THE JORDAN-HÖLDER SERIES OF THE LOCALLY ANALYTIC STEINBERG REPRESENTATION

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ABSTRACT. We determine the composition factors of a Jordan-Hölder series including multiplicities of the locally analytic Steinberg representation. For this purpose, we prove the acyclicity of the evaluated locally analytic Tits complex giving rise to the Steinberg representation. Further we describe some analogue of the Jacquet functor applied to the irreducible principal series representation constructed in [OS2].

1. INTRODUCTION

In this paper we study the locally analytic Steinberg representation V_B^G for a given split reductive p-adic Lie group G. This type of object arises in various fields of Representation Theory, cf. [Hum1, DOR]. In the smooth representation theory of p-adic Lie groups as well as in the case of finite groups of Lie type, it is related to the (Bruhat-)Tits building and has therefore interesting applications [Car, DM, Ca1]. In the locally analytic setting it comes up so far in the p-adic Langlands program with respect to semi-stable, non-crystalline Galois representation [Br]. More precisely, V_B^G coincides with the set of locally analytic vectors in the continuous Steinberg representation which should arise in a possible local p-adic Langlands correspondence on the automorphic side. Our main result gives the composition factors including multiplicities of a Jordan-Hölder series for V_B^G. This answers a question raised by Teitelbaum in [T].

Let G be a split reductive p-adic Lie group over a finite extension L of $\mathbb{Q}_p$ and let $B \subset G$ be a Borel subgroup. The definition of V_B^G is completely analogous to the above mentioned classical cases. It is given by the quotient

$$V_B^G = I_B^G / \sum_{P \supsetneq B} I_P^G,$$

where $I^G_P = C^{an}(G/P, K)$ is the G-representation consisting of locally L-analytic functions on the partial flag manifold G/P with coefficients in some fixed finite extension K of L. In contrast to the smooth situation or that of a finite group of Lie type, the locally analytic Steinberg representation is not irreducible. Indeed, it contains the smooth Steinberg representation $v^G_B = i^G_B / \sum_{P \supsetneq B} i^G_P$, where $i^G_P = C^\infty(G/P, K)$, as a closed subspace. On the other hand, it is a consequence of the construction in [OS2] that V^G_B has a composition series of finite length and therefore the natural question of determining its Jordan-Hölder series comes up. Morita [Mo] proved that for $G = \mathrm{GL}_2$, the topological dual of V^G_B is isomorphic to the space of K-valued sections of the canonical sheaf ω on the Drinfeld half space $\mathcal{X} = \mathbb{P}^1 \setminus \mathbb{P}^1(L)$. In higher dimensions Schneider and Teitelbaum [ST1] construct an injective map from the space of K-valued sections of ω to the topological dual of V^G_B. However, the natural hope that this map is an isomorphism in general turns out not to be correct. In fact, this follows by gluing some results of [Schr] and [O] considering the Jordan-Hölder series of both representations in the case of GL_3.

In order to determine the composition factors of V^G_B in the general case, we apply the machinery constructing locally analytic G-representations $\mathcal{F}^G_P(M, V)$ developed in [OS2]. Here M is an object of type P in the category $\mathcal{O}$ of Lie algebra representations of $\mathrm{Lie}\, G$ and V is a smooth admissible representation of a Levi factor $L_P \subset P$ (we refer to Section 2 for a more detailed recapitulation). It is proved in loc.cit. that $\mathcal{F}^G_P(M, V)$ is topologically irreducible if M and V are simple objects and if furthermore P is maximal for M. On the other hand, it is shown that $\mathcal{F}^G_P$ is bi-exact which allows us to speak of a locally analytic BGG-resolution. This latter aspect is one main ingredient for proving the acyclicity of the evaluated locally analytic Tits complex

$$0 \to I^G_G \to \bigoplus_{\substack{K \subset \Delta \\ |\Delta \setminus K| = 1}} I^G_{P_K} \to \bigoplus_{\substack{K \subset \Delta \\ |\Delta \setminus K| = 2}} I^G_{P_K} \to \cdots \to \bigoplus_{\substack{K \subset \Delta \\ |K| = 1}} I^G_{P_K} \to I^G_B \to V^G_B \to 0.$$

Here Δ is the set of simple roots with respect to B and a choice of a maximal torus $T \subset B$. Hence the determination of the composition factors of V^G_B is reduced to the situation of an induced representation I^G_P which lies in the image of the functor $\mathcal{F}^G_P$. This leads to the question when two irreducible representations of the shape $\mathcal{F}^G_P(M, V)$ with V a composition factor of i^G_B are isomorphic. It turns out that this holds true if and only if all ingredients are the same. Thus we arrive at the stage where Kazhdan-Lusztig theory enters the game.

Now we formulate our main result. For two reflections w, w' in the Weyl group W, let $m(w', w) \in \mathbb{Z}_{\geq 0}$ be the multiplicity of the simple highest weight module $L(w \cdot 0)$ of weight $w \cdot 0 \in \mathrm{Hom}_L(\mathrm{Lie}\, T, K)$ in the Verma module $M(w' \cdot 0)$ of weight $w' \cdot 0$. Let $\mathrm{supp}(w) \subset W$ be the subset of simple reflections which appear in w. In the following theorem we identify the set of simple reflections with the set Δ.

Theorem: *For $w \in W$, let $I \subset \Delta$ be a subset such that the standard parabolic subgroup P_I attached to I is maximal for $L(w \cdot 0)$. Let $v_{P_J}^{P_I}$ be the smooth generalized Steinberg representation of L_{P_I} with respect to a subset $J \subset I$. Then the multiplicity of the irreducible representation $\mathcal{F}_{P_I}^G(L(w \cdot 0), v_{P_J}^{P_I})$ in V_B^G is given by*

$$\sum_{\substack{w' \in W \\ \mathrm{supp}(w') = J}} (-1)^{\ell(w') + |J|} m(w', w),$$

and we obtain in this way all the Jordan-Hölder factors of V_B^G. In particular, the smooth Steinberg representation v_B^G is the only smooth subquotient of V_B^G. Moreover, the representation $\mathcal{F}_{P_I}^G(L(w \cdot 0), v_{P_J}^{P_I})$ appears with a non-zero multiplicity if and only if $J \subset \mathrm{supp}(w)$.

We close this introduction by mentioning that we discuss in our paper more generally generalized locally analytic Steinberg representation V_P^G, as well as their twisted versions $V_P^G(\lambda)$ involving a dominant weight $\lambda \in X^*(T)$.

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Notation: We denote by p a prime number and by $K \supset L \supset \mathbb{Q}_p$ finite extensions of the field of p-adic integers $\mathbb{Q}_p$. Let O_L be the ring of integers in L and fix an uniformizer π of O_L. We let $k_L = O_L/(\pi)$ be the corresponding residue field. For a locally convex K-vector space V, we denote by V' its strong dual, i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence, cf. [S].

For an algebraic group $\mathbf{G}$ over L we denote by $G = \mathbf{G}(L)$ the p-adic Lie group of L-valued points. We use a Gothic letter $\mathfrak{g}$ to indicate its Lie algebra. We denote by $U(\mathfrak{g}) := U(\mathfrak{g} \otimes_L K)$ the universal enveloping algebra of $\mathfrak{g}$ after base change to K. We let $\mathcal{O}$ be the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand in the sense of [OS2].

2. The functors $\mathcal{F}^G_P$

In this first section we recall the definition of the functors $\mathcal{F}^G_P$ constructed in [OS2]. As explained in the introduction they are crucial for the determination of a Jordan-Hölder series of the locally analytic Steinberg representation.

Let $\mathbf{G}$ be a split reductive algebraic group over L. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus and fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ containing $\mathbf{T}$. Let Δ be the set of simple roots, Φ the set of roots and Φ^+ the set of positive roots of $\mathbf{G}$ with respect to $\mathbf{T} \subset \mathbf{B}$. In what follows we assume that our prime number p is odd, if the root system Φ has irreducible components of type B, C or $F4$, and if Φ has irreducible components of type $G2$ we assume that $p > 3$.[1]

We identify the group $X^*(\mathbf{T})$ of algebraic characters of $\mathbf{T}$ via the derivative as a lattice in $\mathrm{Hom}_L(\mathfrak{t}, K)$. Let $\mathbf{P}$ be standard parabolic subgroup (std psgp). Consider the Levi decomposition $\mathbf{P} = \mathbf{L_P} \cdot \mathbf{U_P}$ where $\mathbf{L_P}$ is the Levi subgroup containing $\mathbf{T}$ and $\mathbf{U_P}$ is its unipotent radical. Let $\mathbf{U_P^-}$ be the unipotent radical of the opposite parabolic subgroup. Let $\mathcal{O}^{\mathfrak{p}}$ be the full subcategory of $\mathcal{O}$ consisting of $U(\mathfrak{g})$-modules of type $\mathfrak{p} = \mathrm{Lie}\,\mathbf{P}$. Its objects are $U(\mathfrak{g})$-modules M over the coefficient field K satisfying the following properties:

(1) The action of $\mathfrak{u}_P$ on M is locally finite.

(2) The action of $\mathfrak{l}_P$ on M is semi-simple and locally finite.

(3) M is finitely generated as $U(\mathfrak{g})$-module.

In particular, we have $\mathcal{O} = \mathcal{O}^{\mathfrak{b}}$. Moreover, if $\mathbf{Q}$ is another parabolic subgroup of $\mathbf{G}$ with $\mathbf{P} \subset \mathbf{Q}$, then $\mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$.

[1] This restriction is here to use results of [OS2].

Let $\mathrm{Irr}(\mathfrak{l}_P)^{\mathrm{fd}}$ be the set of finite-dimensional irreducible $\mathfrak{l}_P$-modules. Any object in $\mathcal{O}^{\mathfrak{p}}$ has by property (2) a direct sum decomposition into $\mathfrak{l}_P$-modules

$$M = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathfrak{l}_P)^{\mathrm{fd}}} M_{\mathfrak{a}} \tag{2.1}$$

where $M_{\mathfrak{a}} \subset M$ is the $\mathfrak{a}$-isotypic part of the finite-dimensional irreducible representation $\mathfrak{a}$. We let $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ given by objects such that all $\mathfrak{l}_P$-representations appearing in (2.1) are induced by finite-dimensional algebraic $\mathbf{L_P}$-representations. The above inclusion $\mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$ is compatible with these new subcategories, i.e. we also have $\mathcal{O}^{\mathfrak{q}}_{\mathrm{alg}} \subset \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. In particular, the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ contains all finite-dimensional $\mathfrak{g}$-modules which come from algebraic $\mathbf{G}$-modules. Every object in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ has a Jordan-Hölder series which coincides with the Jordan-Hölder series in $\mathcal{O}$.

Let $\mathrm{Rep}^{\infty,\mathrm{adm}}_K(L_P)$ be the category of smooth admissible L_P-representations with coefficients over K. In [OS2] there is constructed a bi-functor

$$\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}^{\infty,\mathrm{adm}}_K(L_P) \longrightarrow \mathrm{Rep}^{\ell a}_K(G),$$

where $\mathrm{Rep}^{\ell a}_K(G)$ denotes the category of locally analytic G-representations with coefficients in K. It is contravariant in the first and covariant in the second variable. Furthermore, $\mathcal{F}^G_P$ factorizes through the full subcategory of admissible representations in the sense of Schneider and Teitelbaum [ST2]. Let us recall the definition of $\mathcal{F}^G_P$.

Let M be an object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. By the defining axioms (1) - (3) above there is a surjective map

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M \tag{2.2}$$

for some finite-dimensional algebraic P-representation $W \subset M$. Let V be additionally a smooth admissible L_P-representation. We consider V via the trivial action of U_P as a P-representation. Further by equipping V with the finest locally convex topology it becomes a locally analytic P-representation, cf. [ST4, §2]. Hence we may consider the tensor product representation $W' \otimes_K V$ as a locally analytic P-representation. Let $\mathrm{Ind}^G_P : \mathrm{Rep}^{\ell a}_K(P) \to \mathrm{Rep}^{\ell a}_K(G)$ be the locally analytic induction functor [Fe]. Then

$$\mathcal{F}^G_P(M, V) = \mathrm{Ind}^G_P(W' \otimes_K V)^{\mathfrak{d}}$$

denotes the subset of functions $f \in \mathrm{Ind}^G_P(W' \otimes_K V)$ which are killed by the $U(\mathfrak{g})$-submodule $\mathfrak{d} = \ker(\phi) \subset U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ for the action obtained by joining left translation for elements of $U(\mathfrak{g})$ and evaluation in W' for elements of W. Then $\mathcal{F}^G_P(M, V)$

is a well-defined G-stable closed subspace of $\mathrm{Ind}_P^G(W' \otimes_K V)$ and has therefore a natural structure of a locally analytic G-representation. Further the above construction is even functorial. If $V = \mathbf{1}$ is the trivial L_P-representation, we simply write $\mathcal{F}_P^G(M)$ instead of $\mathcal{F}_P^G(M, \mathbf{1})$. The functors $\mathcal{F}_P^G$ satisfy the following properties, cf. [OS2, Prop. 4.9, Thm. 5.8]:

- (exactness) The bi-functor $\mathcal{F}_P^G$ is exact in both arguments.
- (PQ-formula) Let $Q \supset P$ be another parabolic subgroup. We suppose that $L_P \subset L_Q$. If V is a smooth admissible L_P-representation, then we denote by
$$i_P^Q(V) = \mathrm{Ind}_P^{\infty, Q}(V)$$
the smooth induced Q-representation. Note that as L_Q-representation one has the identification $i_P^Q(V) = i_{L_P \cdot (U_P \cap L_Q)}^{L_Q}(V)$. Let $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{q}} \subset \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$. Then there is the following identity:
$$\mathcal{F}_P^G(M, V) = \mathcal{F}_Q^G(M, i_P^Q(V)).$$
- (irreducibility) A std psgp $P \subset G$ is called maximal for an object $M \in \mathcal{O}$ if $M \in \mathcal{O}^{\mathfrak{p}}$ and if $M \notin \mathcal{O}^{\mathfrak{q}}$ for all $\mathfrak{q} \supsetneq \mathfrak{p}$. Let P be a std psgp, maximal for a simple object $M \in \mathcal{O}_{\mathrm{alg}}$. Further let V be an irreducible smooth admissible L_P-representation, then $\mathcal{F}_P^G(M, V)$ is (topologically) irreducible.

In [OS2] it is explained how one can deduce from the previous properties of the bi-functors $\mathcal{F}_P^G$ the Jordan-Hölder series of a representation $\mathcal{F}_P^G(M, V)$. Let us recall this procedure in the case $V = \mathbf{1}$. The smooth generalized Steinberg representation to P is the quotient
$$v_P^G = i_P^G / \sum_{P \subsetneq Q \subset G} i_Q^G.$$
This is an irreducible G-representation and all irreducible subquotients of i_P^G occur in the shape v_Q^G with $Q \supset P$ and with multiplicity one, cf. [BoWa, Ch. X,§4], [Ca2].

Let M be an object of the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$. Then it has a Jordan-Hölder series
$$M = M^0 \supset M^1 \supset M^2 \supset \ldots \supset M^r \supset M^{r+1} = (0)$$
in the same category. By applying the functor $\mathcal{F}_P^G$ to it we get a sequence of surjections
$$\mathcal{F}_P^G(M) \xrightarrow{p_0} \mathcal{F}_P^G(M^1) \xrightarrow{p_1} \mathcal{F}_P^G(M^2) \xrightarrow{p_2} \ldots \xrightarrow{p_{r-1}} \mathcal{F}_P^G(M^r) \xrightarrow{p_r} (0).$$

For any integer i with $0 \leq i \leq r$, we put

$$q_i := p_i \circ p_{i-1} \circ \cdots \circ p_1 \circ p_0$$

and set

$$\mathcal{F}^i := \ker(q_i)$$

which is a closed subrepresentation of $\mathcal{F}^G_P(M)$. We obtain a filtration

$$\mathcal{F}^{-1} = (0) \subset \mathcal{F}^0 \subset \cdots \subset \mathcal{F}^{r-1} \subset \mathcal{F}^r = \mathcal{F}^G_P(M) \tag{2.3}$$

by closed subspaces with

$$\mathcal{F}^i/\mathcal{F}^{i-1} \simeq \mathcal{F}^G_P(M^i/M^{i+1}) \ .$$

Let $Q_i = L_i \cdot U_i \supset P$ a std psgp maximal for M^i/M^{i+1}. Then by the PQ-formula, we get

$$\mathcal{F}^G_P(M^i/M^{i+1}) = \mathcal{F}^G_{Q_i}(M^i/M^{i+1}, i^{Q_i}_P).$$

where $i^{Q_i}_P = i^{L_i}_{L_i \cap P}$. We conclude that the representations

$$\mathcal{F}^G_{Q_i}(M^i/M^{i+1}, v^{Q_i}_R)$$

where R is a std psgp of G with $Q_i \supset R \supset P$ and $v^{Q_i}_R = v^{L_i}_{L_i \cap R}$ are the topologically irreducible constituents of $\mathcal{F}^G_{Q_i}(M^i/M^{i+1}, i^{Q_i}_P)$. By refining the filtration (2.3) we get thus a Jordan-Hölder series of $\mathcal{F}^G_P(M)$.

Finally we recall the parabolic BGG resolution of a finite-dimensional algebraic G-representation [Le]. We let $\langle \, , \, \rangle$ be the natural pairing $X^*(\mathbf{T}) \times \mathbf{X}_*(\mathbf{T}) \to \mathbb{Z}$ defined by $x(u(t)) = t^{\langle x,u \rangle}$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in X_*(\mathbf{T})$ the cocharacter associated to α as in [Sp, §2.2]. Let W be the Weyl group of G and consider the dot action $\cdot$ of W on $X^*(\mathbf{T})$ given by

$$w \cdot \chi = w(\chi + \rho) - \rho,$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. For a character $\lambda \in X^*(\mathbf{T})$, let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K_\lambda$$

be the corresponding Verma module. Clearly $M(\lambda)$ is an object of $\mathcal{O}_{\mathrm{alg}}$. We denote its irreducible quotient by $L(\lambda)$. Let

$$X_+ = \{\lambda \in X^*(\mathbf{T}) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\}$$

be the set of dominant weights in $X^*(\mathbf{T})$. If $\lambda \in X_+$, then $L(\lambda)$ is finite-dimensional and comes from an irreducible algebraic G-representation. In this situation, we also write $V(\lambda)$ for $L(\lambda)$.

For a subset $I \subset \Delta$, let $P = P_I \subset G$ be the attached std psgp and denote by

$$X_I^+ = \{\lambda \in X^*(\mathbf{T}) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in I\}$$

be the set of L_I-dominant weights. Every $\lambda \in X_I^+$ gives rise to a finite-dimensional algebraic L_I-representation

$$V_I(\lambda) = V_P(\lambda). \tag{2.4}$$

We consider $V_I(\lambda)$ as a P_I-module by letting act U_I trivially on it. The generalized (parabolic) Verma module attached to the weight λ is given by

$$M_I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} V_I(\lambda).$$

We have a surjective map

$$q_I : M(\lambda) \to M_I(\lambda),$$

where the kernel is given by the image of $\oplus_{\alpha \in I} M(s_\alpha \cdot \lambda) \to M(\lambda)$, cf. [Le, Prop. 2.1]. If $J \subset I$, then we obtain a transition map $q_{J,I} : M_J(\lambda) \to M_I(\lambda)$ such that $q_I = q_{J,I} \circ q_J$.

Let $W_I \subset W$ be the parabolic subgroup induced by $I \subset \Delta$. Consider the set ${}^IW = W_I \backslash W$ of left cosets which we identify with their representatives of shortest length in W. Let ${}^Iw$ be the element of maximal length in ${}^IW$. If λ is in X_+ and $w \in {}^IW$ then $w \cdot \lambda \in X_I^+$, cf. [Le, p. 502]. The I-parabolic BGG-resolution of $V(\lambda)$, $\lambda \in X_+$, is given by the exact sequence

$$0 \to M_I({}^Iw \cdot \lambda) \xrightarrow{d^I_{\ell({}^Iw)}} \bigoplus_{\substack{w \in {}^IW \\ \ell(w) = \ell({}^Iw)-1}} M_I(w \cdot \lambda) \xrightarrow{d^I_{\ell({}^Iw)-1}} \cdots$$

$$\cdots \xrightarrow{d^I_2} \bigoplus_{\substack{w \in {}^IW \\ \ell(w)=1}} M_I(w \cdot \lambda) \xrightarrow{d^I_1} M_I(\lambda) \to V(\lambda) \to 0.$$

We shall specify the differentials in this complex. For each $w \in W$, fix once and for all an embedding $i_w : M(w \cdot \lambda) \hookrightarrow M(\lambda)$. If $w' \geq w$ (for the Bruhat order $\leq$ on W), $i_{w'}$ factors through the image of i_w and we can define thus a unique map

$i_{w',w} \in \mathrm{Hom}_{\mathfrak{g}}(M(w' \cdot \lambda), M(w \cdot \lambda))$ such that $i_{w'} = i_w \circ i_{w',w}$. These maps satisfy the relations $i_{w'',w} = i_{w',w} \circ i_{w'',w'}$ for all $w'' \geq w' \geq w$. If $I \subset \Delta$ is a subset and $w' \geq w$ are elements of ${}^I W$, then the previous relations and the identification of $M_I(w \cdot \lambda)$ with the cokernel of $\oplus_{\alpha \in I} i_{s_\alpha w, w}$ show that $i_{w',w}$ factors through a map $M_I(w' \cdot \lambda) \to M_I(w \cdot \lambda)$ which we will denote by the same symbol $i_{w',w}$. Now recall that to each pair $(w', w) \in W^2$ such that $w' \geq w$ and $\ell(w') = \ell(w) + 1$, we can associate an element $e(w', w) \in \{\pm 1\}$, with $e(w_2, w_4)e(w_1, w_2) = -e(w_3, w_4)e(w_1, w_3)$ once we have $w_4 < w_2 < w_1$, $w_4 < w_3 < w_1$, $w_2 \neq w_3$ and $\ell(w_1) = \ell(w_4) + 2$ (cf. [BGG, Lemma 10.4]). Finally we define

$$d_k^I = \bigoplus_{\substack{w' \in {}^I W \\ \ell(w') = k}} \sum_{\substack{w \in {}^I W,\, w < w' \\ \ell(w') = \ell(w) + 1}} e(w', w) i_{w',w}.$$

The exactness of the above sequence is a consequence of [Le, Thm. 4.3].

By applying our exact functor $\mathcal{F}_P^G$ to this exact sequence, we get another exact sequence

$$0 \leftarrow \mathcal{F}_P^G(M_I({}^I w \cdot \lambda)) \leftarrow \bigoplus_{\substack{w \in {}^I W \\ \ell(w) = \ell({}^I w) - 1}} \mathcal{F}_P^G(M_I(w \cdot \lambda)) \leftarrow \dots$$
$$\dots \leftarrow \bigoplus_{\substack{w \in {}^I W \\ \ell(w) = 1}} \mathcal{F}_P^G(M_I(w \cdot \lambda)) \leftarrow \mathcal{F}_P^G(M_I(\lambda)) \leftarrow \mathcal{F}_P^G(V(\lambda)) \leftarrow 0$$

which coincides by the very definition of $\mathcal{F}_P^G$ and the PQ-formula with

$$(2.5) \quad 0 \leftarrow \mathrm{Ind}_P^G(V_I({}^I w \cdot \lambda)') \leftarrow \bigoplus_{\substack{w \in {}^I W \\ \ell(w) = \ell({}^I w) - 1}} \mathrm{Ind}_P^G(V_I(w \cdot \lambda)') \leftarrow \dots$$
$$\dots \leftarrow \bigoplus_{\substack{w \in {}^I W \\ \ell(w) = 1}} \mathrm{Ind}_P^G(V_I(w \cdot \lambda)') \leftarrow \mathrm{Ind}_P^G(V_I(\lambda)') \leftarrow V(\lambda)' \otimes_K i_P^G \leftarrow 0.$$

3. Isomorphism between simple objects

In this section we analyse when two simple modules of the shape $\mathcal{F}_{P_I}^G(M, v_{P_J}^{P_I})$, with P_I maximal for M and a subset $J \subset I$, are isomorphic. This will be used in the next section for determining the multiplicities of composition factors of the locally analytic Steinberg representation.

Let's begin by recalling some additional notation of [OS2]. Let $\mathbf{G}_0$ be a split reductive group model of $\mathbf{G}$ over $\mathcal{O}_L$. Let $\mathbf{T}_0 \subset \mathbf{B}_0 \subset \mathbf{G}_0$ be $\mathcal{O}_L$-models of $\mathbf{T}$ and $\mathbf{B}$. Fix a std psgp $\mathbf{P}_0$ and denote by $\mathbf{U}_{P,0}$ its unipotent radical. Let $\mathbf{U}^-_{P,0}$ be its opposite unipotent radical and denote by $\mathbf{L}_{P,0}$ the Levi factor containing $\mathbf{T}_0$. Let $G_0 = \mathbf{G}_0(\mathcal{O}_L) \subset G$, $P_0 = \mathbf{P}_0(\mathcal{O}_L) = G_0 \cap P \subset P$ etc. be the corresponding compact open subgroups consisting of O_L-valued points. We denote by $\mathcal{I} = p^{-1}(\mathbf{B}_0(k_L)) \subset G_0$ the standard Iwahori subgroup where $p: G_0 \to \mathbf{G}_0(k_L)$ is the reduction map.

Consider now for an open subgroup H of G_0 the distribution algebra $D(H) := D(H,K) = C^{an}_L(H,K)'$ which is defined by the dual of the locally convex K-vector space $C^{an}_L(H,K)$ of locally L-analytic functions [ST2]. It has the structure of a Fréchet-Stein algebra. More precisely, for each $\frac{1}{p} < r < 1$, there is a multiplicative norm q_r on the Fréchet algebra $D(H)$ such that if $D(H)_r$ denotes the Banach algebra given by the completion of $D(H)$ with respect to q_r, we have a topological isomorphism of algebras $D(H) \simeq \varprojlim_r D(H)_r$. For the precise definition of a Fréchet-Stein algebra we refer to loc.cit. We set $P_H = P_0 \cap H$ and let $U(\mathfrak{g}, P_H)$ be the subalgebra of $D(H)$ generated by $U(\mathfrak{g})$ and $D(P_H)$. Let $U(\mathfrak{g}, P_H)_r$ be the topological closure of $U(\mathfrak{g}, P_H)$ in $D(H)_r$.

The notion of a coadmissible module on a Fréchet-Stein algebra is defined in [ST2, §3]. If $\mathcal{M}$ is such a coadmissible $D(H)$-module, it comes along with a family $(q_{r,\mathcal{M}})_r$ of seminorms such that if $\mathcal{M}_r$ denotes the completion of $\mathcal{M}$ with respect to $q_{r,\mathcal{M}}$, we have $\mathcal{M} \simeq \varprojlim_r \mathcal{M}_r$ and each $\mathcal{M}_r$ is a finitely-generated $D(H)_r$-module.

Let M be a simple object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ As such an object it has naturally the structure of a $U(\mathfrak{g}, P_0)$-module. By [OS2, Section 4] there is a continuous $D(G_0)$-isomorphism

$$D(G_0) \otimes_{U(\mathfrak{g},P_0)} M \simeq \mathcal{F}^G_P(M)'.$$

Let $\mathcal{M} = D(H) \otimes_{U(\mathfrak{g},P_H)} M$. Then $\mathcal{M}$ is a coadmissible $D(H)$-module in the above sense, cf. [OS2, Prop. 4.4] and $\mathcal{M}_r$ is given by $\mathcal{M}_r = D(H)_r \otimes_{U(\mathfrak{g},P_H)} M$. We denote by $q_{r,M}$ the restriction of the seminorm $q_{r,\mathcal{M}}$ to M via the inclusion $M \hookrightarrow \mathcal{M}$ and by M_r the completion of M for $q_{r,M}$, which we may identify with $U(\mathfrak{g}, P_H)_r \otimes_{U(\mathfrak{g},P_H)} M$. Thus we obtain a $U(\mathfrak{g}, P_H)_r$-equivariant map $M_r \to \mathcal{M}_r$ giving rise to a $D(H)_r$-equivariant isomorphism

$$D(H)_r \otimes_{U(\mathfrak{g},P_H)_r} M_r \xrightarrow{\sim} \mathcal{M}_r.$$

Hence we get a topological isomorphism

$$\mathcal{M} \simeq \varprojlim_r D(H)_r \otimes_{U(\mathfrak{g},P_H)_r} M_r. \tag{3.1}$$

Let $\mathcal{D}$ be the category of topological $U(\mathfrak{t})$-modules M whose topology is metrizable, which are semi-simple with finite-dimensional eigenspaces and such that the topology can be defined by a family of norms $(q_r)_r$ such that

$$q_r(\sum_\mu x_\mu) = \sup_\mu q_r(x_\mu), \tag{3.2}$$

for $x_\mu \in M_\mu$ in a decomposition $M = \bigoplus_{\mu \in \mathrm{Hom}_L(\mathfrak{t},K)} M_\mu$. In this case, the completion M_r of M with respect to the norm q_r is given by

$$M_r = \{\sum_\mu x_\mu \mid q_r(x_\mu) \to 0 \text{ cofinite}\}.$$

A simple consequence of this description is the uniqueness of the expansion $\sum_\mu x_\mu$ and the fact that $(M_r)_\mu = M_\mu$ for all $\mu \in \mathrm{Hom}_L(\mathfrak{t}, K)$. In particular, the $U(\mathfrak{t})$-eigenspaces of M_r are all finite-dimensional.

Lemma 3.1. *If M is an object of $\mathcal{D}$, each $U(\mathfrak{t})$-submodule (resp. quotient) of M with the induced (resp. quotient) topology is an object of $\mathcal{D}$. Moreover, each $U(\mathfrak{t})$-module coming from $\mathcal{O}_{alg}$ is the image of an object of $\mathcal{D}$ under the functor forgetting the topology.*

Proof. A $U(\mathfrak{t})$-submodule S of an object M in $\mathcal{D}$ is clearly contained in $\mathcal{D}$. In particular, we have $S = \bigoplus_\mu S_\mu$ with $S_\mu = S \cap M_\mu$. Since the eigenspaces M_ν are finite-dimensional, each subspace S_μ is closed in M_μ. It follows that S is closed in M. As a consequence the quotient topology on M/S is metrizable. It is given by the induced family of quotient norms $(\overline{q}_r)_r$. We have to check that condition (3.2) is satisfied for each such norm $\overline{q}_r$. For every $\epsilon > 0$, there is an element $y = \sum_\mu y_\mu \in S$ such that

$$\begin{aligned}\overline{q}_r(\sum_\mu x_\mu + S) &\geq q_r(\sum_\mu x_\mu + y) - \epsilon \\ &= \sup_\mu q_r(x_\mu + y_\mu) - \epsilon \\ &\geq \sup_\mu \overline{q}_r(x_\mu + S) - \epsilon.\end{aligned}$$

Hence $\overline{q}_r(\sum_\mu x_\mu + S) \geq \sup_\mu \overline{q}_r(x_\mu + S)$. The other inequality is immediate since $\bar{q}_r$ is non-archimedean.

As explained above, if M is an object of $\mathcal{O}_{\mathrm{alg}}$, then it has the structure of a topological metrizable $U(\mathfrak{t})$-module. It is a consequence of [K, Theorem 1.4.2] that each module of the form $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} W \simeq U(\mathfrak{u}^-) \otimes_K W$, with W a finite-dimensional representation of B, is an object of $\mathcal{D}$. But each object of $\mathcal{O}_{\mathrm{alg}}$ is a quotient of some module of the shape $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} W$, so that we deduce the last assertion. □

The following statement generalizes [OS1, Prop. 3.4.8] in the split case and is again a consequence of [Fe, 1.3.12] and the finiteness of $U(\mathfrak{t})$-eigenspaces in M_r when M is an object of $\mathcal{D}$.

Proposition 3.2. *Let M be an object of $\mathcal{D}$. We have an inclusion preserving bijection*

$$\begin{array}{ccc} \Big\{\textit{closed } U(\mathfrak{t})\textit{-invariant subspaces of } M_r\Big\} & \stackrel{\sim}{\longrightarrow} & \Big\{U(\mathfrak{t})\textit{-invariant subspaces of } M\Big\}. \\ S & \longmapsto & S \cap M \end{array}$$

The inverse map is induced by taking the closure. In particular, any weight vector for the action of $\mathfrak{t}$ lies already in M.

Let $U = U_B$ be the unipotent radical of our fixed Borel subgroup $B \subset G$ and let $\mathfrak{u} = \operatorname{Lie} U$ be its Lie algebra. Recall that if N is a Lie algebra representation of $\mathfrak{g}$, then $H^0(\mathfrak{u}, N) = \{n \in N \mid \mathfrak{x} \cdot n = 0 \ \forall \ \mathfrak{x} \in \mathfrak{u}\}$ denotes the subspace of vectors killed by $\mathfrak{u}$. This is a $U(\mathfrak{t})$-module which is an object of $\mathcal{D}$ if $N \in \mathcal{O}_{\mathrm{alg}}$.

Corollary 3.3. *Let M be an object of $\mathcal{O}_{\mathrm{alg}}$. Then $H^0(\mathfrak{u}, M_r) = H^0(\mathfrak{u}, M)$. In particular, $H^0(\mathfrak{u}, M_r)$ is finite-dimensional.*

Proof. We clearly have $H^0(\mathfrak{u}, M_r) \cap M = H^0(\mathfrak{u}, M)$. As $H^0(\mathfrak{u}, M_r)$ is closed in M_r by the continuity of the action of $\mathfrak{g}$ and as $H^0(\mathfrak{u}, M)$ is finite-dimensional and therefore complete the statement follows by Proposition 3.2. □

Let V be additionally a smooth admissible representation of L_P. Below we compute the first U-homology resp. U-cohomology group of the various representations $\mathcal{F}^G_P(M, V)$. More precisely, we denote by $\overline{H}_0(U, \mathcal{F}^G_P(M, V))$ the quotient of $\mathcal{F}^G_P(M, V)$ by the topological closure of the K-subspace generated by the elements $ux - x$ for

$x \in \mathcal{F}^G_P(M,V)$ and $u \in U$. It is the largest Hausdorff quotient of $\mathcal{F}^G_P(M,V)$ on which U acts trivially. On the other hand, $H^0(\mathfrak{u}, \mathcal{F}^G_P(M,V)')$ is a Fréchet space since it is a closed subspace of $\mathcal{F}^G_P(M,V)'$.

Lemma 3.4. *As Fréchet spaces we have* $H^0(\mathfrak{u}, \mathcal{F}^G_P(M,V)') = H^0(\mathfrak{u}, \mathcal{F}^G_P(M)')\hat{\otimes}_K V'$. *(Note that the tensor product topology on the right hand side is unambiguous, since both factors are Fréchet spaces).*

Proof. Since the action of $\mathfrak{u}$ is trivial on V, there is an identification $\mathcal{F}^G_P(M,V) = \mathcal{F}^G_P(M) \otimes_K V$ as $\mathfrak{u}$-modules. Now we write $V = \varinjlim_n V_n$ as the union of finite-dimensional K-vector spaces. Then $\mathcal{F}^G_P(M,V) = \varinjlim_n \mathcal{F}^G_P(M) \otimes_K V_n$ as locally convex K-vector spaces. By passing to the dual we get

$$\begin{aligned} \mathcal{F}^G_P(M,V)' &= \varprojlim_n (\mathcal{F}^G_P(M) \otimes_K V_n)' \\ &= \varprojlim_n \mathcal{F}^G_P(M)' \otimes_K V_n' = \mathcal{F}^G_P(M)' \hat{\otimes}_K V'. \end{aligned}$$

For the first identity confer [Em, Prop. 1.1.22] resp. [S, Prop. 16.10], the second one follows as V_n is finite-dimensional, the third one is [Em, Prop. 1.1.29]. Now, the space $H^0(\mathfrak{u}, \mathcal{F}^G_P(M)')$ is the kernel of the map

$$d : \mathcal{F}^G_P(M)' \to (\mathcal{F}^G_P(M)')^{\dim \mathfrak{u}}$$

given by $v \mapsto (\mathfrak{x}_i v)$ where $(\mathfrak{x}_i)$ is a basis of $\mathfrak{u}$. By the exactness of the tensor product $- \otimes_K V_n'$ and the left exactness of the projective limit, the space $H^0(\mathfrak{u}, \mathcal{F}^G_P(M)')\hat{\otimes}_K V'$ is the kernel of the map $d\hat{\otimes} 1 : \mathcal{F}^G_P(M)'\hat{\otimes}_K V' \to (\mathcal{F}^G_P(M)'\hat{\otimes}_K V')^{\dim \mathfrak{u}}$. The result follows. □

Now we can prove the main result of this section.

Theorem 3.5. *Let* M *be a simple object of* $\mathcal{O}^{\mathfrak{p}}_{alg}$ *such that* P *is maximal for* M*, and let* V *be a smooth admissible representation of* L_P*. Let* $\lambda \in X^*(T)$ *be the highest weight of* M*, so that* $M \simeq L(\lambda)$*. Then there are* T*-equivariant isomorphisms*

$$H^0(U, \mathcal{F}^G_P(M,V)') = \lambda \otimes_K J_{U\cap L_P}(V)', \tag{3.3}$$

and

$$\overline{H}_0(U, \mathcal{F}^G_P(M,V)) = \lambda' \otimes_K J_{U\cap L_P}(V), \tag{3.4}$$

where $J_{U\cap L_P}$ *is the usual Jacquet functor for the unipotent subgroup* $U \cap L_P \subset L_P$.

Proof. The underlying topological space of $\mathcal{F}^G_P(M,V)$ is of compact type. As the category of locally convex vector spaces of compact type is stable by forming Hausdorff quotients, the underlying topological vector space of $\overline{H}_0(U,\mathcal{F}^G_P(M,V))$ is reflexive. As $H^0(U,\mathcal{F}^G_P(M,V)')$ is the topological dual of $\overline{H}_0(U,\mathcal{F}^G_P(M,V))$, it is sufficient to prove the first isomorphism (3.3).

Let's begin by determining $H^0(\mathfrak{u},\mathcal{F}^G_P(M,V)')$. The Iwahori decomposition $G_0 = \coprod_{w\in W^I}\mathcal{I}wP_0$ induces a decomposition

$$D(G_0)\otimes_{U(\mathfrak{g},P_0)} M \simeq \bigoplus_{w\in W^I} D(\mathcal{I})\otimes_{U(\mathfrak{g},\mathcal{I}\cap wP_0w^{-1})} M^w.$$

Here M^w denotes the module M with the twisted action given by conjugation with w. For each $w\in W^I$, we have

$$H^0(\mathfrak{u}, D(\mathcal{I})\otimes_{U(\mathfrak{g},I\cap wP_0w^{-1})} M^w) \simeq H^0(\mathrm{Ad}(w^{-1})\mathfrak{u}, D(w^{-1}\mathcal{I}w)\otimes_{U(\mathfrak{g},w^{-1}Iw\cap P_0)}\otimes M).$$

Now we apply the precedent discussion with $H = w^{-1}\mathcal{I}w$. Set $\mathfrak{n} = \mathrm{Ad}(w^{-1})\mathfrak{u}$.

First we consider the case $w\neq 1$. By [OS1, Lemma 3.3.2], there is an Iwahori decomposition $w^{-1}\mathcal{I}w = (U^-_{P,0}\cap w^{-1}\mathcal{I}w)(P_0\cap w^{-1}\mathcal{I}w)$, hence there is by [K, 1.4.2] a finite number of elements u in $U^-_{P,0}$ such that $D(H)_r = \bigoplus \delta_u\cdot U(\mathfrak{g},P_H)_r$, so that

$$\mathcal{M}_r \simeq \bigoplus_u \delta_u\otimes M_r$$

and the action of $\mathfrak{x}\in\mathfrak{n}$ is given by

$$\mathfrak{x}\cdot\sum\delta_u\otimes m_u = \sum\delta_u\otimes(\mathrm{Ad}(u^{-1})\mathfrak{x})m_u.$$

Now we can find a non-trivial element $\mathfrak{x}\in\mathfrak{u}_P^-\cap\mathrm{Ad}(w^{-1})\mathfrak{u} = \mathfrak{u}_P^-\cap\mathfrak{n}$. Indeed, the set $\Phi^-\cap w^{-1}(\Phi^+)$ contains an element of $\Phi^-\backslash\Phi_I^-$. For that, choose $\beta\in\Phi^+\backslash\Phi_I^+$ such that $w^{-1}\beta\in\Phi^-$. This is possible since $w\notin W_I$. Then we have $w^{-1}\beta\notin\Phi_I^-$ since W^I is exactly the set of w such that $w(\Phi_I^+)\subset\Phi^+$. Now $\mathrm{Ad}(u^{-1})\mathfrak{x}\in\mathfrak{u}_P^-$ since $u\in U_P^-$. By [OS2, Corollary 8.6], elements of $\mathfrak{u}_P^-$ act injectively on M, and arguing as in Step 1 of the proof of [OS2, Theorem 5.7], they act injectively on M_r, as well. We conclude that $H^0(\mathrm{Ad}(u^{-1})\mathfrak{n},M_r)=0$ and therefore $H^0(\mathfrak{n},\mathcal{M}_r)=0$. By identity (3.1), we get $H^0(\mathfrak{n},\mathcal{M})=0$.

Now consider the case $w=1$. Again we may write $D(\mathcal{I})_r = \bigoplus\delta_u U(\mathfrak{g},P_0)_r$ for a finite number of $u\in U^-_{P,0}$, so that $D(\mathcal{I})_r\otimes_{U(\mathfrak{g},P_0)_r} M_r = \bigoplus_u\delta_u\otimes M_r$. We will show that if $u \notin U^-_{P,0} \cap U(\mathfrak{g}, P_0)_r$, then $H^0(\mathrm{Ad}(u^{-1})\mathfrak{u}, M_r) = 0$. Here we will use Step 2 in the proof of [OS2, Theorem 5.7]. Let $\hat{M}$ be the formal completion of M, i.e. $\hat{M} = \prod_\mu M_\mu$ which is a $\mathfrak{g}$-module. The action of $\mathfrak{u}^-_P$ can be extended to an action of U^-_P as explained in loc.cit. If $\mathfrak{x} \in \mathfrak{g}$ and $u \in U^-_P$, the action of $\mathrm{ad}(u)\mathfrak{x}$ on M_r is the restriction of the composite $u \circ \mathfrak{x} \circ u^{-1}$ on $\hat{M}$. As a consequence, we get

$$H^0(\mathrm{ad}(u^{-1})\mathfrak{u}, M_r) = M_r \cap u^{-1} \cdot H^0(\mathfrak{u}, \hat{M}).$$

But

$$H^0(\mathfrak{u}, \hat{M}) = H^0(\mathfrak{u}, M) = Kv^+ = K_\lambda$$

where v^+ is a highest weight vector of M. So if $H^0(\mathrm{ad}(u^{-1})\mathfrak{u}, M_r) \neq 0$, we have $u^{-1}v^+ \in M_r$. By the proof of [OS2, Theorem 5.7], this gives a contradiction if $u \notin U^-_P \cap U_r(\mathfrak{g}, P_0)$. Hence by using identity (3.1), we obtain finally an isomorphism of Fréchet spaces

$$H^0(\mathfrak{u}, D(\mathcal{I}) \otimes_{U(\mathfrak{g},P_0)} M) \simeq H^0(\mathfrak{u}, M).$$

By combining the result above together with Lemma 3.4, we get thus an isomorphism

$$H^0(\mathfrak{u}, \mathcal{F}^G_P(M, V')) \simeq H^0(\mathfrak{u}, M) \otimes_K V' \simeq K_\lambda \otimes_K V'$$

Now $H^0(U, \mathcal{F}^G_P(M, V)')$ is a subspace of $H^0(\mathfrak{u}, \mathcal{F}^G_P(M, V)')$ and this latter one is stable by the action of U. Thus we deduce that

$$\begin{aligned} H^0(U, \mathcal{F}^G_P(M, V)') &= H^0(U, H^0(\mathfrak{u}, \mathcal{F}^G_P(M, V)')) \\ &= H^0(U, K_\lambda \otimes_K V') \\ &= K_\lambda \otimes_K J_{U \cap L_P}(V)'. \end{aligned}$$

□

Next we can formulate our result about intertwining between subquotients of the principal series. For this recall that by Lemma $X.4.6$ of [BoWa] the smooth induction i^G_B has Jordan-Hölder factors $v^G_{P_I}$ indexed by subsets $I \subset \Delta$ which appear all with multiplicity one. Moreover, by $X.3.2$, (1) to (5), of [BoWa], if Z is an irreducible subquotient of i^G_B, then there is a smooth character δ of T such that Z is the unique non-zero irreducible subrepresentation of $i^G_B(\delta)$ and δ contributes to $J_U(Z)$.

Corollary 3.6. *Let $L(\lambda_1)$ and $L(\lambda_2)$ be two simple objects in the category $\mathcal{O}_{alg}$. For $i = 1, 2$, let P_i be a std psgp maximal for $L(\lambda_i)$ and let V_i be a simple subquotient of the smooth parabolic induction $i_B^{P_i}$. Then the two irreducible representations $\mathcal{F}_{P_1}^G(L(\lambda_1), V_1)$ and $\mathcal{F}_{P_2}^G(L(\lambda_2), V_2)$ are isomorphic if and only if $P_1 = P_2$, $V_1 = V_2$ and $\lambda_1 = \lambda_2$.*

Proof. Suppose that there is a non-trivial isomorphism between the irreducible representations $\mathcal{F}_{P_1}^G(L(\lambda_1), V_1)$ and $\mathcal{F}_{P_2}^G(L(\lambda_2), V_2)$. Let δ_2 be a smooth character of T such that $V_2 \hookrightarrow i_B^{P_2}(\delta_2)$. By the sequence of embeddings

$$\mathcal{F}_{P_2}^G(L(\lambda_2), V_2) \subset \mathcal{F}_{P_2}^G(L(\lambda_2), i_B^{P_2}(\delta_2)) \simeq \mathcal{F}_B^G(L(\lambda_2), \delta_2) \subset \mathrm{Ind}_B^G(\lambda_2' \otimes_K \delta_2),$$

we obtain a non-trivial map $\mathcal{F}_{P_1}^G(L(\lambda_1), V_1) \to \mathrm{Ind}_B^G(\lambda_2' \otimes \delta_2)$ and by Frobenius reciprocity a non-trivial T-equivariant map

$$\overline{H}_0(U, \mathcal{F}_{P_1}^G(L(\lambda_1), V_1)) = \lambda_1' \otimes_K J_{U \cap L_{P_1}}(V_1) \to \lambda_2' \otimes \delta_2.$$

It follows that $\lambda_1 = \lambda_2$ and $P_1 = P_2$. By [BoWa, X.3.2.(1)], we know that $J_{U \cap L_{P_1}}(V_1)$ is a direct sum of smooth characters of T. By Frobenius reciprocity, these are exactly the smooth characters δ such that V_1 is an irreducible subobject of $i_B^{P_2}(\delta)$. As δ_2 is one of them, we can conclude by [BoWa, X.3.2.(4)] that $V_1 = V_2$. $\square$

4. The locally analytic Steinberg representation

The section deals with the proof of our main theorem. Here we start with the proof of the acyclicity of the evaluated locally analytic Tits complex.

As before let $P = P_I$ be a std psgp attached to a subset $I \subset \Delta$. Let

$$I_P^G = I_P^G(\mathbf{1}) = \mathrm{Ind}_P^G(\mathbf{1})$$

be the locally analytic G-representation induced from the trivial representations $\mathbf{1}$. If $Q \supset P$ is another parabolic subgroup, we get an injection $I_Q^G \hookrightarrow I_P^G$. We denote by

$$V_P^G = I_P^G / \sum_{Q \supsetneq P} I_Q^G$$

the generalized locally analytic Steinberg representation associated to P. For $P = G$, we have $V_G^G = I_G^G = \mathbf{1}$.

The next result is the locally analytic analogue of the classical situations, cf. [BoWa, Ch. X,4], [Leh], [DOR].

Theorem 4.1. *Let $I \subset \Delta$. Then the following complex is an acyclic resolution of $V^G_{P_I}$ by locally analytic G-representations,*

$$0 \to I^G_G \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |\Delta\setminus K|=1}} I^G_{P_K} \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |\Delta\setminus K|=2}} I^G_{P_K} \to \cdots \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |K\setminus I|=1}} I^G_{P_K} \to I^G_{P_I} \to V^G_{P_I} \to 0. \tag{4.1}$$

Here the differentials $d_{K',K} : I^G_{P_{K'}} \to I^G_{P_K}$ are defined as follows. We fix a total ordering on Δ. For two subsets $K \subset K'$ of Δ, we let

$$p_{K',K} : I^G_{P_{K'}} \longrightarrow I^G_{P_K}$$

be the natural homomorphism induced by the surjection $G/P_K \to G/P_{K'}$. For arbitrary subsets $K, K' \subset \Delta$, with $|K'| - |K| = 1$ and $K' = \{k_1 < \ldots < k_r\}$, we put

$$d_{K',K} = \begin{cases} (-1)^i p_{K',K} & K' = K \cup \{k_i\} \\ 0 & K \not\subset K' \end{cases}.$$

It is easy to check that $p_{K',K}$ is nothing else but the composite

$$I^G_{P_{K'}} = \mathcal{F}^G_{P_{K'}}(M_{K'}(1), 1) \xrightarrow{\mathcal{F}^G_{P_{K'}}(q_{K,K'},\mathrm{incl.})} \mathcal{F}^G_{P_{K'}}(M_K(1), i^{P_{K'}}_{P_K}) \simeq \mathcal{F}^G_{P_K}(M_K(1), 1) \simeq I^G_{P_K}.$$

More generally, we will prove a variant of the above theorem. For this, let $\lambda \in X_+$ be a dominant weight. For a std psgp $P = P_I \subset G$, we consider the finite-dimensional algebraic P-representation $V_I(\lambda) = V_P(\lambda)$ with highest weight λ. We set

$$I^G_P(\lambda) := \mathrm{Ind}^G_P(V_P(\lambda)').$$

In particular, we get $I^G_G(\lambda) = V(\lambda)'$. If $Q \subset G$ is another parabolic subgroup with $P \subset Q$, then there is a map

$$I^G_Q(\lambda) \to I^G_P(\lambda)$$

similarly as above for $V(\lambda) = \mathbf{1}$. More precisely, by the transitivity of parabolic induction we deduce that $I^G_P(\lambda) \simeq \mathrm{Ind}^G_Q(\mathrm{Ind}^Q_P(V_P(\lambda)'))$. As $V_Q(\lambda)'$ is the space of algebraic vectors in $\mathrm{Ind}^Q_P(V_P(\lambda)')$, we see that the above map is injective and has closed image.

We define analogously as above the twisted generalized locally analytic Steinberg representation by

$$V^G_P(\lambda) := I^G_P(\lambda)/\sum_{Q\supsetneq P} I^G_Q(\lambda).$$

Theorem 4.2. *Let $\lambda \in X_+$ and let $I \subset \Delta$. Then the following complex is an acyclic resolution of $V^G_{P_I}(\lambda)$ by locally analytic G-representations,*

$$0 \to I^G_G(\lambda) \to \bigoplus_{\substack{I\subset K\subset \Delta\\ |\Delta\setminus K|=1}} I^G_{P_K}(\lambda) \to \bigoplus_{\substack{I\subset K\subset \Delta\\ |\Delta\setminus K|=2}} I^G_{P_K}(\lambda) \to \dots$$
$$\dots \to \bigoplus_{\substack{I\subset K\subset \Delta\\ |K\setminus I|=1}} I^G_{P_K}(\lambda) \to I^G_{P_I}(\lambda) \to V^G_{P_I}(\lambda) \to 0. \tag{4.2}$$

Proof. The proof is by induction on the semi-simple rank $\mathrm{rk}_{\mathrm{ss}}(G) = |\Delta|$ of G. We start with the case $|\Delta| = 1$. Then the complex (4.2) coincides with

$$0 \to I^G_G(\lambda) \to I^G_B(\lambda) \to V^G_B(\lambda) \to 0$$

and the claim is trivial.

Now, let $|\Delta| > 1$. We consider for any subset $K \subset \Delta$, the resolution (2.5) :

$$0 \leftarrow \mathrm{Ind}^G_{P_K}(V_K(w\cdot\lambda)') \leftarrow \bigoplus_{\substack{w\in {}^KW\\ \ell(w)=\ell(w^K)-1}} \mathrm{Ind}^G_{P_K}(V_K(w\cdot\lambda)') \leftarrow \dots$$
$$\dots \leftarrow \bigoplus_{\substack{w\in {}^KW\\ \ell(w)=1}} \mathrm{Ind}^G_{P_K}(V_K(w\cdot\lambda)') \leftarrow I^G_{P_K}(\lambda) \leftarrow i^G_{P_K}(\lambda) \leftarrow 0.$$

Here we set $i^G_{P_K}(\lambda) := i^G_{P_K} \otimes_K V(\lambda)'$. We abbreviate for any $w \in {}^KW$ and any integer $i \geq 0$,

$$I^G_{P_K}(w) := \mathrm{Ind}^G_{P_K}(V_K(w\cdot\lambda)'),$$

$$I^G_{P_K}[i] := \bigoplus_{\substack{w\in {}^KW\\ \ell(w)=i}} I^G_{P_K}(w).$$

and

$$V^G_{P_K}(w) = I^G_{P_K}(w)/\sum_{\substack{K'\supsetneq K\\ w\in {}^{K'}W}} I^G_{P_{K'}}(w).$$

The complexes above induce hence a double complex

$$\begin{array}{ccccccccc}
0 \to & i^G_G(\lambda) & \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |\Delta\setminus K|=1}} i^G_{P_K}(\lambda) & \to \cdots \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} i^G_{P_K}(\lambda) & \to & i^G_{P_I}(\lambda) \\
& \| & & \downarrow & & \downarrow & & \downarrow \\
0 \to & I^G_G(\lambda) & \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |\Delta\setminus K|=1}} I^G_{P_K}(\lambda) & \to \cdots \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} I^G_{P_K}(\lambda) & \to & I^G_{P_I}(\lambda) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 0 & \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |\Delta\setminus K|=1}} I^G_{P_K}[1] & \to \cdots \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} I^G_{P_K}[1] & \to & I^G_{P_I}[1] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 0 & \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |\Delta\setminus K|=1}} I^G_{P_K}[2] & \to \cdots \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} I^G_{P_K}[2] & \to & I^G_{P_I}[2] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \vdots & & \vdots & & \vdots & & \vdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 0 & \to & 0 & \to \cdots \to & \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} I^G_{P_K}[\ell({}^I w)-1] & \to & I^G_{P_I}[\ell({}^I w)-1] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 0 & \to & 0 & \to \cdots \to & 0 & \to & I^G_{P_I}({}^I w).
\end{array}$$

To prove the commutativity of this diagram, it suffices to prove the following fact. Let $K' \subset K \subset \Delta$ with $|K'| = |K| - 1$. If $\mathcal{C}(K)$ and $\mathcal{C}(K')$ are the BGG resolutions (2.5) obtained with $P = P_K$ and $P = P_{K'}$, then the maps $p_{K,K'}$ induce a morphism of complexes $\mathcal{C}(K) \to \mathcal{C}(K')$.

Let $w \in {}^K W$. Choose $w' \geq w$ with $\ell(w') = \ell(w) + 1$ such that $w' \in {}^{K'}W$. Clearly, we have $w \in {}^{K'}W$, so that we have to consider two cases, depending on whether $w' \in {}^K W$ or not. If $w' \in {}^K W$, it is tautological from the definitions of $i_{w',w}$ and $q_{K,K'}$ that the following diagram is commutative

$$\begin{array}{ccc}
M_{K'}(w' \cdot \lambda) & \xrightarrow{q_{K',K}} & M_K(w' \cdot \lambda) \\
\downarrow i_{w',w} & & \downarrow i_{w',w} \\
M_{K'}(w \cdot \lambda) & \xrightarrow{q_{K',K}} & M_K(w \cdot \lambda).
\end{array}$$

If $w' \notin {}^K W$, we claim that $w' = s_\alpha w$ with $K = K' \cup \{\alpha\}$. Namely, we must have $\ell(s_\alpha w') < \ell(w')$ and by the exchange condition a reduced expression of w' is of the form $w' = s_\alpha s_2 \cdots s_r$. By [Hum3, Thm. 5.10], w is obtained as a a subexpression of this reduced expression. We have $w \in {}^K W$ so that a reduced expression of w cannot begin by s_α, so that $w' = s_\alpha w$. We conclude from this fact that $M(w' \cdot \lambda)$ is included in the kernel of $M(w \cdot \lambda) \to M_K(w \cdot \lambda)$ and finally that the composite

$$M_{K'}(w' \cdot \lambda) \xrightarrow{i_{w',w}} M_{K'}(w \cdot \lambda) \xrightarrow{q_{w',w}} M_K(w \cdot \lambda)$$

is zero.

Applying the functor $\mathcal{F}^G_{P_{K'}}$, we obtain the commutativity of the following square

$$\begin{array}{ccc} I^G_{P_K}(w \cdot \lambda) & \hookrightarrow & I^G_{P_{K'}}(w \cdot \lambda) \\ \downarrow \mathcal{F}(i_{w',w}) & & \downarrow \mathcal{F}(i_{w',w}) \\ I^G_{P_K}(w' \cdot \lambda) & \hookrightarrow & I^G_{P_{K'}}(w' \cdot \lambda) \end{array}$$

where we have replaced $I^G_{P_K}(w' \cdot \lambda)$ by 0 if $w' \notin {}^K W$. This is enough to prove our assertion on BGG resolutions.

For proving our theorem, it suffices now to show that each row of the big double complex, except from the second one, is exact apart from the very right hand side. Indeed, let

$$C^{p,q} = \bigoplus_{\substack{I \subset K \subset \Delta \\ |K \setminus I| = -p}} I^G_{P_K}[q-1]$$

(with convention that $I^G_{P_K}[-1] = i^G_{P_K}(\lambda)$). The double complex above is a second quadrant double complex. There are two spectral sequences converging towards the cohomology of the total complex associated to $C^{\cdot,\cdot}$. The first is $E^{p,q}_{2,I} = H^p_h H^q_v(C^{\cdot,\cdot}) = 0$. The second is $E^{p,q}_{2,II} = H^p_v H^q_h(C^{\cdot,\cdot})$. By our hypothesis, we have $E^{p,q}_{2,II} = 0$ except in the cases $p = 0$ or $q = 1$. Moreover, the term $E^{0,0}_{2,II}$ is the kernel of the natural map $v^G_{P_I}(\lambda) \to V^G_{P_I}(\lambda)$ which is 0, this last fact can be shown using description of Jordan-Hölder factors of the $I^G_{P_K}(\lambda)$ together with Corollary 3.6. Finally this second spectral sequence degenerates in E_2 and we deduce that $E^{p,1}_{1,II} = 0$ for $p \leq -1$, this shows that all the lines are exact apart form the very right hand side.

Now let's prove that each row, except from the second one, is exact apart from the very right hand side. The upper line satisfies this property, since it is tensor product of the (generalized) smooth Tits complex by the algebraic representation $V(\lambda)'$. For

$w \in {}^I W$, let $I(w) = \{\alpha \in \Delta \mid \ell(s_\alpha w) > \ell(w)\} \subset \Delta$ be the unique maximal subset such that $w \in {}^{I(w)}W$ and set $P_w = P_{I(w)}$. Hence it suffices to show the exactness of the evaluated sequence

$$0 \to I^G_{P_w}(w) \to \cdots \to \bigoplus_{\substack{I\subset K \subset I(w)\\ |K\setminus I|=1}} I^G_{P_K}(w) \to I^G_{P_I}(w) \to V^G_{P_I}(w) \to 0$$

for each $w \in {}^I W$. This complex can be rewritten as follows. We have $I^G_{P_K}(w) = \mathrm{Ind}^G_{P_w}(I^{P_w}_{P_K}(w))$. Since the induction functor $\mathrm{Ind}^G_{P_w}$ is exact, it suffices to show the exactness of the sequence

$$0 \to I^{P_w}_{P_w}(w) \to \cdots \to \bigoplus_{\substack{I\subset K \subset I(w)\\ |K\setminus I|=1}} I^{P_w}_{P_K}(w) \to I^{P_w}_{P_I}(w) \to V^{P_w}_{P_I}(w) \to 0$$

where $V^{P_w}_{P_I}(w) = I^{P_w}_{P_I}(w)/\sum_{K\supsetneq I, w\in {}^K W} I^{P_w}_{P_K}(w)$. But we have $I^{P_w}_{P_K}(w) = I^{L_w}_{L_w\cap P_K}(w)$ and $\mathrm{rk_{ss}}(L_w) < \mathrm{rk_{ss}}(G)$. Thus the complex above is by induction hypothesis exact. Hence the claim of our theorem follows. □

As a byproduct of the proof we get the following result (Note that $V^G_{P_I}({}^I w) = I^G_{P_I}({}^I w)$.):

Corollary 4.3. *For any dominant weight $\lambda \in \Lambda_+$ and any subset $I \subset \Delta$, there is an acyclic complex*

$$0 \to v^G_{P_I}(\lambda) \to V^G_{P_I}(\lambda) \to \bigoplus_{\substack{w\in {}^I W\\ \ell(w)=1}} V^G_{P_I}(w) \to \cdots \to \bigoplus_{\substack{w\in {}^I W\\ \ell(w)=\ell({}^I w)-1}} V^G_{P_I}(w) \to V^G_{P_I}({}^I w) \to 0.$$

□

Example 4.4. Let $G = \mathrm{GL}_2$ and $\lambda = 0$. Then $W = \{1, s\}$ and the above complex is given by

$$0 \to v^G_B \to V^G_B \to I^G_B(s) \to 0.$$

This complex was already considered by Morita [Mo] resp. Schneider and Teitelbaum [ST5] and coincides with the dual of the K-valued "de Rham" complex

$$0 \to \mathcal{O}(\mathcal{X})/K \to \Omega^1(\mathcal{X}) \to H^1_{\mathrm{dR}}(\mathcal{X}) \to 0$$

of the Drinfeld half space $\mathcal{X} = \mathbb{P}^1 \setminus \mathbb{P}^1(L)$.

Lemma 4.5. *The representation $V_P^G(\lambda)$ has a Jordan-Hölder series of finite length.*

Proof. The representation $V_P^G(\lambda)$ is a quotient of $I_P^G(\lambda)$ which coincides by definition of the functor $\mathcal{F}_P^G$ with $\mathcal{F}_P^G(M_P(\lambda))$. As explained in Section 1, the latter object has a finite Jordan-Hölder series, hence the same holds true for $V_P^G(\lambda)$. □

From the previous lemma we see that a Jordan-Hölder series of $V_B^G(\lambda)$ is induced by such a series of the induced representation $I_P^G(\lambda)$. Now we are able to give a recipe for the determination of the composition factors with respect to $V_B^G(\lambda)$. For two reflections w, w' in W, let $m(w', w) \in \mathbb{Z}_{\geq 0}$ be the multiplicity of the simple $U(\mathfrak{g})$-module $L(w \cdot 0)$ in the Verma module $M(w' \cdot 0)$. It is known that $m(w', w) \geq 1$ if and only if $w' \leq w$ for the Bruhat order $\leq$ on W. These multiplicities can be computed using Kazhdan-Lusztig polynomials, as it was conjectured by Kazhdan and Lusztig in [KaLu, Conj. 1.5] and proved independently by Beilinson-Bernstein [BeBe] and Brylinski-Kashiwara [BrKa]. Let's recall that if $w \in W$ is an element of the Weyl group, the support $\mathrm{supp}(w)$ of w is the set of simple reflections appearing in one (and so in any) reduced expression of w. In the following we identify the set of simple reflections with Δ. Then $w \in W_I$ if and only if $\mathrm{supp}(w) \subset I$.

Recall from the proof of Theorem 4.2, that we defined for every $w \in W$ a subset $I(w) \subset \Delta$ with $w \in {}^{I(w)}W$ and that the subset $I(w)$ is maximal with this property. On the other hand, by [Hum2, p. 186 − 187], the simple module $L(w \cdot \lambda)$ lies in $\mathcal{O}^{\mathfrak{p}_I}$ if and only if $w \in {}^I W$. It follows that for $w \in W$, the parabolic Lie Algebra $\mathfrak{p}_{I(w)}$ is maximal for $L(w \cdot \lambda)$.

Theorem 4.6. *Fix $w \in W$ and let $I = I(w) \subset \Delta$ be as above. For a subset $J \subset \Delta$ with $J \subset I$, let $v_{P_J}^{P_I}$ be the generalized smooth Steinberg representation of L_{P_I}. Then the multiplicity of the irreducible G-representation $\mathcal{F}_{P_I}^G(L(w \cdot \lambda), v_{P_J}^{P_I})$ in $V_B^G(\lambda)$ is*

$$\sum_{\substack{w' \in W \\ \mathrm{supp}(w') = J}} (-1)^{\ell(w') + |J|} m(w', w), \tag{4.3}$$

and we obtain in this way all the Jordan-Hölder factors of $V_B^G(\lambda)$. Moreover, this multiplicity is non-zero if and only if $J \subset \mathrm{supp}(w)$. In particular, the locally algebraic representation $v_B^G(\lambda)$ is the only locally algebraic subquotient of $V_B^G(\lambda)$.

Proof. By Theorem 4.2, we get the following formula for the multiplicity of the simple object $\mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})$ in $V^G_B(\lambda)$:

$$[V^G_B(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})] = \sum_{K\subset\Delta}(-1)^{|K|}[I^G_{P_K}(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})].$$

By definition we have $I^G_{P_K}(\lambda) = \mathcal{F}^G_{P_K}(M_K(\lambda))$. Since the Jordan-Hölder series of $M_K(\lambda)$ lies in $\mathcal{O}^{\mathfrak{p}_K}_{\mathrm{alg}}$, as well, we deduce by by Corollary 3.6 and the discussion in Section 2 that $K\subset I$ is a necessary condition for the non-vanishing of the expression $[I^G_{P_K}(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})]$. On the other hand, $v^{P_I}_{P_J}$ is a subquotient of $i^{P_I}_{P_K}$ if and only if $K\subset J$. Again by Corollary 3.6 and Section 2, the term $\mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})$ appears in $I^G_{P_K}(\lambda)$ only if $K\subset J$.

So, let $K\subset J$ and consider the JH-component $Q = L(w\cdot\lambda)$ of $M_K(\lambda)$ in $\mathcal{O}_{\mathrm{alg}}$. Then $\mathcal{F}^G_{P_K}(Q,1) = \mathcal{F}^G_{P_I}(Q, i^{P_I}_{P_K})$ by the PQ-formula. In a Jordan-Hölder series of $\mathcal{F}^G_{P_I}(Q, i^{P_I}_{P_K})$ the term $\mathcal{F}^G_{P_I}(Q, v^{P_I}_{P_J})$ appears with multiplicity one by Corollary 3.6, so that

$$[I^G_{P_K}(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})] = [M_K(\lambda) : L(w\cdot\lambda)].$$

As a consequence, we have

$$[V^G_B(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})] = \sum_{K\subset J}(-1)^{|K|}[M_K(\lambda) : L(w\cdot\lambda)].$$

Now we make use of the character formula

$$\mathrm{ch}M_K(\lambda) = \sum_{w'\in W_K}(-1)^{\ell(w')}\mathrm{ch}M(w'\cdot\lambda),$$

cf. [Hum2, Proposition 9.6]. We obtain

$$\begin{aligned}[V^G_B(\lambda) : \mathcal{F}^G_{P_I}(L(w\cdot\lambda), v^{P_I}_{P_J})] &= \sum_{K\subset J}(-1)^{|K|}\sum_{w'\in W_K}(-1)^{\ell(w')}[M(w'\cdot\lambda) : L(w\cdot\lambda)] \\ &= \sum_{w'\in W}(-1)^{\ell(w')}[M(w'\cdot\lambda) : L(w\cdot\lambda)]\sum_{\mathrm{supp}(w')\subset K\subset J}(-1)^{|K|}.\end{aligned}$$

But the sum

$$\sum_{\mathrm{supp}(w')\subset K\subset J}(-1)^{|K|} = (1-1)^{|J\setminus\mathrm{supp}(w')|}$$

is non-zero if and only if $\mathrm{supp}(w') = J$, so we deduce the formula. The non-vanishing criterion follows from Corollary 5.3 in the appendix. □

Example 4.7. a) Let $G = \mathrm{GL}_3$, $\Delta = \{\alpha_1, \alpha_2\}$. Let s_i be the element of W corresponding to α_i and abbreviate $P_i = P_{\{\alpha_i\}}$. In this case, $m(w', w) = 1$ if and only if $w' \leq w$ for the Bruhat order (cf. [Hum2, 8.3.(c)]). As a consequence, we obtain the following Jordan-Hölder factors all with multiplicity one :

$$v_B^G$$
$$\mathcal{F}_{P_1}^G(L(s_2 \cdot 0), v_B^{P_1}),\ \mathcal{F}_{P_2}^G(L(s_1 \cdot 0), v_B^{P_2})$$
$$\mathcal{F}_{P_1}^G(L(s_2 s_1 \cdot 0), 1),\ \mathcal{F}_{P_1}^G(L(s_2 s_1 \cdot 0), v_B^{P_1})$$
$$\mathcal{F}_{P_2}^G(L(s_1 s_2 \cdot 0), 1),\ \mathcal{F}_{P_2}^G(L(s_1 s_2 \cdot 0), v_B^{P_2})$$
$$\mathcal{F}_B^G(L(s_1 s_2 s_1 \cdot 0), 1).$$

This particular result was already shown in [Schr].

b) Let $G = \mathrm{GL}_4$. Here there are exactly two subquotients of V_B^G with multiplicity greater than one. We fix notation as follows. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ such that s_1 and s_3 commute. We use the more compact notation $P_i = P_{\{\alpha_i\}}$, $P_{i,j} = P_{\{\alpha_i, \alpha_j\}}$, $W^{i,j} = W^{\{\alpha_i, \alpha_j\}}$ etc. The two subquotients with multiplicity greater than one are then $\mathcal{F}_{P_{1,3}}^G(L(s_2 s_3 s_1 s_2 \cdot 0), v_B^{P_{1,3}})$ and $\mathcal{F}_{P_2}^G(L(s_1 s_3 s_2 s_3 s_1 \cdot 0), v_B^{P_2})$ and they have both multiplicity 2. The length of V_B^G is 50 and it contains 48 non-isomorphic Jordan-Hölder components. Let us remark that this example also shows that Kazhdan-Lusztig multiplicities are the reason for unexpected components in V_B^G. For instance, the component $\mathcal{F}_{P_2}^G(L(s_1 s_3 s_2 s_1 s_3 \cdot 0, 1))$ appears in $I_{P_2}^G$, but passing to V_B^G kills only one of its two occurrences in I_B^G.

Here is the idea to carry out this computation. The number of components of the form $\mathcal{F}_{P_I}^G(L(w \cdot \lambda, V)$ for a fixed $w \in W^I$ with $I = I(w)$ maximal and V a smooth irreducible subquotient of $i_{L_I \cap B}^{L_I}$, is exactly the number of subsets of $I \cap \mathrm{supp}(w)$, that is $2^{|I \cap \mathrm{supp}(w)|}$. Let W_p^I be the subset of W^I consisting of elements for which I is maximal, an easy computation shows that

$$|W_p^I| = \sum_{J \subset \Delta \setminus I} (-1)^{|J|} |W^{I \cup J}| = \sum_{J \subset \Delta \setminus I} (-1)^{|J|} |W| / |W_{I \cup J}|. \tag{4.4}$$

If $I = \varnothing$ or $I = \Delta$, then $|W_p^I| = 1$ and $|I \cap \mathrm{supp}(w)| = 0$. Furthermore, we compute easily that

$$|W_p^{1,2}| = |W_p^{2,3}| = |W_p^1| = |W_p^3| = 3$$
$$|W_p^{1,3}| = |W_p^2| = 5.$$

We see for example that $W_p^{1,2} = \{s_3, s_3s_2, s_3s_2s_1\}$ so that $|\{s_1, s_2\} \cap \mathrm{supp}(w)|$ takes the values 0, 1 and 2 when w is varying in $W_p^{1,2}$. This procedure gives us the cardinality of the composition factors appearing in V_B^G. As for the multiplicities, we can use a software package like CHEVIE ([GHLMP]) to see that the only simple modules $L(w \cdot 0)$ having multiplicities greater than one in $M(0)$ are given by $w = s_2s_1s_3s_2$ and $w = s_1s_3s_2s_1s_3$.

5. Appendix

In this appendix, we prove the last assertion of Theorem 4.6 by interpreting the alternate sum (4.3) as the multiplicity of $L(w \cdot \lambda)$ in the H_0 of a subcomplex of the BGG complex. Then we show that the submodule of $M(\lambda)$ generated by the $M(w \cdot \lambda)$ with w a Coxeter element in W_J is a quotient of this H_0. We fix the same notation as before.

Let $\lambda \in X_+$ and recall from Section 1 that the BGG resolution of $L(\lambda) = V(\lambda)$ takes the form

$$\cdots \to C_i(\mathfrak{g}) \xrightarrow{\epsilon_i} C_{i-1}(\mathfrak{g}) \xrightarrow{\epsilon_{i-1}} \cdots \xrightarrow{\epsilon_1} C_0(\mathfrak{g}) \to L(\lambda) \to 0$$

with

$$C_i(\mathfrak{g}) = \bigoplus_{\ell(w)=i} M(w \cdot \lambda).$$

Let I be a subset of Δ. For $i \geq 0$, let $C_i(\mathfrak{g}, I) \subset C_i(\mathfrak{g})$ be defined by

$$C_i(\mathfrak{g}, I) := \bigoplus_{w \in W_I, \ell(w)=i} M(w \cdot \lambda).$$

As $\epsilon_i(C_i(\mathfrak{g}, I)) \subset C_{i-1}(\mathfrak{g}, I)$, we obtain a subcomplex $C_\bullet(\mathfrak{g}, I)$ of $C_\bullet(\mathfrak{g})$. Now we abbreviate $W_I^{(i)}$ for the subset of W_I consisting of elements of length i.

Lemma 5.1. *The complex $C_\bullet(\mathfrak{g}, I)$ is isomorphic to the complex $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} C_\bullet(\mathfrak{l}_I)$, hence it is exact in positive degrees.*

Proof. We can write $\epsilon_i|_{C_i(\mathfrak{g},I)} = \sum_{w_1,w_2} a^i_{w_1,w_2} i_{w_1,w_2}$ where the sum is over all $w_1 \in W_I^{(i)}$ and $w_2 \in W_I^{(i-1)}$ such that $w_2 \leq w_1$. By [Ro, Lemma 10.2] we have $a^i_{w_1,w_2} \neq 0$ for each such pair (w_1, w_2). We fix injections $U(\mathfrak{l}_I) \otimes_{U(\mathfrak{b}\cap\mathfrak{l}_I)} K_{w\cdot\lambda} \hookrightarrow U(\mathfrak{l}_I) \otimes_{U(\mathfrak{b}\cap\mathfrak{l}_I)} K_\lambda$, $w \in W_I$, giving rise to well defined maps

$$i^I_{w,w'} : U(\mathfrak{l}_I) \otimes_{U(\mathfrak{b}\cap\mathfrak{l}_I)} K_{w\cdot\lambda} \hookrightarrow U(\mathfrak{l}_I) \otimes_{U(\mathfrak{b}\cap\mathfrak{l}_I)} K_{w'\cdot\lambda}$$

such that after applying $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} -$, we have the equality $\mathrm{id} \otimes i^I_{w,w'} = i_{w,w'}$. Now consider the complex $\widetilde{C}_\bullet(\mathfrak{l}_I)$ where $\widetilde{C}_\bullet(\mathfrak{l}_I) = C_\bullet(\mathfrak{l}_I)$ and where the differential maps are given by $\tilde{\epsilon}_i = \sum_{w_1 \le w_2 \in W_I} a^i_{w_1,w_2} i^I_{w_1,w_2}$, so that $C_\bullet(\mathfrak{g}, I) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} \widetilde{C}_\bullet(\mathfrak{l}_I)$. As the Bruhat order on W_I is induced by the Bruhat order of W, the squares (see [Ro, Definition 10.3]) of W with elements in W_I are exactly the squares of W_I, so that $\widetilde{C}_\bullet(\mathfrak{l}_I)$ is a complex. By Corollary 10.7 or Lemma 10.5 of [Ro] the complex $\widetilde{C}_\bullet(\mathfrak{l}_I)$ is exact in positive degrees and even isomorphic to the BGG complex $C_\bullet(\mathfrak{l}_I)$. □

For every integer $i \ge 0$, let $C^{\le i}_\bullet(\mathfrak{g}, I)$ be the subcomplex of $C_\bullet(\mathfrak{g}, I)$ defined by

$$C^{\le i}_j(\mathfrak{g}, I) = \bigoplus_{\substack{\exists J \subset I, |J| \le i \\ w \in W^{(j)}_J}} M(w \cdot \lambda)$$

and set $\overline{C}_\bullet(\mathfrak{g}, I) = C_\bullet(\mathfrak{g}, I)/C^{\le |I|-1}_\bullet(\mathfrak{g}, I)$.

Proposition 5.2. *The complex $C^{\le i}_\bullet(\mathfrak{g}, I)$ is exact in degrees $\ge i+1$ and the complex $\overline{C}_\bullet(\mathfrak{g}, I)$ in degree $\ge |I|+1$.*

Proof. We will prove this proposition by induction on i. For $i = 0$, this is clear. Now we can apply induction hypothesis to the long exact sequence coming from the exact sequence of complexes

$$0 \to C^{\le i-1}_\bullet(\mathfrak{g}, I) \to C^{\le i}_\bullet(\mathfrak{g}, I) \to \bigoplus_{J \subset I, |J| = i} \overline{C}_\bullet(\mathfrak{g}, J) \to 0 \tag{5.1}$$

(Here we use the identity $W_{J_1} \cap W_{J_2} = W_{J_1 \cap J_2}$ for the appearance of the direct sum.). In the special case, where $|I| = i$, we can deduce from Lemma 5.1 that $C_\bullet(\mathfrak{g}, I)$ is acyclic in strictly positive degrees, hence by induction on i that $H_j(\overline{C}_\bullet(\mathfrak{g}, I)) = 0$ for $j > i$. In general, when $|I| \ne i$, either $|I| < i$ and there is nothing to prove, or $|I| > i$. But for $J \subset I$ with $|J| = i$, we already know that $H_j(\overline{C}_\bullet(\mathfrak{g}, J)) = 0$ for $j > i$ by what precedes. Hence, using (5.1) and our induction hypothesis on i, we can conclude that $C^{\le i}_\bullet(\mathfrak{g}, I)$ is acyclic in degree $> i$. The claim follows. □

Corollary 5.3. *For $I \subset \Delta$, let M_I be the submodule of $M(\lambda)$ generated by the Verma modules $M(w \cdot \lambda)$ where w is a Coxeter element in W_I. For $w \in W$, we have*

$$[M_I : L(w \cdot \lambda)] \leq \sum_{\mathrm{supp}(w')=I} (-1)^{|I|+\ell(w')}[M(w' \cdot \lambda) : L(w \cdot \lambda)]. \tag{5.2}$$

Hence this sum is non-zero if and only if $I \subset \mathrm{supp}(w)$.

Proof. As the complex $\overline{C}_\bullet(\mathfrak{g}, I)$ is exact in degree $> |I|$ and $\overline{C}_j(\mathfrak{g}, I) = 0$ for $j < |I|$, it is sufficient to prove that M_I is a quotient of $H_{|I|}(\overline{C}_\bullet(\mathfrak{g}, I))$. Remark that $M' = \epsilon_{|I|}(\bigoplus_{\mathrm{supp}(w)=I,\ell(w)=|I|} M(w))$ is a quotient of $H_{|I|}(\overline{C}_\bullet(\mathfrak{g}, I))$. Then the image of M' by $\sum_{\mathrm{supp}(w)=I,\ell(w)=|I|-1} i_{w,1}$ is exactly M_I, and we deduce the result. It is not difficult to see that there exists a Coxeter element w' of W_I such that $w' \leq w$ if and only if $I \subset \mathrm{supp}(w)$. □

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