THE DE RHAM COHOMOLOGY OF DRINFELD'S HALF SPACE

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Abstract. Let $\mathcal{X} \subset \mathbb{P}^d_K$ be Drinfeld's half space over a p-adic field K. The de Rham cohomology of $\mathcal{X}$ was first computed by Schneider and Stuhler [SS]. Afterwards there were given different proofs by Alon, de Shalit, Iovita and Spiess [AdS, dS, IS]. This paper presents yet another approach for the determination of these invariants by analysing the de Rham complex of $\mathcal{X}$ from the viewpoint of results given in [O], [OS]. Moreover, we treat as a generalization the dual BGG complex of a given algebraic representation in the sense of Faltings [Fa] respectively Schneider [S].

1. Introduction

Let p be a prime number and let K be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. We denote by $\mathcal{X} = \mathcal{X}_K^{(d+1)} = \mathbb{P}^d_K \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$ (the complement of all K-rational hyperplanes in projective space) Drinfeld's half space [D] of dimension $d \geq 1$ over K. It is a rigid analytic variety over K which is equipped with an action of the p-adic Lie group $G = \mathrm{GL}_{d+1}(K)$. In [SS] Schneider and Stuhler determined the cohomology of $\mathcal{X}$ for any "good" cohomology theory (e.g. the étale and the de Rham cohomology) as G-representations. Here they make only use of the "good" properties as homotopy invariance, existence of a product structure etc. It turns out that the de Rham cohomology is given by

$$H^*_{\mathrm{dR}}(\mathcal{X}) = \bigoplus_{i=0}^{d} \mathrm{Hom}_K(v^G_{P_{(d+1-i,1,\ldots,1)}}, K)[-i]. \tag{1.1}$$

Here $P_{(d+1-i,1,\ldots,1)}$ is the (lower) standard parabolic subgroup of G which corresponds to the decomposition $(d+1-i, 1, \ldots, 1)$ of $d+1$. Further for a parabolic subgroup $P \subset G$, the smooth generalized Steinberg representation v^G_P is the unique irreducible quotient of the smooth unnormalized induced representation $i^G_P = \mathrm{ind}^G_P(K)$ with respect to the trivial P-representation [BW, Ca]. A few years later there were given different proofs of this result by Alon, de Shalit, Iovita and Spiess [AdS, dS, IS] by relating differential forms on $\mathcal{X}$ with harmonic cochains on the Bruhat-Tits building of G and considering logarithmic forms, respectively.

In this short notice we explain how we can determine the de Rham cohomology of $\mathcal{X}$ from its de Rham complex

$$\Omega^\bullet(\mathcal{X}) :\, 0 \to \mathcal{O}(\mathcal{X}) \to \Omega^1(\mathcal{X}) \to \cdots \to \Omega^d(\mathcal{X}) \to 0 \tag{1.2}$$

by applying some recent results given in [O, OS]. Here for $i = 0, \ldots, d$, the expression $\Omega^i(\mathcal{X}) = H^0(\mathcal{X}, \Omega^i)$ is the space of $\mathcal{X}$-valued sections of the usual homogeneous vector bundle Ω^i on projective space $\mathbb{P}^d_K$. Further the de Rham cohomology of $\mathcal{X}$ is the ordinary homology of the above complex since $\mathcal{X}$ is a Stein space. In contrast to the generalized Steinberg representations v_P^G the contributions $\Omega^i(\mathcal{X})$ in the de Rham complex are much bigger objects. Indeed they are reflexive K-Fréchet spaces with a continuous G-action [ST1]. Their strong duals $\Omega^i(\mathcal{X})'$, $i = 0, \ldots, d$, (i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence) are locally analytic G-representations in the sense of Schneider and Teitelbaum [ST2]. More generally, the same holds true for arbitrary homogeneous vector bundles on $\mathbb{P}_K$. In [O] there is constructed for any such homogeneous vector bundle $\mathcal{E}$, a decreasing filtration by closed G-stable subspaces

$$\mathcal{E}(\mathcal{X})^0 \supset \mathcal{E}(\mathcal{X})^1 \supset \cdots \supset \mathcal{E}(\mathcal{X})^{d-1} \supset \mathcal{E}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{E}) \tag{1.3}$$

on $\mathcal{E}(\mathcal{X})^0 = \mathcal{E}(\mathcal{X})$. As we will see in the next section the filtration behaves functorially in $\mathcal{E}$. Hence we get a filtered de Rham complex

$$\big(0 \to \mathcal{O}(\mathcal{X})^j \to \Omega^1(\mathcal{X})^j \cdots \to \Omega^d(\mathcal{X})^j \to 0\big)_{j=0,\ldots,d}. \tag{1.4}$$

In this paper we analyse its induced spectral sequence

$$E_0^{p,q} = \mathrm{gr}^p(\Omega^{p+q}(\mathcal{X})) \Rightarrow H^{p+q}(\Omega^\bullet(\mathcal{X})),$$

cf. [EGAIII]. In the case of $d = 2$ this was also carried out by Schraen [Sch]. The main theorem of this paper is the following result.

Theorem 1. *The spectral sequence E_0 attached to the filtered de Rham complex (1.4) degenerates at E_1 and yields the cohomology formula (1.1).*

In the final section we replace the de Rham complex by the dual BGG complex attached to an algebraic representation in the sense of Faltings [Fa, FC] respectively Schneider [S]. More precisely, let $\lambda \in \mathbb{Z}^{d+1}$ be a dominant weight with corresponding irreducible algebraic representation $V(\lambda)$. Then we consider the complex

$$0 \to \mathcal{E}_\lambda(\mathcal{X}) \to \mathcal{E}_{w_1\cdot\lambda}(\mathcal{X}) \to \cdots \to \mathcal{E}_{w_d\cdot\lambda}(\mathcal{X}) \to 0$$

where the $\mathcal{E}_{w_i\cdot\lambda}$ are certain homogeneous vector bundles on $\mathbb{P}^d_K$ depending on the weight $w_i \cdot \lambda$ (For a precise description we refer to the final section). It is proved in

[S] that it is quasi-isomorphic to the complex $\Omega^\bullet(\mathcal{X}) \otimes V(\lambda)$. It coincides with the de Rham complex (1.2) for $\lambda = 0$. In particular the determination of the homology of $\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})$ is not a surprising issue. Nevertheless, we get with the same proof:

Theorem. 1'. *Let $\lambda \in X^+$. Then the spectral sequence E_0 attached to the attached filtered complex degenerates at E_1 and we get*

$$H^*(\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})) = \bigoplus_{i=0}^{d} \mathrm{Hom}_K(v^G_{P_{(d+1-i,1,\ldots,1)}}, V(\lambda))[-i].$$

2. The proof of Theorem 1

We begin by recalling some terminology used in [O]. The following lines are an extract of [O, Section 1].

We consider the action of G on projection space $\mathbb{P}^d_K$ given by

$$g \cdot [q_0 : \cdots : q_d] := [q_0 : \cdots : q_d] g^{-1} .$$

We fix a homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}^d_K$ and let $\mathfrak{g} = \mathrm{Lie}\, G$ be the Lie algebra of G. Then $\mathcal{E}$ is naturally a $\mathfrak{g}$-module, i.e., there is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(\mathcal{E})$ which extends to the universal enveloping algebra $U(\mathfrak{g})$. Fix an integer $0 \leq j \leq d-1$ and let

$$\mathbb{P}^j_K = V(X_{j+1}, \ldots, X_d) \subset \mathbb{P}^d_K$$

be the closed K-subvariety defined by the vanishing of the coordinates $X_{j+1}, \ldots, X_d$. Let $P_{\underline{j+1}} = P_{(j+1,d-j)} \subset G$ be the (lower) standard-parabolic subgroup attached to the decomposition $(j+1, d-j)$ of $d+1$. It is clearly the stabilizer of $\mathbb{P}^j_K$ under the above action. Both the Zariski cohomology $H^*(\mathbb{P}^d_K \setminus \mathbb{P}^j_k, \mathcal{E})$ and the algebraic local cohomology $H^*_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E})$ are thus equipped with an action of the semi-direct product $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$. Here the semi-direct product is as usual induced by the adjoint action of $P_{(j+1,d-j)}$ on $\mathfrak{g}$. Further the natural long exact sequence

$$(2.1) \quad \cdots \to H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{E}) \to H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \to H^i(\mathbb{P}^d_K, \mathcal{E}) \to H^i(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{E}) \to \cdots$$

is equivariant with respect to this action. By general arguments in local cohomology theory [Ha2], one deduces that

$$(2.2) \qquad H^i_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) = \begin{cases} 0 & ; \; i < d-j \\ & \\ H^i(\mathbb{P}^d_K, \mathcal{E}) & ; \; i > d-j \end{cases} .$$

In the case $i = d - j$, we have thus an exact sequence

$$\begin{aligned} 0 \to \quad H^{d-j-1}(\mathbb{P}^d_K, \mathcal{E}) \quad &\to H^{d-j-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{E}) \to H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \\ \to \quad H^{d-j}(\mathbb{P}^d_K, \mathcal{E}) \quad &\to 0. \end{aligned}$$

We set

$$\begin{aligned} \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) &:= \ker\Big(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{E})\Big) \\ &\cong \operatorname{coker}\Big(H^{d-j-1}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{E})\Big) \end{aligned} \tag{2.3}$$

which is consequently a $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-module.

For an arbitrary parabolic subgroup $P \subset G$, let $\mathcal{O}^{\mathfrak{p}}$ be the full subcategory of the category $\mathcal{O}$ (in the sense of Bernstein, Gelfand, Gelfand [BGG]) consisting of $U(\mathfrak{g})$-modules of type $\mathfrak{p} = \operatorname{Lie} P$. We let $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ given by objects M such that all $\mathfrak{p}$-representations appearing in M are induced by finite-dimensional algebraic P-representations, cf. [OS].

Lemma 2. *The $U(\mathfrak{g})$-module $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E})$ lies in the category $\mathcal{O}^{\mathfrak{p}_{(j+1,d-j)}}_{\mathrm{alg}}$.*

Proof. This is an easy consequence of [O, Lemma 1.2.1] which states the existence of a finite-dimensional algebraic $P_{(j+1,d-j)}$-module which generates $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E})$ as $U(\mathfrak{g})$-module. □

The next statement is the main result of [O]. For its formulation we need some more notation. Denote by $\mathrm{Rep}^{\ell a}_K(G)$ the category of locally analytic G-representations with coefficients in K. For a parabolic subgroup $P \subset G$, let

$$\mathrm{Ind}^G_P : \mathrm{Rep}^{\ell a}_K(P) \to \mathrm{Rep}^{\ell a}_K(G)$$

be the locally analytic induction functor [F]. Let $\mathrm{St}_{d-j} = v_B^{\mathrm{GL}_{d-j}}$ be the smooth Steinberg representation of $\mathrm{GL}_{d-j}(K)$, $j = 0, \ldots, d$. We consider St_{d-j} as a representation of $P_{(j+1,d-j)}$ via the trivial action of the unipotent radical of $P_{(j+1,d-j)}$ and the factor $\mathrm{GL}_{j+1}(K) \subset L_{(j+1,d-j)}$, respectively. We equip St_{d-j} with the finest locally convex topology so that it becomes a locally analytic P-representation [ST2]. Thus for any algebraic representation N of $P_{(j+1,d-j)}$, the tensor product $N \otimes \mathrm{St}_{d-j}$ is a locally analytic representation. In particular this applies to the G-representation $H^{d-j}(\mathbb{P}^d_K, \mathcal{E})' \otimes v^G_{P_{(j+1,1,\ldots,1)}}$ which we also denote by $v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{E})')$.

Theorem 3. *For $j = 0, \dots, d-1$, there are extensions of locally analytic G-representations*

$$0 \to v^G_{P_{(j+1,1,\dots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{E})') \to (\mathcal{E}(\mathcal{X})^j / \mathcal{E}(X)^{j+1})' \to \mathrm{Ind}^G_{P_{\underline{j+1}}}(U'_j)^{\mathfrak{d}_j} \to 0.$$

Proof. This is [O, Theorem 1]. □

Here the $P_{\underline{j+1}}$-representation U'_j is a tensor product $N'_j \otimes \mathrm{St}_{d-j}$ of an algebraic $P_{\underline{j+1}}$-representation N'_j and St_{d-j}. The symbol $\mathfrak{d}_j$ indicates a system of differential equations depending on N_j. Here the representation N_j is characterized by the property that it generates the kernel of the natural homomorphism $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{E})$ as a module with respect to $U(\mathfrak{g})$.

This is exactly the starting point of the main construction in [OS]. In fact the locally analytic G-representation $\mathrm{Ind}^G_{P_{\underline{j+1}}}(U'_j)^{\mathfrak{d}_j}$ above can be characterized as the image of the object $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \times \mathrm{St}_{d-j}$ under a functor

$$\mathcal{F}^G_{P_{\underline{j+1}}} : \mathcal{O}^{\mathfrak{p}_{\underline{j+1}}}_{\mathrm{alg}} \times \mathrm{Rep}^\infty_K(L_{\underline{j+1}}) \longrightarrow \mathrm{Rep}^{\ell a}_K(G),$$

i.e.

$$\mathrm{Ind}^G_{P_{\underline{j+1}}}(U'_j)^{\mathfrak{d}_j} = \mathcal{F}^G_{P_{\underline{j+1}}}(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}), \mathrm{St}_{d-j}).$$

Here $\mathrm{Rep}^\infty_K(L_{\underline{j+1}})$ is the category of smooth $L_{\underline{j+1}}$-representations with coefficients over K.

Let us briefly recall the definition of this functor for an arbitrary parabolic subgroup $P \subset G$ with Levi decomposition $P = L \cdot U$. Let M be an object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Then there is a surjective map

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M$$

for some finite-dimensional algebraic P-representation $W \subset M$. Let V be a smooth L-representation. We consider V via the trivial action of U as a P-representation. As explained above the tensor product representation $W' \otimes_K V$ (where W' is the dual of W) is a locally analytic P-representation. Then

$$\mathcal{F}^G_P(M, V) = \mathrm{Ind}^G_P(W' \otimes_K V)^{\mathfrak{d}}$$

denotes the subset of functions $f \in \mathrm{Ind}^G_P(W' \otimes_K V)$ which are killed by the submodule $\mathfrak{d} = \ker(\phi)$. In loc.cit. it is shown that this subset is a well-defined G-stable closed subspace of $\mathrm{Ind}^G_P(W' \otimes_K V)$ and has therefore a natural structure of a locally analytic G-representation. The resulting functor is contravariant in the first and covariant in the second variable. It is proved in [OS, Prop. 4.10 a)] that $\mathcal{F}^G_P$ is exact in both arguments.

Now we come to the functoriality aspect concerning the filtration (1.3) mentioned in the introduction.

Lemma 4. *Let $f : \mathcal{E} \to \mathcal{F}$ be a homomorphism of homogeneous vector bundles on $\mathbb{P}^d_K$. Then f is compatible with the filtrations, i.e., f induces G-equivariant homomorphisms $\mathcal{E}(\mathcal{X})^i \to \mathcal{F}(\mathcal{X})^i$, $i \geq 0$.*

Proof. The definition of the filtration involves only the geometry of $\mathcal{X}$ (being the complement of a hyperplane arrangement) and not the homogeneous vector bundle itself. In fact, the K-Fréchet space $\mathcal{E}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{E})$ appears in an exact sequence

$$0 \to H^0(\mathbb{P}^d_K, \mathcal{E}) \to H^0(\mathcal{X}, \mathcal{E}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E}) \to H^1(\mathbb{P}^d_K, \mathcal{E}) \to 0.$$

We consider the K-Fréchet space $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$, where $\mathcal{Y} \subset \mathbb{P}^d_K$ is the "closed" complement of $\mathcal{X}$ in $\mathbb{P}^d_K$. The filtration is induced (by taking the preimage) by a similar one on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$ which we briefly review. Here all geometric objects are considered as pseudo-adic spaces in the sense of [Hu].

Let $\{e_0, \ldots, e_d\}$ be the standard basis of $V = K^{d+1}$ and let Δ be the standard basis of simple roots with respect to the Borel subgroup of lower triangular matrices. For any $\alpha_i \in \Delta$, put $V_{\alpha_i} = V_i = \bigoplus_{j=0}^{i} K \cdot e_j$ and set $Y_{\alpha_i} = Y_i = \mathbb{P}(V_i)$. For any subset $I \subset \Delta$ with $\Delta \setminus I = \{\alpha_{i_1} < \ldots < \alpha_{i_r}\}$, let $Y_I = Y_{\alpha_{i_1}} = \mathbb{P}(V_{i_1})$. Furthermore, let P_I be the lower parabolic subgroup of G, such that I coincides with the set of simple roots appearing in the Levi factor of P_I. Then gY_I is a closed subset of $\mathcal{Y}$ and we denote by $\Phi_{g,I} : gY_I \hookrightarrow \mathcal{Y}$ the corresponding embedding. Let $\mathbb{Z}$ be the constant sheaf on $\mathcal{Y}$ and set $\mathbb{Z}_{g,I} := (\Phi_{g,I})_*(\Phi^*_{g,I}(\mathbb{Z}))$. Then

$$\prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \subset \prod_{g \in G/P_I} \mathbb{Z}_{g,I}$$

denotes the subsheaf of locally constant sections with respect to the topological space G/P_I. In [O, 2.1] we proved that there is an acyclic resolution

$$0 \to \mathbb{Z} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 2}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \cdots$$

$$\cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = d-1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \prod_{g \in G/P_\emptyset}{}' \mathbb{Z}_{g,\emptyset} \to 0$$

of the constant sheaf $\mathbb{Z}$ on $\mathcal{Y}$. Let $i : \mathcal{Y} \hookrightarrow \mathbb{P}^d_K$ be the closed immersion. By applying the functor $\mathrm{Hom}(i_*(\ -\ ), \mathcal{E})$ to this complex, we get a spectral sequence converging to

$H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$. Finally the filtration on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$ is just the one induced by this spectral sequence. It follows now easily from the construction that f is compatible with the filtrations on $\mathcal{E}(\mathcal{X})$ and $\mathcal{F}(\mathcal{X})$. $\square$

The de Rham complex (1.2) together with Lemma 4 induces complexes

$$0 \to \mathcal{O}(\mathcal{X})^j/\mathcal{O}(\mathcal{X})^{j+1} \to \Omega^1(\mathcal{X})^j/\Omega^1(\mathcal{X})^{j+1} \to \cdots \to \Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1} \to 0,$$

$j = 0, \ldots, d-1$, which form just the E_0-term of the spectral sequence attached to the filtered de Rham complex (1.4). Apart from the terms $v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \Omega^i)')$, $i = 0, \ldots, d$, appearing in Theorem 3, this complex coincides by what we observed above with the dual of the complex

$$(2.4)\quad 0 \to \mathcal{F}^G_{P_{\underline{j+1}}}(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^d), \mathrm{St}_{d-j}) \to \cdots$$
$$\cdots \to \mathcal{F}^G_{P_{\underline{j+1}}}(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^1), \mathrm{St}_{d-j}) \to \mathcal{F}^G_{P_{\underline{j+1}}}(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{O}), \mathrm{St}_{d-j}) \to 0$$

Proposition 5. *The above complex is acyclic.*

Proof. By the exactness of the functor $\mathcal{F}^G_P$ in the first entry it suffices to prove that the complex

$$0 \to \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{O}) \to \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^1) \to \cdots \to \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^d) \to 0$$

of $\mathfrak{g}$-modules is acyclic. Hence we have reduced the whole issue to a computation in coherent cohomology of projective space. Set $V = \mathbb{P}^d_K \setminus \mathbb{P}^j_K = \bigcup_{k=j+1}^d D(T_k)$ where we denote as usual by $T_0, \ldots, T_d$ the homogeneous coordinate functions on $\mathbb{A}^{d+1}_K$. Then by identity (2.3) we have the description

$$\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^i) \cong \mathrm{coker}(H^{d-j-1}(\mathbb{P}^d_K, \Omega^i) \to H^{d-j-1}(V, \Omega^i))$$

for all $i \geq 0$. On the other hand, we have the following well-known chain of identities

$$(2.5)\qquad K = H^0(\mathbb{P}^d_K, \mathcal{O}) = H^1(\mathbb{P}^d_K, \Omega^1) = \cdots = H^d(\mathbb{P}^d_K, \Omega^d),$$

cf. [Ha1]. All other cohomology groups vanish. Therefore it is enough to prove that the homology in degree $d-j-1$ of the complex

$$0 \to H^{d-j-1}(V, \mathcal{O}) \to H^{d-j-1}(V, \Omega^1) \to \cdots \to H^{d-j-1}(V, \Omega^d) \to 0$$

induces $H^{d-j-1}(\mathbb{P}^d_K, \Omega^{d-j-1}) = K$ and vanishes elsewhere. For this issue, we consider the double complex

$$
\begin{array}{ccccc}
\bigoplus_{k=j+1}^{d} \Omega^d(D(T_k)) & \rightarrow & \bigoplus_{j+1\leq k<l\leq d} \Omega^d(D(T_k)\cap D(T_l)) & \rightarrow \cdots \rightarrow & \Omega^d(\bigcap_{k=j+1}^{d} D(T_k)) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{k=j+1}^{d} \Omega^i(D(T_k)) & \rightarrow & \bigoplus_{j+1\leq k<l\leq d} \Omega^i(D(T_k)\cap D(T_l)) & \rightarrow \cdots \rightarrow & \Omega^i(\bigcap_{k=j+1}^{d} D(T_k)) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{k=j+1}^{d} \Omega^1(D(T_k)) & \rightarrow & \bigoplus_{j+1\leq k<l\leq d} \Omega^1(D(T_k)\cap D(T_l)) & \rightarrow \cdots \rightarrow & \Omega^1(\bigcap_{k=j+1}^{d} D(T_k)) \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{k=j+1}^{d} \mathcal{O}(D(T_k)) & \rightarrow & \bigoplus_{j+1\leq k<l\leq d} \mathcal{O}(D(T_k)\cap D(T_l)) & \rightarrow \cdots \rightarrow & \mathcal{O}(\bigcap_{k=j+1}^{d} D(T_k))
\end{array}
$$

whose total complex gives rise to the de Rham cohomology of V, cf. [Gr]. Since $H^k_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \Omega^i) = 0$ for all $k < d-j$ by identity (2.2), we see that $H^k(V, \Omega^i) = H^k(\mathbb{P}^d, \Omega^i)$ for all such indices k. Evaluating the double complex along the horizontal lines we get thus the E_1-term:

$$
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & H^{d-j-1}(V, \Omega^d) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & K & \rightarrow & H^{d-j-1}(V, \Omega^{d-j-2}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & \iddots & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & K & \rightarrow \cdots \rightarrow & 0 & \rightarrow & H^{d-j-1}(V, \Omega^1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & H^{d-j-1}(V, \mathcal{O})
\end{array}
$$

But the de Rham cohomology of V is easily computed in another way. In fact, using the comparison isomorphism with Betti cohomology [Gr] and the long exact cohomology sequence for constant coefficients (2.1), we see that $H^*_{\mathrm{dR}}(V) = \bigoplus_{i=0}^{d-j-1} K[-2i]$. The claim follows now easily. □

For the proof of Theorem 1, we recall that $E_0^{p,q} = \mathrm{gr}^p(\Omega^{p+q}(\mathcal{X})) \Rightarrow H^{p+q}(\Omega^\bullet(\mathcal{X}))$ is the induced spectral sequence of our filtered de Rham complex.

Corollary 6. *The E_1-term of the above spectral sequence has the shape*

$$E_1^{p,q} = \begin{cases} \mathrm{Hom}_K(v^G_{P_{(d+1-p,1,\ldots,1)}}, K) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

for $p \geq 0$. Hence it degenerates at E_1 and we get the formula (1.1).

This finishes the proof of Theorem 1. □

3. A generalization: The dual BGG complex

In this final section we consider a generalization of what we have done before. We replace the de Rham complex (1.2) by the dual BGG complex attached to an algebraic representation in the sense of Faltings [Fa, FC] respectively Schneider [S]. For introducing this complex we have to introduce some more notation.

Let $\mathbf{G} = \mathbf{GL_{d+1}}$ considered as a linear algebraic group over K. Let $\mathbf{T} \subset \mathbf{G}$ be the diagonal torus and let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of lower triangular matrices. Denote by $\Phi \subset X^*(\mathbf{T})$ the corresponding set of roots of $\mathbf{G}$. Let $\mathbf{B}^+ \subset \mathbf{G}$ the Borel subgroup of upper triangular matrices and and let $\Delta^+ \subset \Phi$ be the set of simple roots with respect to B^+. We consider the set

$$X^+ = \{\lambda \in X^*(\mathbf{T}) \mid (\lambda, \alpha^\vee) \geq 0 \ \forall \alpha \in \Delta^+\}$$

of dominant weights in $X^*(\mathbf{T})$. For $\lambda \in X^+$, we denote by $V(\lambda)$ the finite-dimensional irreducible algebraic $\mathbf{G}$-representation over K of highest weight λ, cf. [Ja]. We consider $V(\lambda)$ as an G-representation in the sequel.

Let $\mathbf{P_{(1,d)}}$ be the stabilizer of the base point $[1 : 0 : \cdots : 0] \in \mathbb{P}^d_K(K)$ and let $\mathbf{L} = \mathbf{L_{(1,d)}} \subset \mathbf{P_{(1,d)}}$ be the Levi subgroup. Further let

$$X^+_L = \{\lambda \in X^*(\mathbf{T}) \mid (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta^+_L\}$$

be the set of $\mathbf{L}$-dominant weights where $\Delta_L^+ \subset \Delta$ consist of those simple roots which appear in $\mathbf{L}$. Every $\lambda \in X_L^+$ gives rise to a finite-dimensional irreducible algebraic $\mathbf{L}$-representation $V_L(\lambda)$. We consider it as a $\mathbf{P}$-module by letting act the unipotent radical trivially on it. Let

$$\pi : \mathbf{G} \to \mathbf{G}/\mathbf{P}_{(\mathbf{1},\mathbf{d})}$$

be the projection map and identify $\mathbf{G}/\mathbf{P}_{(\mathbf{1},\mathbf{d})}$ with $\mathbb{P}^d_K$. Let V be a finite-dimensional algebraic representation of $\mathbf{P}_{(\mathbf{1},\mathbf{d})}$. For a Zariski open subset $U \subset \mathbb{P}^d_K$, put

$$\mathcal{E}_V(U) := \Big\{ \text{ algebraic morphisms } f: \pi^{-1}(U) \to V \mid f(gp) = p^{-1}f(g) \text{ for all } g \in \mathbf{G}(\overline{K}), p \in \mathbf{P}_{(\mathbf{1},\mathbf{d})}(\overline{K}) \Big\}.$$

Then $\mathcal{E}_V$ defines a homogeneous vector bundle on $\mathbb{P}^d_K$ and every homogeneous vector bundle is of this shape. We consider it at the same time as such an object over the rigid-analytic space $(\mathbb{P}^d_K)^{\mathrm{rig}}$. If $\lambda \in X_L^+$ then we set $\mathcal{E}_\lambda := \mathcal{E}_{V_L(\lambda)}$.

Let W be the Weyl group of $\mathbf{G}$ and consider the dot action $\cdot$ of W on $X^*(\mathbf{T})$ given by

$$w \cdot \chi = w(\chi + \rho) - \rho,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Let $W_L \subset W$ be the Weyl group of L. Consider the set ${}^LW = W_L \backslash W$ of left cosets and the cycles

$$w_i := (1, 2, 3, \ldots, i+1) \in S_{d+1} \cong W,$$

$i = 0, \ldots, d$, which are just the representatives of shortest length in their cosets. If $\lambda \in X^+$ and $w \in {}^LW$ then $w \cdot \lambda \in X_L^+$. The dual BGG-complex of $\lambda \in X^+$ is given by the complex

$$0 \to \underline{V}(\lambda) \to \mathcal{E}_\lambda \to \mathcal{E}_{w_1 \cdot \lambda} \to \cdots \to \mathcal{E}_{w_d \cdot \lambda} \to 0. \tag{3.1}$$

Here $\underline{V}(\lambda)$ is the constant sheaf on $\mathbb{P}^d_K$ with values in $V(\lambda)$. By considering sections in $\mathcal{X}$ we get a complex

$$0 \to V(\lambda) \to \mathcal{E}_\lambda(\mathcal{X}) \to \mathcal{E}_{w_1 \cdot \lambda}(\mathcal{X}) \to \cdots \to \mathcal{E}_{w_d \cdot \lambda}(\mathcal{X}) \to 0. \tag{3.2}$$

It is proved in [S] that the complex $\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})$ is quasi-isomorphic to the complex $\Omega^\bullet(\mathcal{X}) \otimes V(\lambda)$. The classical case is [Fa, FC]. For $\lambda = 0$, we get the usual de Rham complex.

Proof. (of Theorem 1') The proof is the same as above. Instead of the series of identities (2.5) we use this time the Borel-Weil-Bott theorem, cf. [Ja]. Indeed by considering the spectral sequence $(R^m\mathrm{ind}_P^G)(R^n\mathrm{ind}_B^P)(M) \Rightarrow R^n\mathrm{ind}_B^G(M)$, cf. [Ja, Prop. 4.5 c)] we deduce that $H^i(\mathbb{P}^d_K, \mathcal{E}_{w \cdot \lambda}) = H^i(G/B, \mathcal{L}_{w \cdot \lambda})$ since $w \cdot \lambda \in X_L^+$ is

L-dominant. Here $\mathcal{L}_{w\cdot\lambda}$ is the line bundle on G/B attached to the weight λ. Hence we get

$$H^i(\mathbb{P}^d_K, \mathcal{E}_{w_j\cdot\lambda}) = \begin{cases} H^0(\mathbb{P}^d_K, \mathcal{E}_\lambda) & i = j \\ 0 & i \neq j. \end{cases}$$

Moreover, the latter object has the description $H^0(\mathbb{P}^d_K, \mathcal{E}_\lambda) = V(\lambda)$. As for the interpretation of the de Rham cohomology of V we use the fact [Fa, FC] that the complex $\mathcal{E}_{\bullet\cdot\lambda}(V)$ is quasi-isomorphic to $V(\lambda) \otimes \Omega^\bullet(V)$ instead. The claim follows. □

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