,G}_{P_{(d,1)}}\mathbb{Q}_p)'(-1).$$

Considered as $U(\mathfrak{t})$-modules the objects in G_2 consist of copies of the trivial representation. Again as in Lemma 2.3 the contributions of F_2 do not contain the trivial representation. Hence there are no non-trivial homomorphisms between G and F and we get in particular $E_2=E_\infty$.

For any integer $s\geq 1$, let $V_s^\bullet=V_s^{-d+1}\supset V_s^{-d+2}\supset\cdots\supset V_s^{-1}\supset V_s^0\supset V_s^1=(0)$ be the canonical filtration on $V_s^{-d+1}=H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\Omega^{\dagger,s-1}/D^{\dagger,s-1})$ defined by the spectral sequence

$$E_{1,s}^{-p,q}=\mathrm{Ext}^q(\bigoplus_{\substack{I\subset\Delta\\ |\Delta\setminus I|=p+1}} i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}),\Omega^{\dagger,s-1}/D^{\dagger,s-1})\Rightarrow H^{-p+q}_{\mathcal{Y}^{ad}}(\mathbb{P}^d_{\mathbb{C}_p},\Omega^{\dagger,s-1}/D^{\dagger,s-1})$$

induced by the acyclic complex (17) in Theorem 3.1. In [O, Lemma 2.7] we proved that in the case of vector bundles that these subspaces are closed. On the other hand, we proved [O2] that the filtration behaves functorial with respect to morphisms of vector bundles[11]. Applying this fact to the (continuous) morphism $d^s:\Omega^s\to\Omega^{s+1}$ we deduce that the subspaces V_s^j are closed in the K-Fréchet space $H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\Omega^{\dagger,s-1}/D^{\dagger,s-1})$.

Consider finally for $\mathcal{F}^\dagger=\Omega^{\dagger,s-1}/D^{\dagger,s-1}$ the topological exact G-equivariant sequence of K-Fréchet spaces

$$\ldots\to H^0(\mathbb{P}^d_K,\mathcal{F}^\dagger)\to H^0(\mathcal{X},\mathcal{F}^\dagger)\xrightarrow{p} H^1_{\mathcal{Y}}(\mathbb{P}^d_K,\mathcal{F}^\dagger)\to H^1(\mathbb{P}^d_K,\mathcal{F}^\dagger)\to\ldots.$$

By Proposition 1.5 the terms $H^i(\mathbb{P}^d,\mathcal{F}^\dagger)=0$ vanish for $i\geq 0$. Hence p is an isomorphism. For $i=0,\ldots,d$, we set

$$Z_s^i:=p^{-1}(V_s^{-d+i+1}).$$

Thus we get a G-equivariant filtration by closed K-Fréchet spaces

$$Z_s^0\supset Z_s^1\supset\cdots\supset Z_s^{d-1}\supset Z_s^d$$

on $Z_0^d=H^0(\mathcal{X},\mathcal{F}^\dagger)$.

Summarizing the evaluation of the spectral sequence we obtain the following theorems mentioned in the introduction.

[11]In loc.cit. we stated the lemma for morphisms of vector bundles. But as one see without any difficulties from the proof this fact is true for arbitrary sheaf morphisms

Theorem 3.5. *For $j = 1, \ldots, d$, the p-adic pro-étale cohomology groups of $\mathcal{X}_{\mathbb{C}_p}$ are extensions of continuous $G \times \Gamma_K$-representations*

$$0 \to (\Omega^{\dagger,s-1}/D^{\dagger,s-1})(\mathcal{X})\hat{\otimes}_K \mathbb{C}_p(-s) \to H^s(\mathcal{X}_{\mathbb{C}_p}, \mathbb{Q}_p) \to v^G_{P_{(d-s+1,1,\ldots,1)}}(\mathbb{Q}_p)'(-s) \to 0.$$

Proof. In order to complete the proof, we mention that the contributions $G_2^{0,2k}$, $k \geq 2$, are mapped isomorphically to the cohomology groups $H^{2k}(\mathbb{P}^d_{\mathbb{C}_p}, \mathbb{Q}_p)$ in the long exact cohomology sequence (14). For $k = 1$, we have a surjection $G_2^{0,1} \to \mathbb{Q}_p$ whose kernel is isomorphic to $v^G_{P_{(d,1)}}(\mathbb{Q}_p)(-1)'$. □

Proposition 3.6. *Let $s \geq 0$. Then $H^n(\mathcal{X}, D^{\dagger,s}) = 0$ for all $n > 0$.*

Proof. We may compute the local cohomology groups $H^i_{\mathcal{Y}}(\mathbb{P}^d_K, D^{\dagger,s})$ by the spectral sequence

$$E_1^{-p,q} = \mathrm{Ext}^q(\bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=p+1}} i_*(\prod_{g\in G/P_I}{}' \mathbb{Z}_{g,I}), D^{\dagger,s}) \Rightarrow H^{-p+q}_{\mathcal{Y}^{ad}}(\mathbb{P}^d_K, D^{\dagger,s})$$

induced by the acyclic complex (17) in Theorem 3.1. As we learned in section 1 the computation proceeds in the same way as for equivariant vector bundles. Here we saw that in loc.cit. (p. 633) that $H^i_{\mathcal{Y}}(\mathbb{P}^d_K, D^{\dagger,s}) = H^i(\mathbb{P}^d_K, D^{\dagger,s})$ for all $i \geq 2$. Moreover, for $i = 1$ we have an epimorphism $H^i_{\mathcal{Y}}(\mathbb{P}^d_K, D^{\dagger,s}) \to H^1(\mathbb{P}^d_K, D^{\dagger,s})$. By considering the corresponding long exact cohomology sequence we deduce the claim. □

Remark 3.7. It follows that $H^0(\mathcal{X}, \Omega^{\dagger,s}/D^{\dagger,s}) = \Omega^{\dagger,s}(\mathcal{X})/D^{\dagger,s}(\mathcal{X})$ for any integer $s \geq 0$. In particular, our result for the pro-étale cohomology of $\mathcal{X}$ coincides with the formula in [CDN]. Moreover, the strong dual $H^0(\mathcal{X}, \Omega^{\dagger,s}/D^{\dagger,s})'$ is as a closed subspace of $H^0(\mathcal{X}, \Omega^s)'$ a locally analytic G-representation, as well. The same holds henceforth for the quotient $H^0(\mathcal{X}, D^{\dagger,s})'$. In particular, both representations are strongly admissible as an extension of strongly admissible representations and in particular admissible [ST1].

Concerning the structure of the term on the left hand side we can make precise it by [O, OS]. Here we refer to [OS] for the definition of the bi-functors $\mathcal{F}^G_P$. Here we use Remark 1.10 in order to see that algebraic local cohomology $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^{s-1}/D^{\dagger,s-1})$ is an object of the category $\mathcal{O}^P$.

Theorem 3.8. *For any fixed integer $s = 1, \ldots, d$, there is a descending filtration $(Z^i_s)_{i=0,\ldots,d}$ on $Z^0_s = H^0(\mathcal{X}, \Omega^{\dagger,s-1}/D^{\dagger,s-1})$ by closed subspaces together with isomorphisms of locally analytic G-representations*

$$(Z^i_s/Z^{i+1}_s)' \cong \mathcal{F}^G_{P_{(i+1,d-i)}}(H^{d-i}_{\mathbb{P}^i_K}(\mathbb{P}^d_K, \Omega^{s-1}/D^{s-1}), St_{d-i}), i = 0, \ldots, d-1.$$

where St_{d-j} is the Steinberg representation of $\mathrm{GL}_{d-j}(K)$ considered as representation of $L_{(j+1,d-j)}$. □

Remark 3.9. In the case of equivariant vector bundles we used in the above formula rather the reduced local cohomology $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) := \ker\big(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{F})\big)$. Concerning the compatibility we note that for $\mathcal{F} = \Omega^{s-1}/D^{s-1}$ we have $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) = H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F})$ by Lemma 1.5.

Remark 3.10. *The above approach for the determination of the p-adic pro-étale cohomology works also for the ℓ-adic pro-étale cohomology with $\ell \neq p$. In this case one gets as cited in* [CDN, Thm. 1.2] $H^s(\mathcal{X}_{\mathbb{C}_p}, \mathbb{Q}_\ell) = v^G_{P_{(d-s+1,1,\ldots,1)}}(\mathbb{Q}_\ell)'(-s)$ *for all* $s \geq 0$.

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