ON SOME PROPERTIES OF THE FUNCTORS $\mathcal{F}_P^G$ FROM LIE ALGEBRA TO LOCALLY ANALYTIC REPRESENTATIONS

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Abstract. For a split reductive group G over a finite extension L of $\mathbb{Q}_p$, and a parabolic subgroup $P \subset G$ we examine certain properties of the functors $\mathcal{F}_P^G$ introduced in [22]. We discuss the aspects of faithfulness, projective and injective objects, Ext-groups and some kind of adjunction formula.

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1. Introduction

This paper is a continuation of the work done in [22]. In loc.cit. we constructed locally analytic representations in K-vector spaces of a p-adic reductive Lie group G by introducing certain bi-functors $\mathcal{F}_P^G : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}_K^{\infty,a}(L_P) \to \mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$. Here $P \subset G$ is a parabolic subgroup, $\mathfrak{p} = \mathrm{Lie}(P)$ its Lie algebra, and $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is a subcategory of the BGG-category[1] $\mathcal{O}^{\mathfrak{p}}$. Furthermore, $\mathrm{Rep}^{\infty,a}_K(L_P)$ is the category of smooth admissible representations of the Levi group L_P. We proved among others that these functors are exact in both arguments and gave a criterion for the irreducibility of those objects lying in the image of $\mathcal{F}^G_P$. Using these properties one can derive a Jordan-Hölder series of any locally analytic representation $\mathcal{F}^G_P(M,V)$ from the corresponding series of M and V.

In this paper we want to concentrate on properties of these functors for a split group G. We shall show that they behave fully faithful if the objects of the category $\mathcal{O}^{\mathfrak{p}}$ are integral (i.e., they are contained in the subcategory $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ of modules such that all non-zero weight spaces belong to integral weights) or generalized Verma modules. This aspect has been considered by Morita in the case of $G = \mathrm{SL}_2$, cf. [15, 16, 17, 18] and Féaux de Lacroix [7]. Concretely, we shall show:

Theorem 1. *For any $M_1, M_2 \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ the canonical map*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1,M_2) &\to \mathrm{Hom}_G(\mathcal{F}^G_P(M_2),\mathcal{F}^G_P(M_1)) \\ f &\mapsto \mathcal{F}^G_P(f) \end{aligned}$$

is bijective (where $\mathcal{F}^G_P(M) := \mathcal{F}^G_P(M,\mathbf{1})$ for the trivial L_P-representation $\mathbf{1}$).

To prove this statement we make use of the (naive) topological Jacquet functor of locally analytic representations and more generally of an analogue of the Casselman-Jacquet functor $\mathcal{G}^G_P : U \mapsto \varinjlim_k H^0(\mathfrak{u}_P^k, U')$ which behaves almost like a section for $\mathcal{F}^G_P$. This topic is a continuation of the theory started in [20, 2].

By the above theorem we can characterize projective and injective objects which lie in the essential image $\mathcal{F}^P_{\mathrm{alg}}$ of the functor $\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \to \mathrm{Rep}^{\mathrm{loc.an.}}_K(G)$. More precisely, it follows that $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is projective (resp. injective) as an object in $\mathcal{O}^{\mathfrak{p}}$ if and only if $\mathcal{F}^G_P(M)$ is injective (resp. projective) in $\mathcal{F}^P_{\mathrm{alg}}$. Hence if we denote for a given integer $i \geq 0$, by $\mathrm{Ext}^i_{\mathcal{F}^P_{\mathrm{alg}}}$ the corresponding Ext-group then the natural morphism

$$\mathrm{Ext}^i_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1,M_2) \to \mathrm{Ext}^i_{\mathcal{F}^P_{\mathrm{alg}}}(\mathcal{F}^G_P(M_2),\mathcal{F}^G_P(M_1))$$

is bijective. These Ext-groups are of course different from those considered more generally in the category of locally analytic G-representations, cf. [14]. These can be seen as an

[1] Deviating from the classical situation of modules over the enveloping algebra of a complex semisimple Lie algebra, we introduced in [22] a version of this category which consists of modules over the enveloping algebra $U(\mathfrak{g} \otimes_L K)$.

analogue of relating the groups $\mathrm{Ext}^i_{\mathfrak{g}}(M_1, M_2)$ and $\mathrm{Ext}^i_{\mathcal{O}}(M_1, M_2)$ for two objects $M_1, M_2 \in \mathcal{O}$.
For considering also smooth contributions in this context, we extend $\mathcal{F}_P^G$ to a bi-functor $\mathcal{F}_P^G : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}_K^{\infty,\infty}(L_P) \to \mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ where $\mathrm{Rep}_K^{\infty,\infty}(L_P)$ denotes the category of smooth L_P-representations of countable dimension. The latter object has enough injectives and projectives. We let ${}^\infty\overline{\mathcal{F}^P}$ be the smallest abelian subcategory of $\mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ which contains the essential images of all bi-functors $\mathcal{F}_Q^G$ with $Q \supset P$. It turns out that ${}^\infty\overline{\mathcal{F}^P}$ has enough injective and projective objects. More precisely, we deduce this fact from the following statement.

Theorem 2: *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be a projective (resp. injective) object and let V be an injective (resp. projective) smooth L_P-representation of countable dimension. Then $\mathcal{F}_P^G(M,V)$ is injective (resp. projective) in ${}^\infty\overline{\mathcal{F}^P}$.*

As an application we are able to determine extensions of generalized Steinberg representations in the category ${}^\infty\overline{\mathcal{F}^B}$. For a parabolic subgroup $P \subset G$ the associated representation is given by the quotient $V_P^G = \mathrm{Ind}_P^G(\mathbf{1})/\sum_{\mathbf{Q}\supsetneq\mathbf{P}} \mathrm{Ind}_{\mathbf{Q}}^{\mathbf{G}}(\mathbf{1})$ where $\mathrm{Ind}_P^G(\mathbf{1})$ is the locally analytic induction with respect to the trivial P-representation $\mathbf{1}$. For a subset $I \subset \Delta$ of a fixed set of simple roots, let P_I be the corresponding standard parabolic subgroup. The next result has the same structure as in the smooth setting [5, 19].

Theorem 3: Let G be semi-simple. *Let $I, J \subset \Delta$. Then*

$$\mathrm{Ext}^i_{{}^\infty\overline{\mathcal{F}^B}}(V_{P_I}^G, V_{P_J}^G) = \begin{cases} K\,; & |I \cup J \setminus I \cap J| = i \\ (0)\,; & \textit{otherwise} \end{cases} .$$

Finally we deduce from the naive Jacquet functor applied to different Borel subgroups lying in the same apartment an adjunction formula (in the sense of Bernstein). Let U_B be the unipotent radical of a fixed Borel subgroup B. If we denote for a given G-representation V by V_{U_B} its (naive) topological Jacquet module then the map below is defined as follows: For an element f of the LHS, the corresponding element on the RHS is given by the composition of the inclusion $((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0} \hookrightarrow \mathrm{Ind}_{\overline{B}}^G(\chi^{-1})_{U_B}$ with the map $f_{U_B} : \mathrm{Ind}_{\overline{B}}^G(\chi^{-1})_{U_B} \to \mathcal{F}_{B^w}^G(M)_{U_B}$.

Theorem 4: *Let χ be a dominant algebraic character of T and let $M \in \mathcal{O}^{\mathfrak{b}^w}_{\mathrm{alg}}$ be a highest weight module. Then*

$$\mathrm{Hom}_G(\mathrm{Ind}_{\overline{B}}^G(\chi^{-1}), \mathcal{F}_{B^w}^G(M)) = \mathrm{Hom}_T(((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0}, \mathcal{F}_{B^w}^G(M)_{U_B}) .$$

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Notation and conventions. We denote by p a prime number and consider fields $L \subset K$ which are both finite extensions of $\mathbb{Q}_p$. Let O_L and O_K be the rings of integers of L, resp. K, and let $|\cdot|_K$ be the absolute value on K such that $|p|_K = p^{-1}$. The field L is our "base field", whereas we consider K as our "coefficient field". For a locally convex K-vector space V we denote by V'_b its strong dual, i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence. Sometimes, in particular when V is finite-dimensional, we simplify notation and write V' instead of V'_b. All finite-dimensional K-vector spaces are equipped with the unique Hausdorff locally convex topology.

We let $\mathbf{G}_0$ be a split reductive group scheme over O_L and $\mathbf{T}_0 \subset \mathbf{B}_0 \subset \mathbf{G}_0$ a maximal split torus and a Borel subgroup scheme, respectively. We denote by $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ the base change of $\mathbf{G}_0$, $\mathbf{B}_0$ and $\mathbf{T}_0$ to L. By $G_0 = \mathbf{G}_0(O_L)$, $B_0 = \mathbf{B}_0(O_L)$, etc., and $G = \mathbf{G}(L)$, $B = \mathbf{B}(L)$, etc., we denote the corresponding groups of O_L-valued points and L-valued points, respectively. Standard parabolic subgroups of $\mathbf{G}$ (resp. G) are those which contain $\mathbf{B}$ (resp. B). For each standard parabolic subgroup $\mathbf{P}$ (or P) we let $\mathbf{L_P}$ (or L_P) be the unique Levi subgroup which contains $\mathbf{T}$ (resp. T) and $\mathbf{U_P}$ (or U_P) its unipotent radical. Finally, Gothic letters $\mathfrak{g}$, $\mathfrak{p}$, etc., will denote the Lie algebras of $\mathbf{G}$, $\mathbf{P}$, etc.: $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, $\mathfrak{t} = \mathrm{Lie}(\mathbf{T})$, $\mathfrak{b} = \mathrm{Lie}(\mathbf{B})$, $\mathfrak{p} = \mathrm{Lie}(\mathbf{P})$, $\mathfrak{l}_P = \mathrm{Lie}(\mathbf{L_P})$, etc.. Base change to K is usually denoted by the subscript $_K$, for instance, $\mathfrak{g}_K = \mathfrak{g} \otimes_L K$.

We make the general convention that we denote by $U(\mathfrak{g})$, $U(\mathfrak{p})$, etc., the corresponding enveloping algebras, *after base change to* K, i.e., what would be usually denoted by $U(\mathfrak{g}) \otimes_L K$, $U(\mathfrak{p}) \otimes_L K$, and so on. All distribution algebras appearing in this paper are tacitly assumed to be distribution algebras with coefficient field K, and we write $D(H)$ for the distribution algebra $D(H,K)$. For a real number $r<1$ with $r \in p^{\mathbb{Q}}$, and H compact we let $D_r(H)$ be the Banach space completion in the sense of [26] so that $D(H) = \varprojlim_r D_r(H)$. Finally, $\mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ denotes the category of locally analytic representations of G on barreled locally convex Hausdorff K-vector spaces.

2. A review of the categories $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and the functors $\mathcal{F}_P^G$

For the convenience of the reader we recall here the definitions of the categories $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$, as well the functors $\mathcal{F}_P^G$, and state some of the key results about those, cf. [22].

2.1. **The categories $\mathcal{O}^{\mathfrak{p}}$ and $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$.** For a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ we define $\mathcal{O}^{\mathfrak{p}}$ to be the full subcategory of the category of all $U(\mathfrak{g})$-modules consisting of objects M which possess the following properties:

(1) M is a finitely generated $U(\mathfrak{g})$-module.
(2) M decomposes as a direct sum of one-dimensional $\mathfrak{t}_K$-modules.
(3) The action of $\mathfrak{p}$ on M is locally finite, i.e., for every $m \in M$ the K-subspace $U(\mathfrak{p}).m \subset M$ is finite-dimensional.

We also put $\mathcal{O} = \mathcal{O}^{\mathfrak{b}}$. Given $M \in \mathrm{ob}(\mathcal{O})$ and $\lambda \in \mathfrak{t}_K^*$ we set

$$M_\lambda = \{m \in M \mid \forall \mathfrak{x} \in \mathfrak{t} : \mathfrak{x}.m = \lambda(\mathfrak{x})m\} .$$

We call $\lambda \in \mathfrak{t}_K^*$ *algebraic* if it is in the image of the canonical homomorphism

$$X^*(\mathbf{T}) = \mathrm{Hom}_{\mathrm{alg.\ gps}}(\mathbf{T}, \mathbb{G}_{m,L}) \longrightarrow \mathfrak{t}_K^* \ , \ \ \chi \mapsto \mathrm{d}\chi \ .$$

We define $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ to be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ which consists of $M \in \mathrm{ob}(\mathcal{O}^{\mathfrak{p}})$ such that $M_\lambda \neq 0$ implies that λ is algebraic.

2.2. **The functors $\mathcal{F}_P^G$.** We consider an object M of the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. By [22, 3.2], the locally finite action of $\mathfrak{p}$ on M lifts canonically to a locally finite algebraic action of the algebraic group $\mathbf{P}_K$. Since M is finitely generated over $U(\mathfrak{g})$, we can choose a $\mathbf{P}_K$-subrepresentation $W \subset M$ which generates M as a $U(\mathfrak{g})$-module. Thus we get an exact sequence

$$0 \longrightarrow \mathfrak{d} \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \longrightarrow M \longrightarrow 0 \tag{2.2.1}$$

Let $\mathrm{Ind}_P^G(W')$ be the locally analytic induction of the dual space W'. There is a pairing

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle_{C^{an}(G,K)}: \ \big(D(G)\otimes_{D(P)} W\big)\otimes_K \mathrm{Ind}^G_P(W') & \longrightarrow & C^{an}(G,K) \\ (\delta\otimes w)\otimes f & \mapsto & \Big[g\mapsto \delta(x\mapsto f(gx)(w))\Big] \end{array}$$

which extends for any smooth admissible L_P representation, to a pairing

$$\langle\cdot,\cdot\rangle_{C^{an}(G,V)} : \big(D(G)\otimes_{D(P)} (W\otimes_K V')\big)\otimes_K \mathrm{Ind}^G_P(W'\otimes_K V) \longrightarrow C^{an}(G,K)\ . \tag{2.2.2}$$

Here and in the following we always equip an admissible smooth representation V with the finest locally convex topology (the final topology with respect to which all inclusion maps $V_1 \hookrightarrow V$, where $V_1 \subset V$ is finite-dimensional, are continuous, cf. [24, ch. I, §5, E]). As W is finite-dimensional, the topology of the inductive tensor product $W\otimes_{K,\iota} V'$ coincides with that of the projective tensor product $W\otimes_{K,\pi} V'$, and we thus write simply $W\otimes_K V'$ for this topological vector space (cf. [24, §17] for tensor products of topological vector spaces and the notation used here). We set

$$\mathrm{Ind}^G_P(W'\otimes_K V)^{\mathfrak{d}} = \{f\in \mathrm{Ind}^G_P(W'\otimes_K V) \mid \forall \mathfrak{z}\in\mathfrak{d} : \langle \mathfrak{z}, f\rangle_{C^{an}(G,V)} = 0\}\ .$$

Then $\mathrm{Ind}^G_P(W'\otimes_K V)$ carries the structure of a K-vector space of compact type, and it is easily seen that the linear maps $\mathrm{Ind}^G_P(W'\otimes_K V)\to C^{an}(G,K)$ defined by the pairing (2.2.2) are continuous. Hence $\mathrm{Ind}^G_P(W'\otimes_K V)^{\mathfrak{d}}$ is a closed subspace of $\mathrm{Ind}^G_P(W'\otimes_K V)$, and we equip it with its subspace topology. As such it is again of compact type [25, 1.2]. By [6, 2.1.2], $\mathrm{Ind}^G_P(W'\otimes_K V)$ is an admissible locally analytic representation of G, and $\mathrm{Ind}^G_P(W'\otimes_K V)^{\mathfrak{d}}$ is thus again an admissible locally analytic representation of G, cf. [26, 6.4].

If $W_1\subset W_2$ are two finite-dimensional $\mathfrak{p}_K$-stable subspaces which generate M as a $U(\mathfrak{g})$-module, then there is a canonical continuous morphism of G-representations

$$\mathrm{Ind}^G_P(W_2'\otimes_K V)^{\mathfrak{d}_2} \longrightarrow \mathrm{Ind}^G_P(W_1'\otimes_K V)^{\mathfrak{d}_1}\ , \tag{2.2.3}$$

where $\mathfrak{d}_i$ is defined by the corresponding exact sequence 2.2.1. By [22, 4.5] the map 2.2.3 is actually an isomorphism of topological vector spaces. We thus see that the formation of $\mathrm{Ind}^G_P(W'\otimes_K V)^{\mathfrak{d}}$ is independent of the choice of W, and we put

$$\mathcal{F}^G_P(M,V) \ = \ \mathrm{Ind}^G_P(W'\otimes_K V)^{\mathfrak{d}}\ . \tag{2.2.4}$$

Denote by $\mathrm{Rep}_K^{\infty,a}(L_P)$ the category of smooth admissible representations of L_P and by $\mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ the category of locally analytic representations of G on K-vector spaces. Then we have a bi-functor

$$\mathcal{F}_P^G : \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}} \times \mathrm{Rep}_K^{\infty,a}(L_P) \longrightarrow \mathrm{Rep}_K^{\mathrm{loc.an.}}(G) \; . \tag{2.2.5}$$

If $V = \mathbf{1}$ denotes the trivial representation, then we simply write $\mathcal{F}_P^G(M)$ for $\mathcal{F}_P^G(M,V)$.

Theorem 2.2.6. [22, 4.9, 5.3]
a) The bi-functor $\mathcal{F}_P^G$ is exact in both arguments.

b) (PQ-formula) If $Q \supset P$ is a parabolic subgroup, $\mathfrak{q} = \mathrm{Lie}(Q)$, and M an object of $\mathcal{O}^Q$, then

$$\mathcal{F}_P^G(M,V) = \mathcal{F}_Q^G(M, i_{L_P(L_Q \cap U_P)}^{L_Q}(V)) \; ,$$

where $i_{L_P(L_Q \cap U_P)}^{L_Q}(V) = i_P^Q(V) = \mathrm{ind}_P^Q(V)$ denotes the corresponding induced representation in the category of smooth representations.

c) Suppose $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ is simple and that $\mathfrak{p}$ is maximal for M (i.e., if $M \in \mathcal{O}^{\mathfrak{q}}$ with $\mathfrak{q} \supset \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}$). Let V be a smooth and irreducible L_P-representation. Then $\mathcal{F}_P^G(M,V)$ is topologically irreducible as a G-representation[2].

2.3. **A description of the dual space of $\mathcal{F}_P^G(M)$.** By [25, 3.2], M carries a canonical structure of a module over the locally analytic distribution algebra $D(P) = D(P,K)$. Let $D(\mathfrak{g},P)$ be the subring of $D(G)$ generated by $U(\mathfrak{g})$ and $D(P)$ inside $D(G)$. By [22, 3.6], the $U(\mathfrak{g})$-module structure on M and the $D(P)$-module structure on M agree on the subring $U(\mathfrak{p})$, and there is a unique structure of a module over $D(\mathfrak{g},P)$ on M which extends these module structures. By [22, 3.7] there is a canonical isomorphism of $D(G)$-modules

$$\mathcal{F}_P^G(M)' \;\cong\; D(G) \otimes_{D(\mathfrak{g},P)} M \; . \tag{2.3.1}$$

[2] Here we assume that if the root system $\Phi = \Phi(\mathfrak{g},\mathfrak{t})$ has irreducible components of type B, C or F_4, then $p > 2$, and if Φ has irreducible components of type G_2, we assume that $p > 3$.

3. Jacquet functors

3.1. **Jacquet module for irreducible objects $\mathcal{F}^G_P(M,V)$.** The first part of this section deals with a résumé of results formulated in [20, 2], where the Jacquet functor of simple objects $\mathcal{F}^G_P(M,V)$ with $M \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ was discussed.

Let P be a parabolic subgroup of G with Levi decomposition $P = L_PU_P$. For a locally analytic P-representation V, let $V(U_P)$ be the subspace generated by the expressions $uv - v$, with $u \in U_P, v \in V$ and let $\overline{V(U_P)}$ be its topological closure which is a P-stable subspace of V. Denote by

$$\overline{H}_0(U_P,V) := V_{U_P} := V/\overline{V(U_P)}$$

the corresponding quotient (the naive topological Jacquet module). It is the largest Hausdorff quotient of V on which U_P acts trivially.

Lemma 3.1.1. *The space $\overline{H}_0(U_P,V)$ has the canonical structure of a locally analytic P-representation.*

Proof. Since $\overline{V(U_P)}$ is a closed subspace of V, the quotient is a barreled locally convex Hausdorff vector space. Moreover the orbit maps $P \to \overline{H}_0(U_P,V)$ are clearly locally analytic since these are induced by the locally analytic orbit maps $P \to V$. □

On the other hand, if V is of compact type then its dual V' is a K-Fréchet space equipped with a continuous and locally analytic action of P. We let $H^0(U_P,V')$ be the subspace of V' consisting of vectors which are fixed by U_P. This is a closed subspace so that $H^0(U_P,V')$ inherits the structure of a K-Fréchet space equipped with an action of P, as well. Since the action of P is locally analytic we can define the subspace

$$H^0(\mathfrak{u}_P,V') = \{w \in V' \mid x \cdot w = 0, \forall x \in \mathfrak{u}_P\}$$

and the Hausdorff quotient $\overline{H}_0(\mathfrak{u}_P,V) = V/\overline{\mathfrak{u}_PV}$, as well. Then $H^0(\mathfrak{u}_P,V')$ is by the continuity of the $\mathfrak{p}$-action a closed P-equivariant subspace of V' with $H^0(U_P,V') \subset H^0(\mathfrak{u}_P,V')$.

Lemma 3.1.2. *Let V be of compact type. Under the duality pairing $V \times V' \to K$ the subspace $H^0(U_P,V')$ (resp. $H^0(\mathfrak{u}_P,V')$) is the topological dual of $\overline{H}_0(U_P,V)$ (resp. $\overline{H}_0(\mathfrak{u}_P,P)$) as P-representations.*

Let Q be another parabolic subgroup with $P \subset Q$ and let $Q = L_Q \cdot U_Q$ be its Levi decomposition. In this sequel we want to determine for certain objects $M \in \mathcal{O}^{\mathfrak{q}}_{\rm alg}$ and smooth admissible L_Q-representations V the L_P-representations $H^0(U_P,\mathcal{F}^G_Q(M,V)')$.

For a compact open subgroup $H \subset G_0$, let $P_H = H \cap P$ and let $D(P_H, \mathfrak{g}) \subset D(H)$ be the subring generated by $\mathfrak{g}$ and P_H. Moreover, let $D_r(P_H, \mathfrak{g})$ be the Banach space completion of $D(P_H, \mathfrak{g})$ inside $D_r(H)$ and set $M_r := D_r(P_H, \mathfrak{g}) \otimes_{D(P_H,\mathfrak{g})} M$.

Proposition 3.1.3. *Let M be an object of $\mathcal{O}^{\mathfrak{p}}_{\text{alg}}$. We have an inclusion preserving bijection*

$$\begin{array}{ccc} \Big\{\text{closed } U(\mathfrak{l}_P)\text{-invariant subspaces of } M_r\Big\} & \xrightarrow{\sim} & \Big\{U(\mathfrak{l}_P)\text{-invariant subspaces of } M\Big\}. \\ S & \longmapsto & S \cap M \end{array}$$

The inverse map is induced by taking the closure.

Proof. By [20] we have such an inclusion preserving bijection

$$\begin{array}{ccc} \Big\{\text{closed } U(\mathfrak{t})\text{-invariant subspaces of } M_r\Big\} & \xrightarrow{\sim} & \Big\{U(\mathfrak{t})\text{-invariant subspaces of } M\Big\}. \\ S & \longmapsto & S \cap M \end{array}$$

for $U(\mathfrak{t})$-submodules. But for a closed $U(\mathfrak{t})$-submodule $N \subset M_r$ the intersection $N \cap M$ is $\mathfrak{l}_P$-stable if and only if N is $U(\mathfrak{l}_P)$-stable. Indeed, whereas one direction is obvious the other one follows by density arguments. The claim follows. □

Recall that if M is a Lie algebra representation of $\mathfrak{g}$, then $H^0(\mathfrak{u}_Q, M) = \{m \in M \mid \mathfrak{x} \cdot m = 0 \; \forall \, \mathfrak{x} \in \mathfrak{u}_Q\}$ denotes the subspace of vectors killed by $\mathfrak{u}_Q$.

Corollary 3.1.4. *Let M be an object of $\mathcal{O}^{\mathfrak{p}}_{\text{alg}}$. Then $H^0(\mathfrak{u}_P, M_r) = H^0(\mathfrak{u}_P, M)$. In particular, $H^0(\mathfrak{u}_P, M_r)$ is finite-dimensional.*

Proof. We clearly have $H^0(\mathfrak{u}_P, M_r) \cap M = H^0(\mathfrak{u}_P, M)$. As $H^0(\mathfrak{u}_P, M_r)$ is closed in M_r by the continuity of the action of $\mathfrak{g}$ and as $H^0(\mathfrak{u}_P, M)$ is finite-dimensional (!!!!) and therefore complete the statement follows by Proposition 3.1.3. □

Lemma 3.1.5. *Let M be an object of $\mathcal{O}^{\mathfrak{q}}_{\text{alg}}$ where $P \subset Q$ and let V be a smooth admissible L_Q-representation. Then there is an identification*

$$H^0(\mathfrak{u}_P, \mathcal{F}_Q^G(M,V)') = H^0(\mathfrak{u}_P, \mathcal{F}_Q^G(M)')\hat{\otimes}_K V'$$

of Fréchet spaces with P-action (Here the action of P on V' is given by the composite $P \hookrightarrow Q \twoheadrightarrow L_Q$.) .

Proof. The proof is the same as in [20] by replacing $\mathfrak{u}_B$ by $\mathfrak{u}_P$. □

For $M \in \mathcal{O}^{\mathfrak{q}}_{\text{alg}}$, let $W \subset M$ be a finite-dimensional algebraic Q-subrepresentation such that the map (a morphism in $\mathcal{O}^{\mathfrak{q}}_{\text{alg}}$)

$$M(W) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W \to M$$

is surjective. If M is simple so that we may assume that W comes via inflation from an irreducible L_Q-representation then $H^0(\mathfrak{u}_Q, M) = W$. We set in this situation $W_M := H^0(\mathfrak{u}_Q, M)$.

Now we are able to state one of the building blocs of this paper which is an analogue of a statement dealing with representations of real Lie groups and Harish-Chandra modules [9, 8]. Its proof is already contained in [20, 2], so that this result is not really new. Nevertheless, for later use we are going to give a proof of it.

Theorem 3.1.6. *Let M be a simple object of $\mathcal{O}^{\mathfrak{q}}_{\rm alg}$ with Q maximal for M. Let V be a smooth admissible L_Q-representation. Then for $P \subset Q$ there are P-equivariant topological isomorphisms*

$$H^0(U_P, \mathcal{F}^G_Q(M,V)') = H^0(\mathfrak{u}_P, W_M) \otimes_K J_{U_P \cap L_Q}(V)',$$

and

$$\overline{H}_0(U_P, \mathcal{F}^G_Q(M,V)) = H_0(\mathfrak{u}_P, W'_M) \otimes_K J_{U_P \cap L_Q}(V),$$

where $J_{U_P\cap L_Q}$ is the usual Jacquet functor for the unipotent subgroup $U_P \cap L_Q \subset L_Q$.

Proof. By the duality treated in Lemma 3.1.2 it suffices to check the first identity. Here we assume first that $V = \mathbf{1}$ is the trivial representation and that $P = Q$. Write $M = L(\lambda)$ for some algebraic character λ of T.

We follow the proof of [20, Thm. 3.5]. Let $\mathcal{I} \subset G$ be the standard Iwahori subgroup. For $w \in W$, let $M^w = M$ be the $D(\mathfrak{g}, \mathcal{I} \cap wP_0w^{-1})$-module with the twisted action given by conjugation with w. Let $I \subset \Delta$ be a subset with $P = P_I$. The Bruhat decomposition $G_0 = \coprod_{w\in W^I} \mathcal{I}wP_0$ induces a decomposition

$$\begin{aligned} D(G_0)\otimes_{D(\mathfrak{g},P_0)} M &\simeq \bigoplus_{w\in W^I} D(\mathcal{I})\otimes_{D(\mathfrak{g},\mathcal{I}\cap wP_0w^{-1})} M^w \\ &\simeq \bigoplus_{w\in W^I} D(w^{-1}\mathcal{I}w)\otimes_{D(\mathfrak{g},w^{-1}\mathcal{I}w\cap P_0)} M. \end{aligned}$$

For each $w \in W^I$, we have

$$H^0(\mathfrak{u}_P, D(\mathcal{I})\otimes_{D(\mathfrak{g},I\cap wP_0w^{-1})} M^w) \simeq H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), D(w^{-1}\mathcal{I}w)\otimes_{D(\mathfrak{g},w^{-1}Iw\cap P_0)} \otimes M).$$

We can write each summand in the shape

$$\mathcal{M}^w := D(w^{-1}\mathcal{I}w)\otimes_{D(\mathfrak{g},w^{-1}\mathcal{I}w\cap P_0)} M = \varprojlim_r \mathcal{M}^w_r$$

where $\mathcal{M}_r^w = D_r(w^{-1}\mathcal{I}w) \otimes_{D(\mathfrak{g},w^{-1}\mathcal{I}w\cap P_0)} M$. If we denote by M_r^w the topological closure of M in $\mathcal{M}_r^w$, we get by [13, 1.4.2] finitely many elements $u \in U_{P_0}^-$ such that

$$\mathcal{M}_r^w \simeq \bigoplus_u \delta_u \otimes M_r^w$$

and the action of $\mathfrak{x} \in \mathrm{Ad}(w^{-1})(\mathfrak{u}_P)$ is given by

$$\mathfrak{x} \cdot \sum \delta_u \otimes m_u = \sum \delta_u \otimes \mathrm{Ad}(u^{-1}(\mathfrak{x}))m_u.$$

In [20, Thm 3.5] it is explained that for $w \neq 1$, there is a non-trivial element $\mathfrak{x} \in \mathfrak{u}_P^- \cap \mathrm{Ad}(w^{-1})(\mathfrak{u}_P)$. Since P is maximal for M and M is simple we deduce by [21, Corollary 8.6], that elements of $\mathfrak{u}_P^-$ act injectively on M, and as explained in Step 1 of [21, Theorem 5.7] they act injectively on M_r^w, as well. We conclude that $H^0(\mathrm{Ad}(u^{-1})(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P)), M_r^w) = 0$ for $w \neq 1$ since $\mathrm{Ad}(u^{-1})(\mathfrak{x}) \in \mathfrak{u}_P^-$. So $H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), \mathcal{M}_r^w) = 0$. Hence by passing to the limit we get $H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), \mathcal{M}^w) = 0$ for $w \neq 1$.

Now consider the case $w = 1$. Again we may write $D(\mathcal{I})_r = \bigoplus \delta_u D(\mathfrak{g}, P_0)_r$ for a finite number of $u \in U_{P,0}^-$, so that $D(\mathcal{I})_r \otimes_{D(\mathfrak{g},P_0)_r} M_r^1 = \bigoplus_u \delta_u \otimes M_r^1$. We shall show that if $u \notin U_{P,0}^- \cap D(\mathfrak{g}, P_0)_r$, then $H^0(\mathrm{Ad}(u^{-1})\mathfrak{u}_P, M_r^1) = 0$. Here we will use Step 2 in the proof of [21, Theorem 4.7] where we have used that P is maximal for M. Let $\hat{M}$ be the formal completion of M, i.e. $\hat{M} = \prod_\mu M_\mu$ which is a $\mathfrak{g}$-module. The action of $\mathfrak{u}_P^-$ can be extended to an action of U_P^- as explained in loc.cit. If $\mathfrak{x} \in \mathfrak{g}$ and $u \in U_P^-$, the action of $\mathrm{ad}(u)\mathfrak{x}$ on M_r is the restriction of the composite $u \circ \mathfrak{x} \circ u^{-1}$ on $\hat{M}$. Let $\hat{M}$ be the formal completion of M, i.e. $\hat{M} = \prod_\mu M_\mu$ which is a $\mathfrak{g}$-module. The action of $\mathfrak{u}_P^-$ can be extended to an action of U_P^- as explained in loc.cit. If $\mathfrak{x} \in \mathfrak{g}$ and $u \in U_P^-$, the action of $\mathrm{ad}(u)\mathfrak{x}$ on M_r is the restriction of the composite $u \circ \mathfrak{x} \circ u^{-1}$ on $\hat{M}$. As a consequence, we get

$$\begin{aligned} H^0(\mathrm{ad}(u^{-1})\mathfrak{u}_P, M_r^1) &= M_r^1 \cap u^{-1} \cdot H^0(\mathfrak{u}_P, \hat{M}) \\ &= M_r^1 \cap u^{-1} W_M \end{aligned}$$

since $H^0(\mathfrak{u}_P, \hat{M}) = H^0(\mathfrak{u}_P, M) = W_M$ (Here and in the sequel we copy the argumentation of Breuil [2]). Let v^+ be a highest weight vector of M. If the term $H^0(\mathrm{ad}(u^{-1})\mathfrak{u}_P, M_r^1) \neq (0)$ does not vanish, then we have consequently $u^{-1}W_M \cap M_r^1 \neq (0)$. we deduce that $u^{-1}W_M \subset M_r^1$ since W_M is irreducible. In particular $u^{-1}v^+ \in M_r$. By the proof of [21, Theorem 4.7], this gives a contradiction if $u \notin U_P^- \cap D_r(\mathfrak{g}, P_0)$. Hence by passing to the limit and using Corollary 3.1.4 we obtain finally an isomorphism of Fréchet spaces

$$H^0(\mathfrak{u}_P, D(\mathcal{I}) \otimes_{D(\mathfrak{g},P_0)} M) \simeq H^0(\mathfrak{u}_P, M) = W_M.$$

Next we consider the general situation where also a smooth representation is involved and where $P \subset Q$. By Lemma 3.1.5 and what we proved above we get

$$H^0(\mathfrak{u}_P, \mathcal{F}^G_Q(M,V)') = H^0(\mathfrak{u}_P, H^0(\mathfrak{u}_Q, \mathcal{F}^G_Q(M,V)') = H^0(\mathfrak{u}_P, W_M) \otimes_K V',$$

Since $H^0(U_P, \mathcal{F}^G_Q(M,V)')$ is a subspace of $H^0(\mathfrak{u}_P, \mathcal{F}^G_Q(M,V)')$ the latter one is stable by the action of U_P. Thus we deduce by Lemma 3.1.5 that

$$\begin{aligned} H^0(U_P, \mathcal{F}^G_Q(M,V)') &= H^0(U_P, H^0(\mathfrak{u}_P, \mathcal{F}^G_Q(M,V)')) \\ &= H^0(U_P, H^0(\mathfrak{u}_P, W_M) \otimes_K V') \\ &= H^0(\mathfrak{u}_P, W_M) \otimes_K J_{U_P \cap L_Q}(V)'. \end{aligned}$$

The last identity follows from the fact that the action of U_P on W_M is induced by the one of $\mathfrak{u}_P$. □

3.2. **An analogue of the Casselman-Jacquet functor.** The next result generalizes Theorem 3.1.6 by considering non-necessarily simple modules M.

Proposition 3.2.1. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and let V be a smooth admissible L_P-representation. Then there are canonical (topological) identities*

$$H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)') = \bigoplus_{W \subset H^0(\mathfrak{u}_P, M)} W \otimes S_W(V)'$$

and

$$\overline{H}_0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)) = \bigoplus_{W \subset H^0(\mathfrak{u}_P, M)} W' \otimes S_W(V)$$

of P-representations where $S_W(V)$ is a quotient of $i^{P_W}_P(V)_{|P}$ for some standard parabolic subgroup $P_W \supset P$ with $V \subset (S_W(V))_{|P}$. (Here the sum is over all simple L_P-subrepresentations W of $H^0(\mathfrak{u}_P, M)'$ with multiplicities.)

Proof. By Lemma 3.1.2 it is enough to prove the result for the $\mathfrak{u}_P$-invariants of the (topological) dual $\mathcal{F}^G_P(M,V)'$.

For simple objects $M = L(\lambda)$ we use Theorem 3.1.6. If here the considered parabolic subgroups P and Q are identical then the claim is trivial since

$$H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)') = W_M \otimes_K V'.$$

Otherwise, we get

$$H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)') = H^0(\mathfrak{u}_P, W_M) \otimes_K i^Q_P(V)'.$$

Then we apply [12, II, Prop. 2.11]. The latter reference says that for an algebraic simple G-module M the fix space M^{U_P} is a simple L_P-module, as well. Hence the module L_P-module $H^0(\mathfrak{u}_P, W_M) = W_M^{U_P}$ is simple and contributes to the index family of the direct sum.

In general we fix a JH-series of M and apply induction to the number of irreducible subquotients of M. More precisely, we consider an exact sequence

$$0 \to M_1 \to M \xrightarrow{p} M_2 \to 0$$

in our category $\mathcal{O}^{\mathfrak{p}}_{\rm alg}$ where we suppose that M_2 is simple. Hence we get by applying our bi-functor $\mathcal{F}_P^G$ composed with taking the dual an exact sequence

$$0 \to \mathcal{F}_P^G(M_1, V)' \to \mathcal{F}_P^G(M, V)' \to \mathcal{F}_P^G(M_2, V)' \to 0.$$

Next we take $\mathfrak{u}_P$-invariants which gives together with the induction hypothesis and the start of induction a (left) exact sequence

$$0 \to \bigoplus_{W \subset H^0(\mathfrak{u}_P, M_1)} W \otimes S_W(V)' \to H^0(\mathfrak{u}_P, \mathcal{F}_P^G(M, V)') \xrightarrow{g} W_2 \otimes i_P^Q(V)'$$

where Q is maximal for M_2 and $W_2 = H^0(\mathfrak{u}_P, M_2)$. Next we consider the natural map

$$\bar{p} = H^0(\mathfrak{u}_P, p) : H^0(\mathfrak{u}_P, M) \to H^0(\mathfrak{u}_P, M_2)$$

and distinguish the following two cases:

1st case: $\bar{p} = 0$. In this case we claim that the map g also vanishes. Indeed, for seeing this we have to reenter the proof of Theorem 3.1.6. Here[3] we saw the identities

$$H^0(\mathfrak{u}_P, \mathcal{F}_P^G(M)') = \bigoplus_{w \in W^I} H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), D(w^{-1}\mathcal{I}w) \otimes_{U(\mathfrak{g}, w^{-1}Iw \cap P_0)} \otimes M)$$

and

$$H^0(\mathrm{Ad}(w^{-1})\mathfrak{u}_P, D(w^{-1}\mathcal{I}w) \otimes_{U(\mathfrak{g}, w^{-1}Iw \cap P_0)} \otimes M) = \varprojlim_r \sum_u \delta_u \otimes H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_p), M)$$

where u varies in $U_{P,0}^-$ depending on r. The map $H^0(\mathfrak{u}_P, \mathcal{F}_P^G(M)') \to H^0(\mathfrak{u}_P, \mathcal{F}_P^G(M_2)')$ is clearly compatible (graded) with respect to the operations $\bigoplus_{w \in W^I}$, $\sum_r$ and $\varprojlim_r$.

If $w \notin Q$ or $u \notin Q$, then $H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P), D(w^{-1}\mathcal{I}w) \otimes_{U(\mathfrak{g}, w^{-1}Iw \cap P_0)} \otimes M_2) = 0$ by the proof of Theorem 3.2.4. In particular, if $P = Q$ we are done.

If $w \in Q$ and $u \in Q$, then $\mathfrak{u}_Q = \mathrm{Ad}((wu)^{-1})(\mathfrak{u}_Q) \subset \mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P)$. By assumption the contribution $\delta_u \otimes H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_p), M)$ vanishes under g for $w = 1, u = 1$.

[3] which holds for arbitrary modules M.

Suppose on the other hand that for $w \neq 1, u \neq 1$, there is a non-trivial vector $v \in H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P), M), v \neq 0$. As $\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P) \cap \mathfrak{u}_p^- \neq 0$ there is a root α of $\mathfrak{u}_P^-$ such that the corresponding root subspace $\mathfrak{u}_\alpha$ kills v. In particular, $\mathfrak{u}_\alpha$ acts locally finitely on v. The subspace $N \subset M$ on which $\mathfrak{u}_\alpha$ acts locally finitely is a $U(\mathfrak{g})$-submodule, cf. [22, Lemma 8.2] and contains v. Thus $v \in H^0(\mathrm{Ad}((wu)^{-1})\mathfrak{u}_P, N) \subset H^0(\mathrm{Ad}((wu)^{-1})\mathfrak{u}_P, M)$. Further $N \subsetneq M$ is a proper submodule. Indeed, suppose that $N = M$. Since $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ there must be a simple subquotient of M for which P is maximal, cf. [11, 9.3 Prop.] and on which u_α acts locally finitely, as well. But this is not possible by [22, Cor. 8.7].

If N is simple then there are the following 2 cases:

α) Let $N \subset M_1$. Then clearly $g(v) = 0$.

β) Let $N \not\subseteq M_1$. Then N is mapped isomorphically onto M_2 giving rise to a splitting of the surjection $M \to M_2$. A contradiction to the vanishing of the map $\bar{p}$.

In general we argue on induction on the length on M to see that $g(v) = 0$ (Indeed we apply the induction hypothesis to N). It follows that the map $H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P), M) \to H^0(\mathrm{Ad}((wu)^{-1})(\mathfrak{u}_P), M_2)$ vanishes, as well. Thus $g = 0$.

2$^{\text{nd}}$ case: $\bar{p} \neq 0$. In this case $\bar{p}$ is automatically surjective. Hence we see that there is an exact sequence

$$0 \to \bigoplus_{W \subset H^0(\mathfrak{u}_P, M_1)} W \otimes S_W(V)' \to H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)') \stackrel{g}{\to} W_2 \otimes S_{W_2}(V)' \to 0$$

for some quotient $S_{W_2}(V)$ of $i^Q_P(V)$. Since $H^0(\mathfrak{u}_P, M)$ is obviously always contained in $H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M)')$ we see that $V \subset (S_{W_2}(V))_{|P}$. On the other hand, it follows from the definition of the category $\mathcal{O}^{\mathfrak{p}}$ that we have $H^0(\mathfrak{u}_P, M) = H^0(\mathfrak{u}_P, M_1) \bigoplus H^0(\mathfrak{u}_P, M_2)$ as L_P-modules with $H^0(\mathfrak{u}_P, M_2) = W_2$. Since the action of U_P is trivial on these spaces, this identity holds even as P-representations. It follows that $H^0(\mathfrak{u}_P, \mathcal{F}^G_P(M,V)') \subset H^0(\mathfrak{u}_P, D(G) \otimes_{D(\mathfrak{g},P)} H^0(\mathfrak{u}_P, M) \hat{\otimes} V')$. (For simple objects this is trivial and in general use induction together with the argument above). Since $D(G) \otimes_{D(\mathfrak{g},P)} H^0(\mathfrak{u}_P, M) \hat{\otimes} V' = (D(G) \otimes_{D(\mathfrak{g},P)} H^0(\mathfrak{u}_P, M_1) \hat{\otimes} V') \bigoplus (D(G) \otimes_{D(\mathfrak{g},P)} H^0(\mathfrak{u}_P, M_2) \hat{\otimes} V')$ we see that above sequence splits.

. □

We can generalize the previous result as follows. Fix an integer $k \geq 1$. Let $\mathfrak{u}_P^k \subset U(\mathfrak{g})$ be the subspace generated by all the products $x_1 \cdots x_k$ with $x_i \in \mathfrak{u}_P$. With the same proof one checks:

Proposition 3.2.2. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ and let V be a smooth admissible L_P-representation. Then there are canonical (topological) identities*

$$H^0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V)') = \bigoplus_{W \subset H^0(\mathfrak{u}_P^k, M)} W \otimes S_W(V)'$$

and

$$\overline{H}_0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V)) = \bigoplus_{W \subset H^0(\mathfrak{u}_P^k, M)} W' \otimes S_W(V)$$

of P-representations where $S_W(V)$ is a quotient of $i_P^{P_W}(V)_{|P}$ for some standard parabolic subgroup $P_W \supset P$ with $V \subset (S_W)(V)_{|P}$. (Here the sum is over all maximal indecomposable P-subrepresentations W of the finite-dimensional P-representation $H^0(\mathfrak{u}_P^k, M)$ with multiplicities.)

Proof. We start with the remark that Lemma 3.1.2 generalizes to these $\mathfrak{u}_P^k$-invariants so that $\overline{H}_0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V))$ is the topological dual of $H^0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V)')$ as P-representation. As already mentioned the proof coincides with that of Proposition 3.2.1. We list here the corresponding modifications.

For the start of induction which is essentially Theorem 3.1.6 one has to pay attention. Here we follow the proof of loc.cit. where $k=1$. If $w \neq 1$, then some elements of $\mathfrak{u}_P^k$ act injectively on M_r^w, too. Hence all the contributions $H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), D(w^{-1}\mathcal{I}w) \otimes_{U(\mathfrak{g}, w^{-1}Iw \cap P_0)} \otimes M)$ vanish. As for $w=1$ we observe that $H^0(\mathrm{ad}(u^{-1})\mathfrak{u}_P^k, M_r^1) \neq 0$ implies that $H^0(\mathrm{ad}(u^{-1})\mathfrak{u}_P, M_r^1) \neq 0$. Hence we obtain for a simple object M for which P is maximal the identity

$$H^0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V)') = H^0(\mathfrak{u}_P^k, M) \otimes V'.$$

The object $H^0(\mathfrak{u}_P^k, M)$ is an indecomposable P-module which gives the claim in this case. If $Q \supset P$ is maximal for M we get

$$H^0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M,V)') = H^0(\mathfrak{u}_P^k, H^0(\mathfrak{u}_Q^k, M)) \otimes i_P^Q(V)'.$$

But the first factor is again indecomposable since even $H^0(\mathfrak{u}_B^k, M)$ is indecomposable as it coincides with the sum $\sum_{i_1+\cdots+i_d<k} K \cdot y_{\alpha_1}^{i_1} \cdots y_{\alpha_d}^{i_d} \cdot v^+$, where v^+ is a highest weight vector generating $H^0(\mathfrak{u}_P, M)$, $\{\alpha_1, \ldots, \alpha_d\}$ is a root basis and $y_{\alpha_1}, \ldots, y_{\alpha_d} \in \mathfrak{u}_P^-$ are the usual generators of the weight spaces.).

As for the induction step we note that $\bar{p} : H^0(\mathfrak{u}_p, M) \to H^0(\mathfrak{u}_p, M_2)$ is surjective iff $H^0(\mathfrak{u}_p^k, M) \to H^0(\mathfrak{u}_p^k, M_2)$ is surjective for all k (by considering the epimorphisms $U(\mathfrak{u}_p^-) \otimes H^0(\mathfrak{u}_p, M) \to M$ and $U(\mathfrak{u}_p^-) \otimes H^0(\mathfrak{u}_p, M_2) \to M_2$, respectively.). $\square$

Remark 3.2.3. By the proof of the above propositions we deduce the following fact for two integers $l > k \geq 1$. If $W_k \subset H^0(\mathfrak{u}_P^k, M)$ and $W_l \subset H^0(\mathfrak{u}_P^l, M)$ are two maximal indecomposable P-subrepresentations such that $W_k \subset W_l$ then $S_{W_k}(V) = S_{W_l}(V)$.

For a locally analytic T-representation V and a locally analytic character $\lambda : T \to K^*$ we denote by

$$V_\lambda := \{v \in V \mid tv = \lambda(t)v \,\forall t \in T\}$$

the λ-eigenspace of V. We set

$$V_{\rm alg} := \bigoplus_{\lambda \in X^*(T)} V_\lambda.$$

Corollary 3.2.4. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ and $k \geq 1$. Then $\overline{H}_0(U_P, \mathcal{F}^G_P(M))_{\rm alg} = H^0(\mathfrak{u}_P, M)'$ and $\overline{H}_0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M))_{\rm alg} = H^0(\mathfrak{u}_P^k, M)'$.*

Proof. Since the weight spaces of M are algebraic we see that $(W \otimes S_W(\mathbf{1}))_{\rm alg} = W$ for all contributions W in $H^0(\mathfrak{u}_P^k, M)$. Hence the claim follows. $\square$

In the case of generalized Verma modules we can give a more precise statement.

Proposition 3.2.5. *Let $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ be a generalized Verma module for some parabolic subgroup P and let V be a smooth admissible L_P-representation. Then $\overline{H}_0(U_P, \mathcal{F}^G_P(M, V)) = H^0(\mathfrak{u}_P, M)' \otimes V$ and $\overline{H}_0(\mathfrak{u}_P^k, \mathcal{F}^G_P(M, V)) = H^0(\mathfrak{u}_P^k, M)' \otimes V$ for all $k \geq 1$.*

Proof. We may suppose that V is trivial. The start of the proof is the same as in Theorem 3.1.6. For $w \neq 1$ one checks that the contributions $H^0(\mathfrak{u}_P^k, D(\mathcal{I}) \otimes_{U(\mathfrak{g}, I \cap wP_0w^{-1})} M^w)$ vanishes well since a generalized Verma module is free over $U(\mathfrak{u}_P^-)$ and consequently elements of $\mathfrak{u}_P^-$ act injectively on M.
Now consider the case $w = 1$. Here we shall show that if $u \in U^-_{P,0} \setminus \{1\}$, then we have $H^0({\rm Ad}(u^{-1})(\mathfrak{u}_P), M) = 0$. Indeed, let $u \neq 1$. Since the normalisator of $\mathfrak{u}_P$ under the adjoint action of G is the parabolic subgroup P, there is some $v \in \mathfrak{u}_P$ such that $uvu^{-1} \notin \mathfrak{u}_P$. Write $uvu^{-1} = v_- + v_+$ where $v_- \in \mathfrak{u}_P^-$ and $v_+ \in \mathfrak{p}$. Let $m \in M_\chi, m \neq 0$ for some weight χ. As we have already used above the action of $\mathfrak{u}_P^-$ is injective on M. Hence $v_- \cdot m \neq 0$. But the elements v_-, v_+ shift the weights of M in opposite directions. Any identity $(v_- + v_+) \cdot m = 0$ would imply $0 \neq v_- m = -v_+ m$ which yields thus for weight reasons a contradiction. In general we decompose any element $m \in M$ into its weight components. For simplicity let $m = m_1 + m_2$ where $m_i \in M_{\chi_i}$ and $\chi_1 \neq \chi_2$ are weights. Again we consider the sequence $0 \neq v_- m = v_- m_1 + v_- m_2 = -v_+ m_1 - v_+ m_2$. Comparing

weights and that the action of $\mathfrak{u}_P^-$ on M is injective we see that this is not possible. Hence $H^0(\mathrm{Ad}(u^{-1})(\mathfrak{u}_P), M) = 0$ since uvu^{-1} acts injectively on M. With the same proof as in Step 1 of [21, Thm. 5.7] one checks that v_- acts injectively on M_r. As the weights for the action of v_+ on M_r are different from those of v_-, we see that uvu^{-1} acts injectively on M_r as well. Thus $H^0(\mathrm{Ad}(u^{-1})(\mathfrak{u}_P), M_r) = 0$. By repeating the arguments in Theorem 3.1.6 we obtain an isomorphism of Fréchet spaces

$$H^0(\mathfrak{u}_P, D(\mathcal{I}) \otimes_{U(\mathfrak{g},P_0)} M) \simeq H^0(\mathfrak{u}_P, M).$$

The claim follows moreover for all $k \geq 1$ inductively since $\mathrm{Ad}(u^{-1})(\mathfrak{u}_P^k) = \mathrm{Ad}(u^{-1})(\mathfrak{u}_P)^k$. Indeed, set $x = \mathrm{Ad}(u^{-1})(\mathfrak{u}_P)$ and suppose that there is an $m \in M$ with $x^k m = 0$ for all $x \in \mathrm{Ad}(u^{-1})(\mathfrak{u}_P)$. Then $x^{k-1}m = 0$ by what we have shown above. By induction hypothesis it follows $m = 0$. $\square$

Example 3.2.6. Let $G = \mathrm{GL}_2$ and consider the exact sequence $0 \to M(s \cdot 0) \to M(0) \to \mathbf{1} \to 0$. By applying of $\mathcal{F}_B^G$ and dualizing we get an exact sequence

$$0 \to D(G) \otimes_{D(\mathfrak{g},B)} M(s \cdot 0) \to D(G) \otimes_{D(\mathfrak{g},B)} M(0) \to (i_B^G(\mathbf{1}))' \to 0.$$

Taking $H^0(\mathfrak{u}_B, -)$-invariants we get a left exact sequence

$$0 \to K_{s\cdot 0} \to K_{s\cdot 0} \bigoplus K_0 \to (i_B^G(\mathbf{1}))'$$

which is not exact.

Remark 3.2.7. The same statement holds true (with the same proof) for objects $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ of the shape $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ where W is an arbitrary finite-dimensional algebraic P-representation. In particular, it holds for objects M which are projective in the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ since such an object it is free as a $U(\mathfrak{u}_P^-)$-module [11].

Next there is the following variant of the above proposition concerning the other parabolic subgroups of type P lying in the same apartment induced by T. Let $P = P_I = L_P U_P$ and set for $w \in W^I$, $P^w = w^{-1}Pw, L_P^w = w^{-1}L_P w, U_P^w = w^{-1}U_P w$. Here for a L_P^w-module V, we let V^w be the L_P-module twisted by w, i.e. we consider the action induced by composing the given action with the homomorphism $L_P \to w^{-1}L_P w, g \mapsto w^{-1}gw$.

Proposition 3.2.8. *With the above notation, let $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}^w}$ be a generalized Verma module with respect to P^w or a simple module such that P^w is maximal for M. Let V be a smooth admissible L_P^w-representation. Then $\overline{H}_0(U_P, \mathcal{F}_{P^w}^G(M,V)) = (H_0(\mathfrak{u}_P^w, M)')^w \otimes V^w$ and $\overline{H}_0(\mathfrak{u}_P^k, \mathcal{F}_{P^w}^G(M,V)) = (H_0((\mathfrak{u}_P^w)^k, M)')^w \otimes V^w$ for all $k \geq 1$.*

Proof. The proof is the same as above. The difference is that this time all contributions $H^0(\mathrm{Ad}(x^{-1})\mathfrak{u}_P^k, D(x^{-1}\mathcal{I}x) \otimes_{U(\mathfrak{g},x^{-1}\mathcal{I}x\cap P_0^w)} \otimes M)$ with $x \neq w$ vanish. □

Next we consider an analogue of the Casselman-Jacquet functor [4], i.e., limits of the above functors $H^0(\mathfrak{u}_P^k, -)$ (resp. $\overline{H_0}(\mathfrak{u}_P^k, -)$ by duality) with varying k. For a locally analytic G-representation U, the expression $\varinjlim_k H^0(\mathfrak{u}_P^k, U')$ is a $\mathfrak{g} \rtimes P$-module as the same reasoning as in loc.cit. applies. We denote by

$$\mathcal{G}_P^G : \mathrm{Rep}_K(G)^{\mathrm{loc.an.}} \to \mathrm{Mod}_{\mathfrak{g}\rtimes P}$$

the induced functor. As before let M be an object of $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ and let V be a smooth admissible L_P-representation. Then the object $\varinjlim_k H^0(\mathfrak{u}_P^k, \mathcal{F}_P^G(M,V)')$ is even a $D(\mathfrak{g}, P)$-module since M is an inductive limit of finite-dimensional P-representations. In this way we get in some sense a right adjoint to the globalisation functor $\mathcal{F}_P^G$. Moreover, it defines a "section" of it for some objects in $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}} \times \mathrm{Rep}_K^{\infty,a}(L_P)$ (i.e. $\mathcal{G}_P^G(\mathcal{F}_P^G(M,V)) = M \otimes V'$) , cf. Proposition 3.2.5 and Theorem 3.1.6. In general we can deduce the following statement.

Proposition 3.2.9. *Let $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ and let V be a smooth admissible L_P-representation. Then there is a canonical (topological) identity*

$$\mathcal{G}_P^G(\mathcal{F}_P^G(M,V)) = \bigoplus_{N\subset M} N \otimes S_N(V)'$$

of P-representations where $S_N(V)$ is a quotient of $i_P^{P_N}(V)_{|P}$ with $V \subset (S_N)(V)_{|P}$ for the uniquely standard parabolic subgroup $P_N \supset P$ which is maximal for N. (Here the sum is over all simple constituents N of M with multiplicities.)

Proof. This follows from Proposition 3.2.2 by taking the inductive limit. Note that $\varinjlim_k H^0(\mathfrak{u}_p^k, M) = M$. □

Proposition 3.2.10. *Let U be some irreducible subquotient of some $\mathcal{F}_P^G(M,V)$ with $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$. Then $\mathcal{G}_P^G(U)$ is a simple $D(\mathfrak{g}, Q)$-module for some parabolic subgroup $Q \subset G$ with $P \subset Q$.*

Proof. Since U is simple it must coincide by the JH-theorem applied to $\mathcal{F}_P^G(M,V)$ with some object of the shape $\mathcal{F}_Q^G(N,W)$ where N is a simple subquotient of M, Q is maximal for N and W is an irreducible subquotient of $i_P^Q(V)$. But for these objects we deduce by Proposition 3.2.9 that $\mathcal{G}_Q^G(U) = N \otimes W'$. On the other hand, we have $\mathcal{G}_Q^G(U) = \mathcal{G}_P^G(U)$ since any element of N is killed by weight reasons by some $\mathfrak{u}_P^k$, $k \geq 1$. Hence we get the claim. □

As a by-product we get the following statement by applying the functor $\mathcal{G}_P^G$ and Proposition 3.2.2. One part of it was already given by Breuil [2, Cor. 2.5].

Corollary 3.2.11. *Let U be an irreducible subobject (quotient) of some $\mathcal{F}_P^G(M,V)$ with $M \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$. Then U has the shape $\mathcal{F}_Q^G(N,W)$ where $P \subset Q$ and N is a simple quotient of M (submodule) and W is an irreducible subrepresentation (quotient) of $i_P^Q(V)$.*

Proof. In the proof of the foregoing proposition we saw that U has the shape $\mathcal{F}_Q^G(N,W)$ where N is a simple subquotient of M, Q is maximal for N and W is an irreducible subquotient of $i_P^Q(V)$. If U is a quotient then we get by the left exactness of the functor $\mathcal{G}_P^G$ an injection $\mathcal{G}_P^G(U) \hookrightarrow \mathcal{G}_P^G(\mathcal{F}_P^G(M,V))$ of $D(\mathfrak{g},P)$-modules. In particular, we see that N is a submodule of M since $\mathcal{G}_P^G(U) = N \otimes W'$. Further we see, e.g., by Proposition 3.2.9 that $W' \subset S_N(V) = i_P^Q(V)$. The claim follows.
If $i : U \hookrightarrow \mathcal{F}_P^G(M,V)$ is a subobject we get a morphism $\mathcal{G}_P^G(i) : \mathcal{G}_P^G(\mathcal{F}_P^G(M,V)) \to \mathcal{G}_P^G(U)$. The dual of i is a homomorphism $i' : D(G) \otimes_{D(\mathfrak{g},P)} M \hat{\otimes} V' \to D(G) \otimes_{D(\mathfrak{g},Q)} N \hat{\otimes} W'$ which gives rise by the very definition (taking $\varinjlim_k$ of $\mathfrak{u}_p^k$-invariants) to the morphism $\mathcal{G}_P^G(i)$. Since $M \otimes V' = \varinjlim_k H^0(\mathfrak{u}_P^k, M) \otimes V'$ we see that the composite of the natural map $M \otimes V' \to \mathcal{G}_P^G(\mathcal{F}_P^G(M,V))$ with $\mathcal{G}_P^G(i)$ is a map $M \otimes V' \to N \otimes W'$. The latter one induces by taking the composite of $D(G) \otimes_{D(\mathfrak{g},P)} -$ with the natural map $D(G) \otimes_{D(\mathfrak{g},P)} N \otimes W' \to D(G) \otimes_{D(\mathfrak{g},Q)} N \otimes W'$ the map i'. In particular the map $M \otimes V' \to N \otimes W'$ is non-trivial. By Frobenius reciprocity we get a non-trivial homomorphism $M \otimes i_P^Q(V)' \to N \otimes W'$ of $\mathfrak{g} \times Q$-modules. As the latter module is simple we obtain surjections $M \to N$ and $i_P^Q(V)' \to W'$. The claim follows. $\square$

4. Are the functors $\mathcal{F}_P^G$ faithful?

In this section we want to address the question whether the functors $\mathcal{F}_P^G$ are faithful resp. fully faithful. This aspect was discussed for $G = \mathrm{SL}_2$ already in the series of papers by Morita [15, 16, 17].

Theorem 4.1.1. *Let $M_1, M_2 \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$. Then the map*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\rm alg}}(M_1,M_2) & \to & \mathrm{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) \\ f & \mapsto & \mathcal{F}_P^G(f) \end{array}$$

is bijective.

Proof. The proof is divided into several steps.

1) Let $M_1 = M(Z)$ be a generalized Verma module for some finite-dimensional algebraic L-representation Z. Then $\mathcal{F}^G_P(M_1) = \mathrm{Ind}^G_P(Z')$ and U acts trivially on Z. Now we have $\overline{H}_0(U, \mathcal{F}^G_P(M_2))_{\mathrm{alg}} = H^0(\mathfrak{u}, M_2)'$ by Lemma 3.2.4. We consider the identities induced by Frobenius reciprocity and the previous observations

$$\begin{aligned}
\mathrm{Hom}_G(\mathcal{F}^G_P(M_2), \mathcal{F}^G_P(M_1)) &= \mathrm{Hom}_P(\mathcal{F}^G_P(M_2), Z') \\
&= \mathrm{Hom}_L(\overline{H}_0(U, \mathcal{F}^G_P(M_2)), Z') \\
&= \mathrm{Hom}_L(\overline{H}_0(U, \mathcal{F}^G_P(M_2))_{\mathrm{alg}}, Z') \\
&\cong \mathrm{Hom}_L(H^0(\mathfrak{u}, M_2)', Z') \\
&\cong \mathrm{Hom}_L(Z, H^0(\mathfrak{u}, M_2)) \\
&= \mathrm{Hom}_P(Z, M_2) \\
&= \mathrm{Hom}_{D(\mathfrak{g},P)}(M_1, M_2).
\end{aligned}$$

Here the third equality follows from that fact that Z is algebraic and $\mathrm{Hom}_G(i^G_P, \mathbf{1}) = 0$ for all parabolic subgroups $P \subsetneq G$. .

2) Let M_1 be a quotient of some generalized Verma module, i.e., there is a surjective homomorphism $M(Z) \to M_1$ for some finite-dimensional algebraic L-representation Z. Let $\mathfrak{d}$ be its kernel. Then by definition we have $\mathcal{F}^G_P(M_1) = \mathcal{F}^G_P(M(Z))^{\mathfrak{d}}$. We consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathfrak{g},P)}(M_1, M_2) & \hookrightarrow & \mathrm{Hom}_{D(\mathfrak{g},P)}(M(Z), M_2) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_G(\mathcal{F}^G_P(M_2), \mathcal{F}^G_P(M_1)) & \hookrightarrow & \mathrm{Hom}_G(\mathcal{F}^G_P(M_2), \mathcal{F}^G_P(M(Z))).
\end{array}$$

By Step 1) the right vertical map is an isomorphism. It follows that the left vertical map is injective. To show surjectivity we consider the dual objects, i.e. the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathfrak{g},P)}(M_1, M_2) & \hookrightarrow & \mathrm{Hom}_{D(\mathfrak{g},P)}(M(Z), M_2) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(G)}(M_1^{D(G)}, M_2^{D(G)}) & \hookrightarrow & \mathrm{Hom}_{D(G)}(M(Z)^{D(G)}, M_2^{D(G)}).
\end{array}$$

where we abbreviate $M^{D(G)} := D(G) \otimes_{D(\mathfrak{g},P)} M$ for $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Moreover the vertical maps are the obvious ones, i.e. induced by base change. For the surjectivity, let $f \in \mathrm{Hom}_{D(G)}(M_1^{D(G)}, M_2^{D(G)})$ and consider it via the injection as an element in the set $\mathrm{Hom}_{D(G)}(M(Z)^{D(G)}, M_2^{D(G)})$. Hence there is some morphism $\check{f} : M(Z) \to M_2$ with $\check{f} \otimes \mathrm{id} = f$. We need to show that $\check{f}(\mathfrak{d}) = 0$. By assumption we have $f(\mathfrak{d}) = 0$. But we proved in [21, (3.7.6)] that if $M \in \mathcal{O}_{\mathrm{alg}}, M \neq 0$ then $D(G) \otimes_{D(\mathfrak{g},B)} M \neq 0$. By applying this fact to $M = \check{f}(\mathfrak{d})$ the claim follows.

3) Let $M_1 = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ for some finite dimensional algebraic P-representation W. We may view it as a successive extension of generalized Verma modules considered in Step 1). The proof of the statement is by dimension on $\dim W$. Here Step 1) serves as the start of induction. Write down an exact sequence

$$0 \to M(Z) \to M_1 \to M_1' \to 0$$

where $M_1' = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W'$ is a generalized Verma module for some algebraic P-representation W' with $\dim W' < \dim W$. We get an induced exact sequence

$$0 \to \mathcal{F}_P^G(M_1') \to \mathcal{F}_P^G(M_1) \to \mathcal{F}_P^G(M(Z)) \to 0.$$

We consider the resulting diagram of long exact sequences

$$\begin{array}{cccccc}
0 \to & \mathrm{Hom}_{D(\mathfrak{g},P)}(M_1', M_2) & \to & \mathrm{Hom}_{D(\mathfrak{g},P)}(M_1, M_2) & \to & \mathrm{Hom}_{D(\mathfrak{g},P)}(M(Z), M_2) \\
 & \downarrow f' & & \downarrow f & & \downarrow f_Z \\
0 \to & \mathrm{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1')) & \to & \mathrm{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) & \to & \mathrm{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M(Z))) \\
 & & & & & \\
\stackrel{\delta}{\to} & \mathrm{Ext}^1(M_1', M_2) & \to & \mathrm{Ext}^1(M_1, M_2) & \to & \mathrm{Ext}^1(M(Z), M_2) \\
 & \downarrow & & \downarrow & & \downarrow \\
\stackrel{\delta_{\mathcal{F}}}{\to} & \mathrm{Ext}^1(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1')) & \to & \mathrm{Ext}^1(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) & \to & \mathrm{Ext}^1(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M(Z)))
\end{array}$$

Here we consider the Ext groups as Yoneda-Ext groups. The maps f' and f_Z are by induction isomorphisms of finite-dimensional vector spaces. By diagram chase, it suffices to check that $\delta(g) \neq 0$ if and only if $\delta_{\mathcal{F}}(\mathcal{F}_P^G(g)) \neq 0$. Concretely we have to show that if $\delta(g) \neq 0$ then $\delta_{\mathcal{F}}(\mathcal{F}_P^G(g)) \neq 0$ since the other direction follows directly by diagram chase again. If $\delta_{\mathcal{F}}(\mathcal{F}_P^G(g)) = 0$, then the extension

$$0 \to \mathcal{F}_P^G(M_1') \to E_{\mathcal{F}_P^G(g)} \to \mathcal{F}_P^G(M_2) \to 0$$

induced by $\mathcal{F}^G_P(g) \in \mathrm{Hom}_G(\mathcal{F}^G_P(M_2), \mathcal{F}^G_P(M(Z))$ splits. Then we apply Remark 3.2.7 to deduce that

$$\begin{aligned} H^0(\mathfrak{u}, E_{\mathcal{F}^G_P(g)}) &= H^0(\mathfrak{u}, \mathcal{F}^G_P(M_1')) \oplus H^0(\mathfrak{u}, \mathcal{F}^G_P(M_2)) \\ &= H^0(\mathfrak{u}, M_1') \oplus H^0(\mathfrak{u}, M_2). \end{aligned}$$

Since $H^0(\mathfrak{u}, E_g) \subset H^0(\mathfrak{u}, E_{\mathcal{F}^G_P(g)})$ we deduce that $H^0(\mathfrak{u}, E_g) = H^0(\mathfrak{u}, E_{\mathcal{F}^G_P(g)})$ and that the extension

$$0 \to M_2 \to E_g \to M_1' \to 0$$

splits as well. Indeed suppose for simplicity that W' is induced via inflation by a L_P-representation. Then $W' = H^0(\mathfrak{u}, M_1')$ appears in E_g so that we get a section of $E_g \to M_1'$.

4) Let M_1 be arbitrary. Then there is a surjective homomorphism $M(Z) \to M$ for some finite dimensional algebraic P-representation Z. Then we proceed as in Step 2).

□

Next we consider the situation where also smooth admissible representations are involved.

Proposition 4.1.2. *Let $M_1, M_2 \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and let V_1, V_2 be smooth admissible L_P-representations. Assume that $Z \subset M_1$ is a finite-dimensional algebraic P-representation which generates M_1 as a $U(\mathfrak{g})$-module. Then the natural map*

$$\mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1, M_2) \otimes \mathrm{Hom}_{L_P}(V_2, V_1) \;\to\; \mathrm{Hom}_G(\mathcal{F}^G_P(M_2, V_2), \mathcal{F}^G_P(M_1, V_1))$$

induced by the functor $\mathcal{F}^G_P$ is injective and extends to a bijection

$$\bigoplus_{W \subset H^0(\mathfrak{u}^k_P, M_2)} \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1, M_2)_W \otimes \mathrm{Hom}_P(S_W(V_2)_{|P}, V_1) \to \mathrm{Hom}_G(\mathcal{F}^G_P(M_2, V_2), \mathcal{F}^G_P(M_1, V_1))$$

where W ranges over all maximal indecomposable P-modules of $H^0(\mathfrak{u}^k_P, M_2)$ such that there is a non-trivial homomorphism $Z \to W$ and $\mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1, M_2)_W \subset \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1, M_2)$ is just the subspace consisting of those maps which are induced by this homomorphism.

Proof. Indeed we consider Steps 1) and 3) in the modified situation. Then we argue as in Steps 2) and 4) for the general case. As for Steps 1) and 3) we choose this time a slightly different approach by way of variation. So, let $M_1 = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Z$ for some finite-dimensional P-module Z. Let $k \geq 1$ be an integer such that $H^0(\mathfrak{u}^k_P, Z) = Z$. Then we apply Proposition 3.2.2 to deduce that

$$\begin{aligned}
\mathrm{Hom}_G(\mathcal{F}_P^G(M_2,V_2),\mathcal{F}_P^G(M_1,V_1)) &= \mathrm{Hom}_P(\mathcal{F}_P^G(M_2,V_2), Z'\otimes V_1)\\
&= \mathrm{Hom}_{D(P)}(Z\otimes V_1', \mathcal{F}_P^G(M_2,V_2)')\\
&\cong \mathrm{Hom}_{D(P)}(Z\otimes V_1', H^0(\mathfrak{u}_P^k, \mathcal{F}_P^G(M_2,V_2))')\\
&\cong \mathrm{Hom}_{D(P)}(Z\otimes V_1', \bigoplus_{W\subset H^0(\mathfrak{u}_P^k,M_2)} W\otimes S_W(V_2)'_{|P})\\
&= \bigoplus_{W\subset H^0(\mathfrak{u}_P^k,M_2)} \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1,M_2)_W\otimes \mathrm{Hom}_P(S_Z(V_2)_{|P},V_1).
\end{aligned}$$

□

Remark 4.1.3. The statement above is also true (with the same proof) if we consider additionally a parabolic subgroup $Q\supset P$ such that $M_2\in\mathcal{O}^{\mathfrak{q}}_{\mathrm{alg}}$, $V_2\in \mathrm{Rep}^\infty(L_Q)$ i.e. we have a bijection

$$\bigoplus_{W\subset H^0(\mathfrak{u}_P^k,M_2)} \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1,M_2)_W\otimes \mathrm{Hom}_P(S_Z(V_2)_{|P},V_1)\to \mathrm{Hom}_G(\mathcal{F}_Q^G(M_2;V_2),\mathcal{F}_P^G(M_1,V_1)).$$

The following example shows that in the general case of objects in $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$, the first map in Proposition 4.1.2 need not to be surjective.

Example 4.1.4. Let $G=\mathrm{SL}_2$, $B\subset G$ the Borel subgroup of upper triangular matrices and let $T=\{\mathrm{diag}(a,a^{-1})\mid a\in L^*\}$ be the diagonal torus. We consider the smooth character χ of T given by

$$\chi(\mathrm{diag}(a,a^{-1}) = |a|(-1)^{val_\pi(a)}$$

where π is our fixed uniformizer of O_L and v is the normalized valuation, i.e. $v(\pi)=1$. Let M be the one-dimensional trivial Lie(G)-representation. Then the object $\mathcal{F}_B^G(M,\chi)$ is just the smooth representation $i_B^G(\chi)$. But the character χ is chosen in such a way that $i_B^G(\chi)$ decomposes as a direct sum of two irreducible representations [3, Cor. 9.4.6 (b)]. Hence $\mathrm{Hom}_G(\mathcal{F}_B^G(M),\mathcal{F}_B^G(M))$ is two-dimensional whereas $\mathrm{Hom}_{\mathcal{O}_{\mathrm{alg}}}(M,M)\otimes\mathrm{Hom}_T(\chi,\chi)$ is one-dimensional.

Recall the definitions before Proposition 3.2.8. For $w\in W$, we denote by P^w the conjugated parabolic subgroup $w^{-1}Pw$. If Z is a finite-dimensional locally analytic representation of L we let $M_w(Z)$ be the corresponding generalized Verma module with respect to P^w, i.e. $M_w(Z)=U(\mathfrak{g})\otimes_{U(\mathfrak{p}^w)} Z$.

Proposition 4.1.5. *Let Z be a finite-dimensional algebraic L_P-representation and let $w \in W$. Then for any finite-dimensional algebraic L_P^w-representation Y there is an identity*

$$\mathrm{Hom}_G\big(\mathrm{Ind}^G_{P^w}(Y'), \mathrm{Ind}^G_P(Z')\big) = \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}^w}}(M_w(Z^{w^{-1}}), M_w(Y)).$$

Proof. We argue as in Step 1) in the proof of Theorem 4.1.1 and use additionally Proposition 3.2.8:

$$\begin{aligned}
\mathrm{Hom}_G\big(\mathrm{Ind}^G_{P^w}(Y'), \mathrm{Ind}^G_P(Z')\big) &= \mathrm{Hom}_P\big(\mathrm{Ind}^G_{P^w}(Y'), Z'\big)\\
&= \mathrm{Hom}_L(\overline{H}_0(U_P, \mathrm{Ind}^G_{P^w}(Y')), Z')\\
&\cong \mathrm{Hom}_L(H_0(\mathfrak{u}_P^w, (M_w(Y)')^w), Z')\\
&\cong \mathrm{Hom}_L(Z, H^0(\mathfrak{u}_P^w, M_w(Y)^w)\\
&= \mathrm{Hom}_L(Z^{w^{-1}}, H^0(\mathfrak{u}_P^w, M_w(Y)))\\
&= \mathrm{Hom}_{P^w}(Z^{w^{-1}}, M_w(Y))\\
&= \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}^w}}(M_w(Z^{w^{-1}}), M_w(Y)).
\end{aligned}$$

□

5. Applications

In the remaining paper we discuss some applications of the material collected in the previous sections.

5.1. **The category $\mathcal{F}^P_{\mathrm{alg}}$.** We begin to recall a definition of [21]. Let $\lambda, \mu : T \to K^*$ be two algebraic characters with derivatives $d\lambda$, $d\mu$, respectively. We write $\mu \uparrow_B \lambda$ if and only if $d\mu \uparrow_{\mathfrak{b}} d\lambda$ in the sense of [11]. Then one has

$$\dim_K \mathrm{Hom}_{\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}}(M(\mu), M(\lambda)) = \begin{cases} 1 & \mu \uparrow_B \lambda \\ 0 & \textit{otherwise} \end{cases}. \tag{5.1.0}$$

For the remainder, we denote for an element $w \in W$ and an algebraic character by $w \cdot_B \lambda$ the usual "dot"-operation with respect to B. If λ is B-dominant, then $w \cdot_B \lambda \uparrow_B \lambda$ for all $w \in W$.

On the other hand, we let $\lambda^w := w(\lambda)$ be the character given by the ordinary action of W.

Corollary 5.1.1. *Let $P = B$ and let $\lambda, \mu : T \to K^*$ be algebraic characters. Then*

$$\dim_K \mathrm{Hom}_G\big(\mathcal{F}_B^G(M(\lambda)), \mathcal{F}_B^G(M(\mu))\big) = \begin{cases} 1 & \mu \uparrow_B \lambda \\ 0 & otherwise \end{cases}$$

Proof. This follows from Theorem 4.1.1 together with identity (5.1.0). □

For a standard parabolic subgroup $P \subset G$, we let $\mathcal{F}^P_{\mathrm{alg}}$ be the full subcategory of $\mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ consisting of locally analytic representations which lie in the essential image of the functor $\mathcal{F}_P^G : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \to \mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$.

Corollary 5.1.2. *i) The category $\mathcal{F}^P_{\mathrm{alg}}$ is abelian and has enough injective and projective objects. For a morphism $f : N \to M$ we have $\mathcal{F}_P^G(\mathrm{coker}(f)) = \ker(\mathcal{F}_P^G(f))$ and $\mathcal{F}_P^G(\ker(f)) = \mathrm{coker}(\mathcal{F}_P^G(f))$.*

ii) Let M be a projective (resp. injective) object in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Then $\mathcal{F}_P^G(M)$ is injective (resp. projective) in the category $\mathcal{F}^P_{\mathrm{alg}}$.

Proof. The category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is abelian and has enough projective and injective objects. This follows for $\mathcal{O}^{\mathfrak{p}}$ from [11]. But the proof shows that for an object $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ the construction of a projective cover N of M, that N is again in the subcategory $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. hence the claim is true for the category $\mathcal{F}^P_{\mathrm{alg}}$. Since the functor $\mathcal{F}_P^G$ induces an equivalence of categories between $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and $\mathcal{F}^P_{\mathrm{alg}}$ we get the first part of i) and ii). The remaining statements follow directly be the exactness of $\mathcal{F}_P^G$. □

We define a dual object for objects lying in the functor. In light of Theorem 4.1.1 it is well-defined.

Definition 5.1.3. Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and let $M^\vee \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be its dual object. Set

$$\mathcal{F}_P^G(M)^\vee := \mathcal{F}_P^G(M^\vee).$$

It follows from the previous corollary that for an object $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ the locally analytic G-representation $\mathcal{F}_P^G(M)$ is projective (resp. injective) object in $\mathcal{F}^P_{\mathrm{alg}}$ if and only if $\mathcal{F}_P^G(M)^\vee$ is injective (resp. projective) object in $\mathcal{F}^P_{\mathrm{alg}}$.

Definition 5.1.4. Let $V_1, V_2 \in \mathcal{F}^P_{\mathrm{alg}}$ be two locally analytic representations. We denote by $\mathrm{Ext}^i_{\mathcal{F}^P_{\mathrm{alg}}}(V_1, V_2)$ the corresponding Ext-group in degree i.

These Ext-groups are of course different from those considered more generally in the category of locally analytic G-representations, cf. [14]. This can be seen as an analogue of relating the groups $\mathrm{Ext}^i_{\mathfrak{g}}(M_1, M_2)$ and $\mathrm{Ext}^i_{\mathcal{O}}(M_1, M_2)$ for two objects $M_1, M_2 \in \mathcal{O}$ as the next statement confirms.

Corollary 5.1.5. *Let $M_1, M_2 \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. The natural map*

$$\mathrm{Ext}^i_{\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}}(M_1, M_2) \;\; \to \;\; \mathrm{Ext}^i_{\mathcal{F}^P_{\mathrm{alg}}}(\mathcal{F}^G_P(M_2), \mathcal{F}^G_P(M_1))$$

is bijective. □

At this point one can derive many consequences on the above defined Ext-groups. Here we exemplary mention only the following:

Corollary 5.1.6. *Let λ be a dominant algebraic character and let $w, w' \in W$.*

a) Unless $w' \cdot \lambda \uparrow w \cdot \lambda$ we have for all $n > 0$,

$$\mathrm{Ext}^n_{\mathcal{F}^B_{\mathrm{alg}}}(\mathcal{F}^G_B(M(w\cdot\lambda)), \mathcal{F}^G_B(M(w'\cdot\lambda)) = 0 = \mathrm{Ext}^n_{\mathcal{F}^B_{\mathrm{alg}}}(\mathcal{F}^G_B(\underline{L}(w\cdot\lambda)), \mathcal{F}^G_B(M(w'\cdot\lambda)).$$

b) If $w' \cdot \lambda \leq w \cdot \lambda$, then for all $n > \ell(w') - \ell(w)$

$$\mathrm{Ext}^n_{\mathcal{F}^B_{\mathrm{alg}}}(\mathcal{F}^G_B(M(w\cdot\lambda)), \mathcal{F}^G_B(M(w'\cdot\lambda)) = 0 = \mathrm{Ext}^n_{\mathcal{F}^B_{\mathrm{alg}}}(\mathcal{F}^G_B(\underline{L}(w\cdot\lambda)), \mathcal{F}^G_B(M(w'\cdot\lambda)).$$

Proof. This is a consequence of [11, Proposition 6.11]. □

5.2. **The category $\overline{{}^\infty\mathcal{F}^P_{\mathrm{alg}}}$.** Next we consider additionally smooth representations as arguments in the functor $\mathcal{F}^G_P$. So let V be a smooth G-representation. We supply V with the finest locally convex topology. This approach is compatible with the one of Schneider and Teitelbaum for admissible smooth representations. [23, Section 2]. Equivalently, if we write $V = \bigcup_n V^{G_n}$ for a system of compact open subgroups $G_n \subset G$ and supply each V^{G_n} with the finest locally convex topology, then the topology on V coincides with the induced locally convex limit topology. It is Hausdorff [24, Prop. 5.5 ii)] and barreled [24, Cor. 6.16, Examples iii)] (see also the construction in [6, 7.1]). Moreover, for any $v \in V$ the orbit map $G \to V$ is locally constant and gives rise to an element of $C^{an}(G; V)$. Hence we may and will consider V with the structure of a locally analytic G-representation. Then $\mathcal{F}^G_P$ extends with the same definition as in (2.2.4) to a bi-functor

$$\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}^\infty_K(L_P) \longrightarrow \mathrm{Rep}^{\mathrm{loc.an.}}_K(G).$$

where $\mathrm{Rep}^\infty_K(L_P)$ is the category of smooth L_P-representations.

Lemma 5.2.1. *Let V be of countable dimension. Then $\mathcal{F}^G_P(M, V)$ is of compact type.*

Proof. We have the following inclusions of closed subspaces

$$\mathcal{F}^G_P(M,V) \subset \mathrm{Ind}^G_P(W'\otimes V) \cong C^{an}(H, W'\otimes V).$$

Here $H \subset G$ is a locally analytic section of the projection $G \to G/P$, i.e. $H \stackrel{\sim}{\longrightarrow} G/P$ and W is as usual a finite-dimensional algebraic P-module which generates M. Since a closed subspace of a space of compact type is again of compact type. it suffices to show that this property holds true for $C^{an}(G, `W'\otimes V)$. Now we may write $V = \varinjlim_n V_n$ as a locally convex limit with finite-dimensional vector spaces. Hence $C^{an}(H, W'\otimes V) = \varinjlim_n C^{an}(H, W'\otimes V_n)$. But each subspace $C^{an}(H, W'\otimes V_n)$ is of compact type. But the inductive limit of compact type spaces with injective transition maps is of compact type again. □

Remark 5.2.2. We stress that apart possible from Proposition 3.2.10 and Corollary 3.2.11 (since the proofs do not apply) all results of the previous sections are also valid for objects lying in the image of this enhanced functor.

We denote by $\mathrm{Rep}_K^{\infty,\infty}(G)$ the full subcategory of $\mathrm{Rep}_K^{\infty}(G)$ whose objects are of countable dimension. This is clearly an abelian subcategory of $\mathrm{Rep}_K^{\infty}(G)$ which is closed under (smooth) duals.

Lemma 5.2.3. *The category* $\mathrm{Rep}_K^{\infty,\infty}(G)$ *has enough injective and projective objects.*

Proof. Let V be an object of $\mathrm{Rep}_K^{\infty,\infty}(G)$. Since the smooth dual of an injective object is projective and vice versa, it suffices to check that V has an embedding into an injective object of countable dimension. For this we consider the injective object $\mathrm{Ind}^{\infty,G}_{\{e\}}(V|\{e\})$ in the larger category $\mathrm{Rep}_K^{\infty}(G)$ together with the embedding $V \hookrightarrow \mathrm{Ind}^{\infty,G}_{\{e\}}(V|\{e\}), v \mapsto [g \mapsto gv]$. As G is a second countable group and V has a countable basis, we deduce that $\mathrm{Ind}^{\infty,G}_{\{e\}}(V|\{e\})$ is of countable dimension, hence the claim. □

We define ${}^\infty\mathcal{F}^P_{\mathrm{alg}}$ to be the full subcategory of $\mathrm{Rep}_K^{\mathrm{loc.an.}}(G)$ consisting of locally analytic representations which lie in the essential image of the functor

$$\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}_K^{\infty,\infty}(L_P) \longrightarrow \mathrm{Rep}_K^{\mathrm{loc.an.}}(G).$$

The category ${}^\infty\mathcal{F}^P_{\mathrm{alg}}$ is not abelian as we saw for instance in Example 4.1.4. Concerning the latter aspect, we consider the smallest abelian subcategory[4] $\overline{{}^\infty\mathcal{F}^P_{\mathrm{alg}}}$ containing all categories ${}^\infty\mathcal{F}^Q_{\mathrm{alg}}$ where $Q \supset P$ is a parabolic subgroup.

[4]which can be seen as a kind of Satake compactification of ${}^\infty\mathcal{F}^P_{\mathrm{alg}}$

Lemma 5.2.4. *Let $M_1, M_2, M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and $V_1, V_2, V \in \mathrm{Rep}^{\infty}_K(L_P)$ such that M_1, M_2 are quotients of M and V_1, V_2 are subrepresentations of V. Then*

$$\mathcal{F}^G_P(M_1, V_1) \cap \mathcal{F}^G_P(M_2, V_2) = \mathcal{F}^G_P(M_1 \oplus_M M_2, V_1 \cap V_2)$$

Proof. We have

$$\mathcal{F}^G_P(M_1, V_1) \cap \mathcal{F}^G_P(M_2, V_2) = \mathcal{F}^G_P(M_1, V_1 \cap V_2) \cap \mathcal{F}^G_P(M_2, V_1 \cap V_2)$$

$$= \mathcal{F}^G_P(M_1 \oplus_M M_2, V_1 \cap V_2)$$

□

Lemma 5.2.5. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be a simple object such that $\mathfrak{p}$ is maximal for M and let V be a smooth L_P-representation. Then any subquotient of $\mathcal{F}^G_P(M, V)$ has the shape $\mathcal{F}^G_P(M, W)$ for some smooth subquotient W of V.*

Proof. By the exactness of $\mathcal{F}^G_P$ it suffices to prove this statement for subobjects. Let $U \subset \mathcal{F}^G_P(M, V)$ be a subobject. We recall a construction of [22, Thm. 5.8] which uses the simplicity of M. Set $U_{sm} = \varinjlim_H \mathrm{Hom}_H(\mathcal{F}^G_P(M)|_H, U|_H)$ where the limit is over all compact open subgroups H of G. It is proved that U_{sm} is a subrepresentation of $\mathcal{F}^G_P(M, V)_{sm}$ and that the latter object identifies with the smooth induction $i^G_P(V) = \mathrm{ind}^G_P(V)$ (for V irreducible, but this holds also true in this general setting). Moreover, the natural map $\mathcal{F}^G_P(M) \otimes \mathrm{ind}^G_P(V) \to \mathcal{F}^G_P(M, V)$ is surjective giving rise by the very definition of this map to a surjection $\phi : \mathcal{F}^G_P(M) \otimes U_{sm} \to U$. Considering U_{sm} as a subrepresentation of $i^G_P(V)$ as above we set $W := \{f(1) \mid f \in U_{sm}\}$. This is a smooth L_P-representation and the map ϕ factorizes over $\mathcal{F}^G_P(M, W)$. It follows that the image of the map ϕ coincides with $\mathcal{F}^G_P(M, W)$. Hence $U = \mathcal{F}^G_P(M, W)$. □

The next statement is clear by the Jordan-Hölder principle for representation of the shape $\mathcal{F}^G_P(M, V)$ where V is admissible smooth.

Proposition 5.2.6. *Every object U in $\overline{{}^{\infty}\mathcal{F}^P_{\mathrm{alg}}}$ is a successive extension of objects of the shape $\mathcal{F}^G_Q(N, W)$ with $P \subset Q$.*

Proof. As the direct sum of two objects of the kind $\mathcal{F}^G_{Q_i}(M_i, V_i)$, $i = 1, 2$, is contained in such an object we may suppose that U is some subquotient of $\mathcal{F}^G_P(M, V)$. Indeed $\mathcal{F}^G_{Q_i}(M_i, V_i) \subset \mathcal{F}^G_P(M, (V_i)_{|P})$ so that it suffices to treat the case $Q_1 = Q_2 = P$. But then $\mathcal{F}^G_P(M_1, V_1) \oplus \mathcal{F}^G_P(M_2, V_2) \subset \mathcal{F}^G_P(M_1 \oplus M_2, V_1 \oplus V_2)$.

The proof is by induction on the length on M. If M is simple (where we may assume that P is maximal for M by the PQ-formula) then the statement follows from the above lemma. Otherwise, let $M_1 \subset M$ be some proper submodule and consider the exact sequence

$$0 \to \mathcal{F}^G_P(M/M_1, V) \to \mathcal{F}^G_P(M, V) \xrightarrow{p} \mathcal{F}^G_P(M_1, V) \to 0.$$

So let $U = U_1/U_2$ be some subquotient of $\mathcal{F}^G_P(M, V)$. We consider the induced exact sequence

$$0 \to \mathcal{F}^G_P(M/M_1, V) \cap U_1/\mathcal{F}^G_P(M/M_1, V) \cap U_2 \to U_1/U_2 \to p(U_1)/p(U_2) \to 0.$$

If $\mathcal{F}^G_P(M/M_1, V) \cap U_1/\mathcal{F}^G_P(M/M_1, V) \cap U_2 \in \{(0), U_1/U_2\}$ we may apply induction hypothesis to prove the claim. But also in the other case the inductive hypothesis applies. $\square$

Proposition 5.2.7. *Let $U \in \overline{{}^\infty\mathcal{F}^P_{\mathrm{alg}}}$ and suppose that there exist $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and $V \in \mathrm{Rep}^{\infty,\infty}_K(L_P)$ such that $U \subset \mathcal{F}^G_P(M, V)$. Suppose further that M is minimal, i.e. there is no proper quotient N of M such that $U \subset \mathcal{F}^G_P(N, V)$. Then we have $\mathcal{G}^G_P(U) = \bigoplus_L L \otimes W'_L$ as P-representations where L goes through all simple constituents of M and W_L is a quotient of $i^{Q_L}_P(V)$ for some parabolic subgroup Q_L depending on L.*

Proof. The proof is by induction on the length of M. If M is simple then we use Lemma 5.2.5 and the claim follows from Proposition 3.2.2. In general let $M_1 \subset M$ be a proper submodule such that M/M_1 is simple. We consider as above the induced exact sequence

$$0 \to \mathcal{F}^G_P(M/M_1, V) \to \mathcal{F}^G_P(M, V) \xrightarrow{p} \mathcal{F}^G_P(M_1, V) \to 0.$$

If $p(U) = 0$ we get a contradiction to the minimality of M. If $U \cap \mathcal{F}^G_P(M/M_1, V) = 0$, then $U \cong p(U) \subset \mathcal{F}^G_P(M_1, V)$. By the induction hypothesis we see that M_1 appears in $\mathcal{G}^G_P(U)$ as described above. By considering the composition $U \hookrightarrow \mathcal{F}^G_P(M, V) \to \mathcal{F}^G_P(M_1, V)$ we get by applying of $\mathcal{G}^G_P$ a splitting of the inclusion $M_1 \hookrightarrow M$ and thus a surjection $M \to M_1$ giving rise to a contradiction by the minimality of M.
Hence we obtain a non-trivial exact sequence

$$0 \to \mathcal{F}^G_P(M/M_1, V) \cap U \to U \to p(U) \to 0.$$

Thus the claim follows by induction once we have proved that $\mathcal{F}^G_P(M_1, V)$ satisfies again the minimality condition with respect to $p(U)$. But if $N_1 \subset M_1$ is a proper submodule with $p(U) \subset \mathcal{F}^G_P(M_1/N_1, V)$ then it would follow that $U \subset \mathcal{F}^G_P(M/N_1, V)$. This is again a contradiction to the minimality of M. $\square$

Proposition 5.2.8. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ be projective (resp. injective) and let $V \in {\rm Rep}^{\infty,\infty}_K(L_P)$ be an injective (resp. projective) object. Then $\mathcal{F}^G_P(M,V)$ is injective (resp. projective) in the category $\overline{{}^\infty\mathcal{F}^P_{\rm alg}}$.*

Proof. We consider here the case of injective objects. The case of projective objects is treated in a dual way. We consider thus a monomorphism $Z_1 \hookrightarrow Z_2$ in our category $\overline{{}^\infty\mathcal{F}^P_{\rm alg}}$ together with a morphism $Z_1 \to \mathcal{F}^G_P(M,V)$. Since any object in $\overline{{}^\infty\mathcal{F}^P_{\rm alg}}$ is a subquotient of an object lying in the image of a functor $\mathcal{F}^G_Q$ with $P \subset Q$ we may suppose by enlarging Z_2 that is has for simplicity the shape $\mathcal{F}^G_Q(N,W)$. Indeed if Z_2 is a submodule of $\mathcal{F}^G_Q(N,W)$ this is clear. If on the other hand, Z_2 is a quotient of $\mathcal{F}^G_Q(N,W)$ then we consider the preimage $\tilde{Z}_1 \hookrightarrow \tilde{Z}_2$ of $Z_1 \hookrightarrow Z_2$ in $\mathcal{F}^G_Q(N,W)$. We get an induced map $f : \tilde{Z}_1 \to \mathcal{F}^G_P(M,V)$ and if this extends to $\tilde{Z}_2$ then also to Z_2 since $\ker(\tilde{Z}_1 \to Z_1) = \ker(\tilde{Z}_2 \to Z_2)$ is mapped to zero under f. By the PQ-formula we see that $\mathcal{F}^G_Q(N,W) \hookrightarrow \mathcal{F}^G_Q(N, i^Q_P(W|L_P)) = \mathcal{F}^G_P(N, W|L_P)$. Hence we may even suppose that $P = Q$. Thus we arrived at the situation where we assume that $Z_2 = \mathcal{F}^G_P(N_2,W_2)$ for $N_2 \in \mathcal{O}^{\mathfrak{p}}_{\rm alg}$ and $W_2 \in {\rm Rep}^{\infty,\infty}_K(G)$.
On the other hand, we may also suppose that Z_1 has also the shape $\mathcal{F}^G_Q(N,W)$. Indeed, by dividing out the kernel of the morphism $Z_1 \to \mathcal{F}^G_P(M,V)$ (from the very beginning) we may assume that it is injective as well. By using Lemma 5.2.4 we see that there are $N \in \mathcal{O}^{\mathfrak{q}}_{\rm alg}$ and $W \in {\rm Rep}^{\infty,\infty}_K(L_Q)$ such that $\mathcal{F}^G_Q(N,W) \subset \mathcal{F}^G_Q(N_2,W_2)$ is a minimal object containing Z_1. By Proposition 5.2.7 we deduce that N and W appear in $\mathcal{G}^G_Q(Z_1)$. Hence the morphism $Z_1 \to \mathcal{F}^G_P(M,V)$ extends automatically to a morphism $\mathcal{F}^G_Q(N,W) \to \mathcal{F}^G_P(M,V)$.
Hence we may think that our embedding $Z_1 \hookrightarrow Z_2$ is of the shape $\mathcal{F}^G_Q(N_1,W_1) \hookrightarrow \mathcal{F}^G_P(N_2,W_2)$. It follows by Proposition 4.1.2, the bi-exactness of $\mathcal{F}^G_P$ and the exactness of the induction functor for smooth representations that it is induced by a surjection $N_2 \to N_1$ and a monomorphism $(W_1)_{|P} \hookrightarrow W_2$. For this note that $\mathcal{F}^G_Q(N_1,W_1) = \mathcal{F}^G_Q(N_1)\hat{\otimes}W_1$ and $\mathcal{F}^G_P(N_2,W_2) = \mathcal{F}^G_Q(N_2)\hat{\otimes}W_2$.
So for proving that $\mathcal{F}^G_P(M,V)$ is injective let $\mathcal{F}^G_Q(N_1,W_1) \to \mathcal{F}^G_P(M,V)$ be any morphism. Again it corresponds to a tuple of morphisms $M \twoheadrightarrow N_1$ and $(W_1)_{|P} \hookrightarrow V$. Since V is injective we see that there is an extension $W_2 \to V$. Further as M is projective we have a lift $M \to N_2$. The claim follows easily. □

Corollary 5.2.9. *The category $\overline{{}^\infty\mathcal{F}^P_{\rm alg}}$ has enough injective and projective objects.*

Proof. As above we consider here only the case of injectives. Let $U \in \overline{{}^\infty\mathcal{F}^P_{\rm alg}}$. Suppose first that it has the shape $\mathcal{F}^G_P(M,V)$. We choose a projective cover N of M and an embedding $V \hookrightarrow W$ into a smooth injective L_P-representation W. Then we have a topological

embedding $\mathcal{F}^G_P(M,V) \hookrightarrow \mathcal{F}^G_P(N,W)$ and by the result above the object $\mathcal{F}^G_P(N,W)$ is injective.
In general we know by Proposition 5.2.6 that it is a successive extension of such objects. As such it has an injective envelope, as well (Indeed, suppose that $0 \to A_1 \to U \to A_2 \to 0$ is exact and that $A_i \to I_i, i = 1,2$ are monomorphism into injective objects. Then we get an exact sequence $0 \to I_1 \to I_1 \oplus_{A_1} U \to A_2 \to 0$ and the middle term is isomorphic to $I_1 \oplus A_2$ by injectivity of I_1. But $I_1 \oplus A_2$ embeds into the injective object $I_1 \oplus I_2$. Therefore U embeds into $I_1 \oplus I_2$, as well.). □

5.3. **Extensions of generalized locally analytic Steinberg representations.** For a parabolic subgroup $P \subset G$, we abbreviate $I^G_P := \mathrm{Ind}^G_P(\mathbf{1})$ and denote by i^G_P the subspace of smooth vectors. The attached Steinberg representation is given by the quotient $V^G_P = \mathrm{Ind}^G_P(\mathbf{1})/\sum_{\mathbf{Q} \supsetneq \mathbf{P}} \mathrm{Ind}^{\mathbf{G}}_{\mathbf{Q}}(\mathbf{1})$. We shall determine the Ext-groups of these objects in our compactified categories.

We recall a result from [19]. Here we denote by ${}^\infty\mathrm{Ext}^*$ the corresponding Ext-groups in the category of smooth representations.

Proposition 5.3.1. *Let $I \subset \Delta$. Then we have*

$$ {}^\infty\mathrm{Ext}^*_{L_I}(\mathbf{1},\mathbf{1}) = \Lambda^*(X^*(\mathbf{L}_I)). $$

□

The next statement is contained in [10, Thm. 9.8].

Lemma 5.3.2. *For a parabolic subgroup Q of G, let $M = M_Q(0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} K$ be the generalized Verma module with respect to the trivial Q-module. Then M is projective in $\mathcal{O}^{\mathfrak{q}}_{\mathrm{alg}}$.* □

Proposition 5.3.3. *Let G be semi-simple and let $I, J \subset \Delta$. Then we have*

$$ Ext^*_{{}^\infty\overline{\mathcal{F}^B_{\mathrm{alg}}}}(I^G_{P_I}, I^G_{P_J}) = \begin{cases} \Lambda^*(X^*(\mathbf{L}_J)) & : \quad \text{if } J \subset I \\ 0 & : \quad \text{otherwise} \end{cases} $$

Proof. We set $P = P_I$ and $Q = P_J$.
1. Case. Suppose that $J \not\subset I$. Let $I^\bullet$ be an injective resolution of the trivial L_Q-representation in the category $\mathrm{Rep}^{\infty,\infty}_K(L_Q)$. Then by Lemma 5.3.2 and Proposition 5.2.8, $\mathcal{F}^G_Q(M_Q(0), I^\bullet)$ is an injective resolution of I^G_Q. Let $J^\bullet$ be an injective resolution of the trivial T-representation in the category $\mathrm{Rep}^{\infty,\infty}_K(T)$. Then $i^Q_B(J^\bullet)$ is an injective resolution of i^Q_B (in the category of smooth representations) since the induction functor is exact

and has with the Jacquet functor an exact left adjoint. Hence the embedding $\mathbf{1}_Q \to i_B^Q$ extends to a morphism of complexes $I^\bullet \to i_B^Q(J^\bullet)$. Here we may suppose by standard arguments that the maps in each degree are injective. We consider the induced (injective) maps $\mathcal{F}_Q^G(M_Q(0), I^\bullet) \to \mathcal{F}_Q^G(M_Q(0), i_B^Q(J^\bullet)) = \mathcal{F}_B^G(M_Q(0), J^\bullet)$. We shall see that any map $I_P^G \to \mathcal{F}_B^G(M_Q(0), J^i), i \geq 0$, vanishes which is enough for our claim. Indeed by Remark 4.1.3 it is induced on the Lie algebra part by a map $M_Q(0) \to M_P(0)$. Any such map vanishes if $Q \not\subset P$.

2. Case. Suppose that $J \subset I$. Then by applying Frobenius reciprocity any map $I_P^G \to \mathcal{F}_Q^G(M_Q(0), I^i) = \mathrm{Ind}_Q^G(I^i)$ is given by a map $(I_P^G)_{U_Q} = H^0(U_Q, I_P^G)' \to I^i$. The left hand side coincides by Proposition 3.2.5 with $H^0(\mathfrak{u}_Q, M_P(0))'$ which is a sum of algebraic representations and which contains the trivial representation. Since any map between an algebraic representation different from the trivial one and a smooth representation vanishes we see that any map $(I_P^G)_{U_Q} \to I^i$ corresponds to a map $\mathbf{1} \to I^i$. Hence the series of maps determines ${}^\infty\mathrm{Ext}^*_{L_J}(\mathbf{1}, \mathbf{1})$ which coincides with $\Lambda^*(X^*(\mathbf{L}_J))$ by Proposition 5.3.1. □

Theorem 5.3.4. *Let G be semi-simple. Let $I, J \subset \Delta$. Then*

$$\mathrm{Ext}^i_{\infty\overline{\mathcal{F}^B}}(V_{P_I}^G, V_{P_J}^G) = \begin{cases} K & |I \cup J \setminus I \cap J| = i \\ (0) & \textit{otherwise} \end{cases}.$$

Proof. In [20] we proved that the following complex is an acyclic resolution of $V_{P_I}^G$ by locally analytic G-representations,

$$(5.3.4) \qquad 0 \to I_G^G \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta\setminus K|=1}} I_{P_K}^G \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta\setminus K|=2}} I_{P_K}^G \to \cdots \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |K\setminus I|=1}} I_{P_K}^G \to I_{P_I}^G \to V_{P_I}^G \to 0.$$

The smooth version of this complex was used in [19] together with the smooth version of Proposition 5.3.3 to get by formal arguments the smooth version of our theorem. Hence the rest of the proof is the same as in loc.cit. □

If G is not necessarily semi-simple, then we have as in the smooth case a contribution of the center $Z(G)$. By using a Hochschild-Serre argument (cf. loc.cit.) we conclude:

Corollary 5.3.5. *Let G be reductive with center $Z(G)$ of rank d. Let $I, J \subset \Delta$. Then we have*

$$Ext^i_{\infty\overline{\mathcal{F}^B}}(V_{P_I}^G, V_{P_J}^G) = \begin{cases} K^{\binom{d}{j}} & : \quad i = |I \cup J| - |I \cap J| + j, \ j = 0, \ldots, d \\ 0 & : \quad otherwise \end{cases}$$

□

5.4. **Adjunction.** As a last application we want to discuss some adjunction formulas. For this we need some preparations.

Lemma 5.4.1. *Let $x, w \in W$ and let $\chi : T \to K^*$ be an algebraic character. Then*

$$(x \cdot_B \chi)^w = \mathrm{Ad}(w)(x) \cdot_{B^{w^{-1}}} \chi^w.$$

Proof. We compute

$$\begin{aligned} (x \cdot_B \chi)^w &= w(x(\chi + \rho_B) - \rho_B) \\ &= \mathrm{Ad}(w)(x)(w(\chi + \rho_B) - w\rho_B) \\ &= \mathrm{Ad}(w)(x)((\chi^w + \rho_{B^{w^{-1}}}) - \rho_{B^{w^{-1}}}) \\ &= \mathrm{Ad}(w)(x) \cdot_{B^{w^{-1}}} \chi^w \ . \end{aligned}$$

□

Let $M = M_{\overline{B}}(\chi) \in \mathcal{O}_{\mathrm{alg}}^{\overline{\mathfrak{b}}}$ be a Verma module with respect to the opposite Borel subgroup $\overline{B}$. Then $\mathcal{F}_{\overline{B}}^G(M) = \mathrm{Ind}_{\overline{B}}^G(\chi^{-1})$ and $\overline{H}_0(U_B, \mathcal{F}_{\overline{B}}^G(M)) = (H^0(\mathfrak{u}_{\overline{B}}, M)')^{w_0}$ by Lemma 3.2.8. Hence there is a natural homomorphism $(\chi^{-1})^{w_0} \to \overline{H}_0(U_B, \mathcal{F}_{\overline{B}}^G(M))$. If further χ is $\overline{B}$-dominant, then we have moreover a natural homomorphism

$$((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0} \to \overline{H}_0(U_B, \mathcal{F}_{\overline{B}}^G(M)).$$

These maps lead by composing with the functor $V \mapsto V_{U_B} = \overline{H}_0(U_B, V)$ to the following statements.

Theorem 5.4.2. *Let χ be a $\overline{B}$-dominant algebraic character. Then for any $w \in W$ and any highest weight module $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{b}^w}$ one has the identity*

$$\mathrm{Hom}_G(\mathrm{Ind}_{\overline{B}}^G(\chi^{-1}), \mathcal{F}_{B^w}^G(M)) = \mathrm{Hom}_T(((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0}, \mathcal{F}_{B^w}^G(M)_{U_B})$$

Proof. i) First let $M = M_{B^w}(\lambda)$ be a Verma module for some algebraic character λ of T. We start with the observation that both sides are at most one-dimensional. Indeed as for the LHS this follows from Proposition 4.1.5. As for the RHS we can identify it (see below) with the anti-dominant eigenspace in $H^0(\mathfrak{u}_{B^w}, M)$. This eigenspace is one-dimensional, as well.

Now we check, that the LHS does not vanish iff the RHS does. Since χ is $\overline{B}$-dominant we see that $\chi^{w^{-1}w_0}$ is B^w-dominant. The LHS does not vanish by Proposition 4.1.5 if and

only if $\lambda^{w_0 w} \uparrow_{\overline{B}} \chi$. This is equivalent to $\lambda \uparrow_{B^w} \chi^{w^{-1}w_0}$ by Lemma 5.4.1. Since $\chi^{w^{-1}w_0}$ is B^w-dominant and $w^{-1}w_0 w$ is the longest Weyl group element in W with respect to B^w, we see that the latter condition is equivalent to $(w^{-1}w_0 w)\cdot_{B^w} (\chi^{w^{-1}w_0}) \uparrow_{B^w} \lambda$.
On the other hand, the Jacquet module $\mathrm{Ind}^G_{B^w}(\lambda^{-1})_U$ coincides by Proposition 3.2.8 with $(H^0(\mathfrak{u}_{B^w}, M)')^w$. Its weights are given by the characters $(\mu^{-1})^w$ with $\mu \uparrow_{B^w} \lambda$. Moreover, $(w_0 \cdot_{\overline{B}} \chi)^{w_0} = ww^{-1}w_0(w_0 \cdot_{\overline{B}} \chi) = w(w^{-1}w_0 w \cdot_{B^w} \chi^{w^{-1}w_0})$ by Lemma 5.4.1. Thus the RHS does not vanish iff the LHS does not vanish. To see that the natural map between these one-dimensional spaces is an isomorphism follows in principal from Theorem 4.1.1 .

ii) Now let M be a quotient of $M_{B^w}(\lambda)$. Then $\mathcal{F}^G_{B^w}(M) \subset \mathrm{Ind}^G_{B^w}(\lambda^{-1})$ so that both vector spaces in the above stated formula are at most one-dimensional. Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_G(\mathrm{Ind}^G_{\overline{B}}(\chi^{-1}), \mathrm{Ind}^G_{B^w}(\lambda^{-1})) & = & \mathrm{Hom}_T(((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0}, \mathrm{Ind}^G_{B^w}(\lambda^{-1})_{U_B}) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_G(\mathrm{Ind}^G_{\overline{B}}(\chi^{-1}), \mathcal{F}^G_{B^w}(M)) & \to & \mathrm{Hom}_T(((w_0 \cdot_{\overline{B}} \chi)^{-1})^{w_0}, \mathcal{F}^G_{B^w}(M)_{U_B})
\end{array}$$

The upper line is an isomorphism by the first case. The LHS is an injection. In particular the lower line is an injection, as well. Since the spaces in question are at most one-dimensional the statement follows easily in this case. Note that if M is a proper quotient of $M_{B^w}(\lambda)$, then the objects in the lower line vanishes and the claim is trivial.

□

Remark 5.4.3. In [2] and [1] are presented adjunction formulas which use on the RHS Emerton Jacquet functor and which have a different style.

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