EQUIVARIANT VECTOR BUNDLES ON DRINFELD'S HALFSPACE OVER A FINITE FIELD

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Abstract. Let $\mathcal{X} \subset \mathbb{P}^d_k$ be Drinfeld's halfspace over a finite field k and let $\mathcal{E}$ be a homogeneous vector bundle on $\mathbb{P}^d_k$. The paper deals with two different descriptions of the space of global sections $H^0(\mathcal{X}, \mathcal{E})$ as $\mathrm{GL}_{d+1}(k)$-representation. This is an infinite dimensional modular G-representation. Here we follow the ideas of [O2, OS] treating the p-adic case. As a replacement for the universal enveloping algebra we consider both the crystalline universal enveloping algebra and the ring of differential operators on the flag variety with respect to $\mathcal{E}$.

Introduction

Let k be a finite field and denote by $\mathcal{X}$ Drinfeld's halfspace of dimension $d \geq 1$ over k. This is the complement of all k-rational hyperplanes in projective space $\mathbb{P}^d_k$, i.e.,

$$\mathcal{X} = \mathbb{P}^d_k \setminus \bigcup\nolimits_{H \subsetneq k^{d+1}} \mathbb{P}(H).$$

This object is equipped with an action of $G = \mathrm{GL}_{d+1}(k)$ and can be viewed as a Deligne-Lusztig variety, as well as a period domain over a finite field [OR]. In particular we get for every homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}^d_k$ an induced action of G on the space of global sections $H^0(\mathcal{X}, \mathcal{E})$ which is an infinite-dimensional modular G-representation.

In [O2] we considered the same problem for the Drinfeld halfspace over a p-adic field K. We constructed for every homogeneous vector bundle $\mathcal{E}$ a filtration by closed $\mathrm{GL}_{d+1}(K)$-subspaces and determined the graded pieces in terms of locally analytic G-representations in the sense of Schneider and Teitelbaum [ST1]. The definition of the filtration above involves the geometry of $\mathcal{X}$ being the complement of an hyperplane arrangement. In the p-adic case $H^0(\mathcal{X}, \mathcal{E})$ is a "bigger" object, it is a reflexive K-Fréchet space with a continuous G-action. Its strong dual is a locally analytic G-representation. The interest here for studying those objects lies in the connection to the p-adic Langland correspondence.

In his thesis [Ku] Kuschkowitz adapts the strategy of the p-adic case to the situation considered here.

Theorem *(Kuschkowitz):* Let $\mathcal{E}$ be a homogeneous vector bundle on $\mathbb{P}^d_k$. There is a filtration

$$\mathcal{E}(\mathcal{X})^0 \supset \mathcal{E}(\mathcal{X})^1 \supset \cdots \supset \mathcal{E}(\mathcal{X})^{d-1} \supset \mathcal{E}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{E})$$

on $\mathcal{E}(\mathcal{X})^0 = H^0(\mathcal{X}, \mathcal{E})$ such that for $j = 0, \ldots, d-1$, there is an extension of G-representations

$$0 \to \mathrm{Ind}^G_{P_{(j+1,d-j)}}(\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d, \mathcal{E}) \otimes St_{d+1-j}) \to \mathcal{E}(\mathcal{X})^j/\mathcal{E}(X)^{j+1} \to v^G_{P_{(j+1,1,\ldots,1)}} \otimes H^{d-j}(\mathbb{P}^d_k, \mathcal{E}) \to 0.$$

Here the module $v^G_{P_{(j+1,1,\ldots,1)}}$ is a generalized Steinberg representation corresponding to the decomposition $(j+1, 1, \ldots, 1)$ of $d+1$. Further $P_{\underline{j}} = P_{(j,d+1-j)} \subset G$ is the (lower) standard-parabolic subgroup attached to the decomposition $(j, d+1-j)$ of $d+1$ and St_{d+1-j} is the Steinberg representation of $\mathrm{GL}_{d+1-j}(k)$. Here the action of the parabolic is induced by the composite

$$P_{(j,d+1-j)} \to L_{(j,d+1-j)} = \mathrm{GL}_j(k) \times \mathrm{GL}_{d+1-j}(k) \to \mathrm{GL}_{d+1-j}(k).$$

Finally we have the reduced local cohomology

$$\tilde{H}^{d-j}_{\mathbb{P}^j_k}(\mathbb{P}^d_k, \mathcal{E}) := \ker\left(H^{d-j}_{\mathbb{P}^j_k}(\mathbb{P}^d_k, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_k, \mathcal{E})\right)$$

which is a $P_{(j+1,d-j)}$-module.

In the p-adic setting the substitute of the LHS of this extension has the structure of an admissible module over the locally analytic distribution algebra. Here we were able to give a description of the dual representation in terms of a series of functors

$$\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}^{\infty,adm}_K(P) \to \mathrm{Rep}^{\ell a}_K(G)$$

where $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ consist of the algebraic objects of type $\mathfrak{p}$ in the category $\mathcal{O}$, $\mathrm{Rep}^{\infty,adm}_K(P)$ is the category of smooth admissible P-representations and $\mathrm{Rep}^{\ell a}_K(G)$ denotes the category of locally analytic G-representations.

In positive characteristic Lie algebra methods do not behave so nice. E.g. the local cohomology groups are not finitely generated over the universal enveloping algebra of the Lie algebra of GL_{d+1} so that the same machinery does not apply. Our goal in this paper is to concentrate on the latter aspect and to present two candidates for a substitution in this situation. The first approach considers the crystalline universal enveloping algebra $\dot{\mathcal{U}}(\mathfrak{g})$ (or Kostant form) which coincides with the distribution algebra of G, cf. [Ja]. The action of $\mathfrak{g}$ extends to one of $\dot{\mathcal{U}}(\mathfrak{g})$, so that $H^0(\mathcal{X}, \mathcal{E})$ becomes a module over the smash product $k[G]\#\dot{\mathcal{U}}(\mathfrak{g})$. We define a positive characteristic version of $\mathcal{F}^G_P$ and prove analogously properties of them as in the p-adic case, e.g. we give an irreducibility criterion, cf. [OS].

The second approach uses instead of $\dot{\mathcal{U}}(\mathfrak{g})$ the ring of distributions $D^{\mathcal{E}}$ on the flag variety with respect to $\mathcal{E}$. The important point is that the natural map $\dot{\mathcal{U}}(\mathfrak{g}) \to D^{\mathcal{E}}$ is in contrast to the field of complex numbers not surjective as shown by Smith [Sm]. We will show that the above local cohomology modules are finitely generated leading to a category $\mathcal{O}_{D^{\mathcal{E}}}$ where we can define similar our functors $\mathcal{F}^G_P$.

Notation: We let p be a prime number, $q = p^n$ some power and let $k = \mathbb{F}_q$ the corresponding field with q elements. We fix an algebraic closure $\mathbb{F} := \overline{\mathbb{F}}_q$ and denote by $\mathbb{P}^d_{\mathbb{F}}$ the projective space of dimension d over $\mathbb{F}$. If $Y \subset \mathbb{P}^d_{\mathbb{F}}$ is a closed algebraic $\mathbb{F}$-subvariety and $\mathcal{F}$ is a sheaf on $\mathbb{P}^d_{\mathbb{F}}$ we write $H^*_Y(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F})$ for the corresponding local cohomology. We consider the algebraic action $\mathbf{G} \times \mathbb{P}^d_{\mathbb{F}} \to \mathbb{P}^d_{\mathbb{F}}$ of $\mathbf{G}$ on $\mathbb{P}^d_{\mathbb{F}}$ given by

$$g \cdot [q_0 : \cdots : q_d] := m(g, [q_0 : \cdots : q_d]) := [q_0 : \cdots : q_d]g^{-1}.$$

We use bold letters $\mathbf{H}$ to denote algebraic group schemes over $\mathbb{F}_q$, whereas we use normal letters H for their $\mathbb{F}_q$-valued points. We denote by $\mathbf{H}_{\mathbb{F}} := \mathbf{H} \times_{\mathbb{F}_q} \mathbb{F}$ their base change to $\mathbb{F}$. We use Gothic letters $\mathfrak{h}$ for their Lie algebras (over $\mathbb{F}$). The corresponding enveloping algebras are denoted as usual by $U(\mathfrak{h})$.

We denote by $\mathbf{G}_{\mathbb{Z}}$ a split reductive algebraic group over $\mathbb{Z}$. We fix a Borel subgroup $\mathbf{B}_{\mathbb{Z}} \subset \mathbf{G}_{\mathbb{Z}}$ and let $\mathbf{U}_{\mathbb{Z}}$ be its unipotent radical and $\mathbf{U}^-_{\mathbb{Z}}$ the opposite radical. Let $\mathbf{T}_{\mathbb{Z}} \subset \mathbf{G}_{\mathbb{Z}}$ be a fixed split torus and denote the root system by Φ and its subset of simple roots by Δ.

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1. The theorem of Kuschkowitz

In this section we recall shortly the strategy for proving the theorem of Kuschkowitz. Here we consider for $\mathbf{G}$ the general linear group $\mathbf{GL_{d+1}}$ and for $\mathbf{B} \subset \mathbf{G}$ the Borel subgroup of lower triangular matrices and $\mathbf{T}$ the diagonal torus. Denote by $\overline{\mathbf{T}}$ its image in $\mathbf{PGL}_{d+1}$. For $0 \leq i \leq d$, let $\epsilon_i : \mathbf{T} \to \mathbb{G}_{\mathbf{m}}$ be the character defined by $\epsilon_i(\mathrm{diag}(t_1, \ldots, t_d)) = t_i$. Put $\alpha_{i,j} := \epsilon_i - \epsilon_j$ for $i \neq j$. Then $\Delta := \{\alpha_{i,i+1} \mid 0 \leq i \leq d-1\}$ are the simple roots and $\Phi := \{\alpha_{i,j} \mid 0 \leq i \neq j \leq d-1\}$ are the roots of $\mathbf{G}$ with respect to $\mathbf{T} \subset \mathbf{B}$. For a decomposition $(i_1, \ldots, i_r)$ of $d+1$, let $\mathbf{P}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ be the corresponding standard-parabolic subgroup of $\mathbf{G}$, $\mathbf{U}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ its unipotent radical and $\mathbf{L}_{(\mathbf{i_1},\ldots,\mathbf{i_r})}$ its Levi component.

Let $\mathcal{E}$ be a homogeneous vector bundle on $\mathbb{P}^d_{\mathbb{F}}$. Our finite group G stabilizes $\mathcal{X}$. Therefore, we obtain an induced action of G on the $\mathbb{F}$-vector space of global sections $\mathcal{E}(\mathcal{X})$. Further $\mathcal{E}$ is naturally a $\mathfrak{g}$-module, i.e., there is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(\mathcal{E})$. For the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{P}^d_{\mathbb{F}}}$ with its natural $\mathbf{G}$-linearization we can describe the action of $\mathfrak{g}$ on $\mathcal{O}(\mathcal{X})$. Indeed, for a root $\alpha = \alpha_{i,j} \in \Phi$, let

$$L_\alpha := L_{(i,j)} \in \mathfrak{g}_\alpha$$

be the standard generator of the weight space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$. Let $\mu \in X^*(\overline{\mathbf{T}})$ be a character of the torus $\overline{\mathbf{T}}$. Write μ in the shape $\mu = \sum_{i=0}^d m_i \epsilon_i$ with $\sum_{i=0}^d m_i = 0$. Define $\Xi_\mu \in \mathcal{O}(\mathcal{X})$ by

$$\Xi_\mu(q_0, \ldots, q_d) = q_0^{m_0} \cdots q_d^{m_d}.$$

For these functions, the action of $\mathfrak{g}$ is given by

$$L_{(i,j)} \cdot \Xi_\mu = m_j \cdot \Xi_{\mu + \alpha_{i,j}} \tag{1.1}$$

and

$$t \cdot \Xi_\mu = (\sum\nolimits_i m_i t_i) \cdot \Xi_\mu, \ t \in \mathfrak{t}.$$

Fix an integer $0 \leq j \leq d-1$. Let

$$\mathbb{P}^j_{\mathbb{F}} = V(X_{j+1}, \ldots, X_d) \subset \mathbb{P}^d_{\mathbb{F}}$$

be the closed subvariety defined by the vanishing of the coordinates $X_{j+1}, \ldots, X_d$. The algebraic local cohomology modules $H^i_{\mathbb{P}^j_{\mathbb{F}}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}),\ i \in \mathbb{N}$, sit in a long exact sequence

$$\cdots \to H^{i-1}(\mathbb{P}^d_{\mathbb{F}} \setminus \mathbb{P}^j_{\mathbb{F}}, \mathcal{F}) \to H^i_{\mathbb{P}^j_{\mathbb{F}}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F}) \to H^i(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F}) \to H^i(\mathbb{P}^d_{\mathbb{F}} \setminus \mathbb{P}^j_{\mathbb{F}}, \mathcal{F}) \to \cdots$$

which is equivariant for the induced action of $\mathbf{P}_{(\mathbf{j+1},\mathbf{d-j})} \ltimes U(\mathfrak{g})$. Here the semi-direct product is defined via the adjoint action of $\mathbf{P}_{(\mathbf{j+1},\mathbf{d-j})}$ on $\mathfrak{g}$. We set

$$\tilde{H}^{d-j}_{\mathbb{P}^j_{\mathbb{F}}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) := \ker\left(H^{d-j}_{\mathbb{P}^j_{\mathbb{F}}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E})\right)$$

which is consequently a $\mathbf{P}_{(\mathbf{j+1},\mathbf{d-j})} \ltimes U(\mathfrak{g})$-module.

Consider the exact sequence of $\mathbb{F}$-vector spaces with G-action

$$0 \to H^0(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \to H^0(\mathcal{X}, \mathcal{E}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \to H^1(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \to 0.$$

Note that the higher cohomology groups $H^i(\mathcal{X}, \mathcal{E}),\ i > 0$, vanish since $\mathcal{X}$ is an affine space. The G-representations $H^0(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F}),\ H^1(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F})$ are finite-dimensional algebraic. Let $i : \mathcal{Y} \hookrightarrow (\mathbb{P}^d_{\mathbb{F}})$ denote the closed embedding and let $\mathbb{Z}$ be constant sheaf on $\mathcal{Y}$. Then by [SGA2, Proposition 2.3 bis.], we conclude that

$$\mathrm{Ext}^*(i_*(\mathbb{Z}_{\mathcal{Y}}), \mathcal{E}) = H^*_{\mathcal{Y}}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}).$$

The idea is now to plug in a resolution of the sheaf $\mathbb{Z}$ on the boundary and works as follows.

Let $\{e_0, \ldots, e_d\}$ be the standard basis of $V = \mathbb{F}^{d+1}$. For any $\alpha_i \in \Delta$, put

$$V_i = \bigoplus_{j=0}^{i} \mathbb{F} \cdot e_j \ \text{ and } \ Y_i = \mathbb{P}(V_i)$$

For any subset $I \subset \Delta$ with $\Delta \setminus I = \{\alpha_{i_1} < \ldots < \alpha_{i_r}\}$, set $Y_I = \mathbb{P}(V_{i_1})$ and consider it as a closed subvariety of $\mathbb{P}^d_{\mathbb{F}}$. Furthermore, let P_I be the lower parabolic subgroup of G, such that I coincides with the simple roots appearing in the Levi factor of P_I. Hence the group P_I stabilizes Y_I. We obtain

$$\mathcal{Y} = \bigcup_{g \in G} g \cdot Y_{\Delta \setminus \{\alpha_{d-1}\}}. \tag{1.2}$$

Consider the natural closed embeddings

$$\Phi_{g,I} : gY_I \longrightarrow \mathcal{Y}$$

and put

$$\mathbb{Z}_{g,I} := (\Phi_{g,I})_*(\Phi^*_{g,I}\,\mathbb{Z}).$$

We obtain the following complex of sheaves on $\mathcal{Y}$:

$$\begin{aligned} 0 \to \mathbb{Z} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 2}} \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I} \to \cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = i}} \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I} \to \cdots \\ \cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = d-1}} \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I} \to \bigoplus_{g \in G/P_\emptyset} \mathbb{Z}_{g,\emptyset} \to 0. \end{aligned} \tag{1.3}$$

which is acyclic by [O1].

In a next step one considers the spectral sequence which is induced by this complex applied to $\mathrm{Ext}^*(i_*(-),\ \mathcal{E})$. Here one uses that for all $I \subset \Delta$, there is an isomorphism

$$\mathrm{Ext}^*(i_*(\bigoplus_{g\in G/P_I} \mathbb{Z}_{g,I}), \mathcal{E}) = \bigoplus_{g\in G/P_I} H^*_{gY_I}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}).$$

By evaluating the spectral sequence Kuschkowitz arrives in [Ku] at the theorem mentioned in the introduction.

2. First approach

In this section we replace $U(\mathfrak{g})$ by its crystalline version and transform the results of [OS] to this setting.

Let $\mathbf{G}_{\mathbb{Z}}$ be a split reductive algebraic group over $\mathbb{Z}$ and let $\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $\mathbf{G}_{\mathbb{Z}}(\mathbb{C})$. On the other hand let $D(\mathbf{G}_{\mathbb{F}})$ be the distribution algebra of $\mathbf{G}_{\mathbb{F}} = \mathbf{G}_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F}$. We identify $D(\mathbf{G}_{\mathbb{F}})$ with the universal crystalline enveloping algebra (Kostant form) $\dot{\mathcal{U}}(\mathfrak{g})$. Thus $\dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{g})_{\mathbb{Z}} \otimes \mathbb{F}$ where $\dot{\mathcal{U}}(\mathfrak{g})_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g}_{\mathbb{C}})$ generated by the expressions

$$x_\alpha^{[n]} := x_\alpha^n/n!,\ y_\alpha^{[n]} := y_\alpha^n/n!,\ \alpha \in \Phi^+, n \in \mathbb{N}$$
$$\text{and } \binom{h_\alpha}{n},\ \alpha \in \Delta, n \in \mathbb{N},$$

where $x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}$ are generators and $h_\alpha = [x_\alpha, y_\alpha]$ for all $\alpha \in \Delta$. We have a PBW-decomposition

$$\dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{u}) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{t}) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{u}^-)$$

where the crystalline enveloping algebras for $\mathfrak{u}, \mathfrak{u}^-$ and $\mathfrak{t}$ are defined analogously.

We mimic the definition of the category $\mathcal{O}$ in the sense of BGG.

Definition 2.1. Let $\dot{\mathcal{O}}$ be the full subcategory of all $\dot{\mathcal{U}}(\mathfrak{g})$-modules such that

i) M is finitely generated as $\dot{\mathcal{U}}(\mathfrak{g})$-module

ii) $\dot{\mathcal{U}}(\mathfrak{t})$ acts semisimple with finite-dimensional weight spaces.

iii) $\dot{\mathcal{U}}(\mathfrak{u})$ acts locally finite-dimensional, i.e., for all $m \in M$ we have $\dim \dot{\mathcal{U}}(\mathfrak{u}) \cdot m < \infty$.

Remark 2.2. In [Hab, Def. 3.2] Haboush calls $\dot{\mathcal{U}}(\mathfrak{g})$-modules satisfying properties i) and ii) admissible. The category $\dot{\mathcal{O}}$ has been also recently considered by Andersen [An] and Fiebig [Fi] (even more generally for weight modules) discussing among others the structure of these objects.

Similarly, for a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L_P} \cdot \mathbf{U_P}$ (induced by one over $\mathbb{Z}$), we let $\dot{\mathcal{O}}^{\mathfrak{p}}$ be the full subcategory of $\dot{\mathcal{O}}$ consisting of objects which are direct sums of finite-dimensional $\dot{\mathcal{U}}(\mathfrak{l}_P)$-modules. We let $\dot{\mathcal{O}}_{\mathrm{alg}}$ be the full subcategory of $\dot{\mathcal{O}}$ such that the action of $\dot{\mathcal{U}}(\mathfrak{t})$ is induced on the weight spaces by algebraic characters $X^*(T_{\mathbb{F}})$ of $T_{\mathbb{F}}$. Finally we set

$$\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}} := \dot{\mathcal{O}}_{\mathrm{alg}} \cap \dot{\mathcal{O}}^{\mathfrak{p}}.$$

As in the classical case there is for every object $M \in \dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$ some finite-dimensional algebraic P-representation[1] $W \subset M$ which generates M as a $\dot{\mathcal{U}}(\mathfrak{g})$-module, i.e., there is a surjective homomorphism $\dot{\mathcal{U}}(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}(\mathfrak{p})} W \to M$. Again there is a PBW-decomposition $\dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{u}_P) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{l}_P) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{u}_P^-)$ such that the latter homomorphism can be seen as a map $\dot{\mathcal{U}}(\mathfrak{u}_P^-) \otimes_{\mathbb{F}} W \to M$.

According to [Hab] there is the notion of maximal vectors, highest weights, highest weight module etc. and we may define Verma modules, cf. Def. 3.1 in loc.cit. In fact let λ be a one-dimensional $\dot{\mathcal{U}}(\mathfrak{t})$-module. Then we consider it as usual via the trivial $\dot{\mathcal{U}}(\mathfrak{u})$-action as a one-dimensional $\dot{\mathcal{U}}(\mathfrak{b})$-module $\mathbb{F}_\lambda$. Then

$$M(\lambda) = \dot{\mathcal{U}}(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}(\mathfrak{b})} \mathbb{F}_\lambda$$

is the attached Verma module of weight λ. As in the classical case Theorem of [Hu, 1.2] holds true for our highest weight modules. In particular it has a unique maximal proper submodule and therefore a unique simple quotient $L(\lambda)$, cf. [Hab, Prop. 4.4], [An, Thm 2.3], [Fi, Prop. 2.3].

Proposition 2.3. *The simple modules in $\dot{\mathcal{O}}_{\mathrm{alg}}$ are exactly of the shape $L(\lambda)$ for $\lambda \in X^*(\mathbf{T}_{\mathbb{F}})$.*

Proof. We need to show that every simple object in $\dot{\mathcal{O}}_{\mathrm{alg}}$ is of this form. But by [Hab, Thm 4.9 i)] simple admissible highest weight modules are of the form $L(\lambda)$ for a one-dimensional $\dot{\mathcal{U}}(\mathfrak{t})$-module λ. The algebraic condition forces λ to be an algebraic character $\lambda \in X^*(\mathbf{T}_{\mathbb{F}})$. □

We also consider the full subcategory $M^d_{\dot{\mathcal{U}}(\mathfrak{g})}$ for all $\dot{\mathcal{U}}(\mathfrak{g})$-modules which satisfy condition ii) in the definition of $\dot{\mathcal{O}}$. For any such object M we define a dual object M' (the graded dual) following the classical concept: consider the weight space decomposition $M = \bigoplus_\lambda M_\lambda$ where λ is as above a one-dimensional $\dot{\mathcal{U}}(\mathfrak{t})$-module. Then the

[1]Meaning that we restrict an algebraic $\mathbf{P}$-representation to the its rational points P.

underlying vector space of M' is the direct sum $\bigoplus_\lambda \mathrm{Hom}(M_\lambda, K)$. The $\dot{\mathcal{U}}(\mathfrak{g})$-structure on it is given by the natural one[2]. Clearly one has $(M')' = M$.

We consider the natural action of $\mathfrak{u}_P^-$ on $\mathcal{O}(\mathbf{U}_{\mathbf{P}^-,\mathbb{F}})$. This extends to a non-degenerate pairing

$$\dot{\mathcal{U}}(\mathfrak{u}_P^-) \otimes \mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \;\; \to \;\; \mathbb{F} \tag{2.1}$$

such that $\mathcal{O}(\mathbf{U}_{\mathbf{P}^-,\mathbb{F}})$ identifies with the graded dual of $\dot{\mathcal{U}}(\mathfrak{u}_P^-)$. Moreover we pull back via this identification the action of P on $(\dot{\mathcal{U}}(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}(\mathfrak{p})} 1)'$ to $\mathcal{O}(\mathbf{U}_{\mathbf{P}^-,\mathbb{F}})$. By construction we obtain the following statement.

Lemma 2.4. *There is an isomorphism of $P \ltimes \dot{\mathcal{U}}(\mathfrak{g})$-modules $\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \cong (\dot{\mathcal{U}}(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}(\mathfrak{p})} 1)'$.* □

The pairing (2.1) extends for every algebraic P-representation W to a pairing

$$(\dot{\mathcal{U}}(\mathfrak{u}_P^-) \otimes W') \otimes (\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W) \;\; \to \;\; \mathbb{F} \tag{2.2}$$

such that $\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W$ becomes isomorphic to $\dot{\mathcal{U}}(\mathfrak{u}_P^-)' \otimes W'$ as $P \ltimes \dot{\mathcal{U}}(\mathfrak{g})$-modules.

Let $\dot{\mathbb{F}}[G, \mathfrak{g}] := \mathbb{F}[G]\#\dot{\mathcal{U}}(\mathfrak{g})$ be the smash product of $\dot{\mathcal{U}}(\mathfrak{g})$ and the group algebra $\mathbb{F}[G]$ of G. Recall that this $\mathbb{F}$-algebra has as underlying vector space the tensor product $\mathbb{F}[G] \otimes \dot{\mathcal{U}}(\mathfrak{g})$ and where the multiplication is induced by $(g_1 \otimes z_1) \cdot (g_2 \otimes z_2) = g_1 g_2 \otimes Ad(g_2)(z_1) z_2$ for elements $g_i \in G, z_i \in \dot{\mathcal{U}}(\mathfrak{g}), i = 1, 2$.

Definition 2.5. i) We denote by $Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$ be the full subcategory of all $\dot{\mathbb{F}}[G, \mathfrak{g}]$-modules for which the action of $\dot{\mathcal{U}}(\mathfrak{t})$ is diagonalisable into finite-dimensional weight spaces.

ii) We denote by $Mod^{fg,d}_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$ be the full subcategory of $Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$ which are finitely generated.

For an object $\mathcal{M}$ of $Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$ we define the dual $\mathcal{M}'$ as the graded dual of the underlying $\dot{\mathcal{U}}(\mathfrak{g})$-module together with the contragradient action of G.

Let M be an object of $\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$. Then there is a surjection

$$p : \dot{\mathcal{U}}(\mathfrak{u}_P^-) \otimes W \to M$$

[2] Without the composition with the Cartan involution.

for some finite-dimensional algebraic P-module W. Let $\mathfrak{d} := \ker(p)$ be its kernel. Then set

$$\mathcal{F}_P^G(M) := \mathrm{Ind}_P^G((\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W)^{\mathfrak{d}})$$

where $(\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W)^{\mathfrak{d}}$ is the orthogonal complement of $\mathfrak{d}$ with respect to the pairing (2.2). The latter submodule can be interpreted as the graded dual of M. In particular we get

$$\mathcal{F}_P^G(M)' = \mathrm{Ind}_P^G(M).$$

Lemma 2.6. *Let M be an object of $\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$. Then $\mathcal{F}_P^G(M)$ is an object of the category $Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$. Its dual $\mathcal{F}_P^G(M)'$ is an object of the category $Mod^{fg,d}_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$.*

Proof. It suffices to show the second assertion. As G/P is a finite set, we need only to show that $\mathcal{F}_P^G(M)'$ has a decomposition into finite-dimensional weight spaces. Let $M = \bigoplus_\lambda M_\lambda$. We write $\mathcal{F}_P^G(M) = \bigoplus_{g\in G/P} \delta_g \star M$ where $\delta_g \star M$ is the $\dot{\mathcal{U}}(\mathfrak{g})$-module with the same underlying vector space but where the Lie algebra action is twisted by $Ad(g)$. We consider the Bruhat decomposition $G/P = \bigcup_{w\in W_P} U_{B,w}wP/P$ where $U_{B,w} = U \cap wU^-w^{-1}$ and take the obvious representatives for G/P. Thus we have

$$\mathcal{F}_P^G(M)' = \bigoplus_{w\in W_P} \bigoplus_{u\in U^-_{B,w}} \delta_{uw} \star M.$$

In the case of $\delta_w, w \in W$, the grading of $\delta_w \star M$ is given by $\bigoplus_\lambda M_{w\lambda}$. In the case of $\delta_u, u \in U_{B,w}$ the grading is given by $\bigoplus_\lambda u \cdot M_\lambda$ (Note that we have an action of U on M). In general we consider the mixture of these cases. □

Let V be additionally a finite-dimensional P-module. Then we set

$$\mathcal{F}_P^G(M, V) := \mathrm{Ind}_P^G((\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W')^{\mathfrak{d}} \otimes V).$$

This is an object of $Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$ by a slight generalization of the above lemma. In this way we get a bi-functor

$$\mathcal{F}_P^G : \dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}(P) \to Mod^d_{\dot{\mathbb{F}}[G,\mathfrak{g}]}.$$

By the following statement the dual $\mathcal{F}_P^G(M, V)'$ is an object of $Mod^{fg,d}_{\dot{\mathbb{F}}[G,\mathfrak{g}]}$.

Lemma 2.7. *The dual of $\mathcal{F}_P^G(M, V)$ is given by*

$$\mathcal{F}_P^G(M, V)' = \dot{\mathbb{F}}[G, \mathfrak{g}] \otimes_{\dot{\mathbb{F}}[P,\mathfrak{g}]} (M \otimes V').$$

Proof. We have $\mathcal{F}_P^G(M, V)' = \mathrm{Ind}_P^G(M' \otimes V)' = \mathrm{Ind}_P^G((M')' \otimes V') = \mathrm{Ind}_P^G(M \otimes V')$. □

Proposition 2.8. *The functor $\mathcal{F}^G_P$ is exact in both arguments.*

Proof. We start to prove that the functor is exact in the first argument. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Then the sequence $0 \to \mathrm{Ind}^G_P M_1 \to \mathrm{Ind}^G_P M_2 \to \mathrm{Ind}^G_P M_3 \to 0$ is also exact. But the graded dual of this sequence is exactly $0 \to \mathcal{F}^G_P(M_3) \to \mathcal{F}^G_P(M_2) \to \mathcal{F}^G_P(M_1) \to 0$.

As for exactness in the second argument let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of P-representations. As

$$\mathcal{F}^G_P(M,V) := \mathrm{Ind}^G_P((\mathcal{O}(\mathbf{U}^-_{\mathbf{P},\mathbb{F}}) \otimes W')^{\mathfrak{d}} \otimes V_i)$$

and Ind^G_P is an exact functor we see easily the claim. □

Now let $\mathbf{Q} \supset \mathbf{P}$ be a parabolic subgroup and let $M \in \dot{\mathcal{O}}^{\mathfrak{q}}_{\mathrm{alg}}$. Then we may consider it also as an object of $\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$.

Proposition 2.9. *If $\mathbf{Q} \supset \mathbf{P}$ is a parabolic subgroup, M an object of $\dot{\mathcal{O}}^{\mathfrak{q}}_{\mathrm{alg}}$ and V a finite-dimensional P-module, then*

$$\mathcal{F}^G_P(M,V) = \mathcal{F}^G_Q(M, \mathrm{Ind}^Q_P(V)) .$$

Proof. We have

$$\begin{aligned}\mathcal{F}^G_P(M,V) &= \mathrm{Ind}^G_P(M' \otimes V) = \mathrm{Ind}^G_Q(\mathrm{Ind}^Q_P(M' \otimes V)) \\ &= \mathrm{Ind}^G_Q(M' \otimes \mathrm{Ind}^Q_P(V)) = \mathcal{F}^G_Q(M, \mathrm{Ind}^Q_P(V))\end{aligned}$$

by the projection formula. Hence we deduce the claim. □

As in [OS] a parabolic Lie algebra $\mathfrak{p}$ is called *maximal* for an object $M \in \dot{\mathcal{O}}^{\mathfrak{p}}$ if there does not exist a parabolic Lie algebra $\mathfrak{q} \supsetneq \mathfrak{p}$ with $M \in \dot{\mathcal{O}}^{\mathfrak{q}}$.

Theorem 2.10. *Let $p > 3$. Let M be an simple object of $\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$ such that $\mathfrak{p}$ is maximal for M. Then $\mathcal{F}^G_P(M)$ is a simple $\dot{\mathbb{F}}[G,\mathfrak{g}]$-module.*

Proof. The proof follows the idea of loc.cit. and is even simpler. We start with the observation that by duality $\mathcal{F}^G_P(M,V)$ is simple as $\dot{\mathbb{F}}[G,\mathfrak{g}]$-module iff $\mathcal{F}^G_P(M,V)'$ is simple as $\dot{\mathbb{F}}[G,\mathfrak{g}]$-module. We consider again the Bruhat decomposition $G/P = \bigcup_{w\in W_P} U^-_{B,w} wB/B$ and the induced decomposition

$$\mathcal{F}^G_P(M)' = \bigoplus_{w\in W_P} \bigoplus_{u\in U^-_{B,w}} \delta_{uw} \star M.$$

We denote (with respect to $\delta_{uw} \star M$) for elements $\mathfrak{z} \in \dot{\mathcal{U}}(\mathfrak{g})$ and $m \in M$ the action of $\mathfrak{z}$ on m by $\mathfrak{z} \cdot_{uw} m$. Now each summand $\delta_{uw} \star M$ is simple since M is simple. Thus it suffices to show that the summands are pairwise non isomorphic as $\dot{\mathcal{U}}(\mathfrak{g})$-modules. Suppose that there is an isomorphism $\phi : \delta_g \star M \to \delta_h \star M$ for some elements g, h as above. We may suppose that $h = e$. Write $g = u^{-1}w$. Then such an isomorphism is equivalent to an isomorphism $\phi : \delta_w \star M \to \delta_u \star M \cong M$. The latter isomorphism is given by the mapping $m \mapsto u^{-1} \cdot m$.

We show that this can only happen if $w \in W_P$. Let $\lambda \in X(\mathbf{T})^*$ be the highest weight of M, i.e. $M = L(\lambda)$, and $P = P_I$ is the standard parabolic subgroup induced by $I = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$, cf. [Hu]. Suppose w is not contained in $W_I = W_P$. Then there is a positive root $\beta \in \Phi^+ \setminus \Phi_I^+$ such that $w^{-1}\beta < 0$, hence $w^{-1}(-\beta) > 0$. Consider a non-zero element element $y \in \mathfrak{g}_{-\beta}$, and let $v^+ \in M$ be a weight vector of weight λ. Then we have for $n \in \mathbb{N}$, the following identity

$$y^{[n]} \cdot_w v^+ = \mathrm{Ad}(w^{-1})(y^{[n]}) \cdot v^+ = 0$$

as $\mathrm{Ad}(w^{-1})(y^{[n]}) \in \mathfrak{g}_{-w^{-1}\beta}$ annihilates v^+. We have $\phi(v^+) = v$ for some nonzero $v \in M$. And therefore

$$0 = \phi(y^{[n]} \cdot_w v^+) = y^{[n]} \cdot \phi(v^+) = y^{[n]} \cdot v \; .$$

But y is an element of $\mathfrak{u}_P^-$, hence we get a contradiction by Proposition 2.13 since n was arbitrary. □

Theorem 2.11. *Let $p > 3$. Let M be an simple object of $\dot{\mathcal{O}}^{\mathfrak{p}}_{\mathrm{alg}}$ such that $\mathfrak{p}$ is maximal for M and let V be an irreducible P-representation. Then $\mathcal{F}^G_P(M,V)$ and its dual $\mathcal{F}^G_P(M,V)'$ are simple as $\dot{\mathbb{F}}[G,\mathfrak{g}]$-module.*

Proof. Again by duality it is enough to check the assertion for $\mathcal{F}^G_P(M,V)'$. So let $U \subset \mathcal{F}^G_P(M,V)'$ be a non-zero G-invariant subspace. Recall that $\mathcal{F}^G_P(M)' = \bigoplus_{\gamma \in G/P} \delta_\gamma \star L(\lambda)$ so that

$$\mathcal{F}^G_P(M,V) = \bigoplus_{\gamma \in G/P} \delta_\gamma \star L(\lambda)' \otimes V^\gamma.$$

Considered as $\dot{\mathcal{U}}(\mathfrak{g})$-module $\mathcal{F}^G_P(M,V)$ is isomorphic to $(\bigoplus_{\gamma \in G/P} \delta_\gamma \star L(\lambda)') \otimes V$. Hence by the simplicity of M and since the summands $\delta_\gamma \star L(\lambda)'$ are pairwise not isomorphic the $\dot{\mathcal{U}}(\mathfrak{g})$-module U is equal to

$$\bigoplus_{\gamma \in G/P} \delta_\gamma \star L(\lambda)' \otimes_{\mathbb{F}} V_\gamma \, ,$$

with subspaces, V_γ, γ, of V. Here $\delta_1 \star L(\lambda)' \otimes V_1 = L(\lambda)' \otimes V_1$ is a $\dot{\mathbb{F}}[P,\mathfrak{g}]$-submodule of $L(\lambda)' \otimes V$. Since V ist irreducible the latter object is irreducible, as well. Hence $V_1 = V$. But since G permutes the summands of U we see that $U = \mathcal{F}^G_P(M,V)'$. □

In the following statement we merely consider elements in a root space by the very definition of $\dot{\mathcal{U}}(\mathfrak{g})$.

Lemma 2.12. *Let $p > 3$. Let $x \in \mathfrak{g}_\gamma$ some element for $\gamma \in \Phi$. Let M be a $\dot{\mathcal{U}}(\mathfrak{g})$-module and $v \in M$.*

(i) If x acts locally finitely[3] *on v (i.e., the K-vector space generated by $(x^{[i]}.v)_{i\geq 0}$ is finite-dimensional), then x acts locally finitely on $\dot{\mathcal{U}}(\mathfrak{g}).v$.*

(ii) If $x.v = 0$ and $[x,[x,y]] = 0$ for some $y \in \mathfrak{g}_\beta$, where $\beta \in \Phi$ then

$$x^{[n]}y^{[n]}.v = [x,y]^{[n]}.v \ .$$

Proof. (i) The idea is to apply Lemma 8.1 of loc.cit. which gives in characteristic 0 the formula

$$x^k \cdot z_1 z_2 \ldots z_n = \sum_{i_1+\ldots+i_{n+1}=k} \frac{k!}{i_1!\ldots i_{n+1}!}[x^{(i_1)},z_1]\cdot\ldots\cdot[x^{(i_n)},z_n]x^{i_{n+1}}.$$

Here the expression $[x^{(i)},z]$ means $ad(x)^i(z)$. We may rewrite this as

$$x^{[k]} \cdot z_1 z_2 \ldots z_n = \sum_{i_1+\ldots+i_{n+1}=k} \frac{1}{i_1!\ldots i_n!}[x^{(i_1)},z_1]\cdot\ldots\cdot[x^{(i_n)},z_n]x^{[i_{n+1}]}.$$

Indeed we consider the PBW-decomposition $\dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{u}) \otimes \dot{\mathcal{U}}(\mathfrak{t}) \otimes \dot{\mathcal{U}}(\mathfrak{u})$ and assume that the elements z_i lie without loss of generality in one of these factors. For any element z in some root space it follows from [Hu, 0.2] that $[x^{(k)},z] = 0$ for all $k \geq 4$. Since we avoid the situation $p = 2,3$ we my divide my the denominators 2! and 3!.

Now in contrast to loc.cit. we have again to consider z_i as elements of $\dot{\mathcal{U}}(\mathfrak{g})$ instead of elements in $\mathfrak{g}$. Let d_i be the order of the differential z_i. Then $[x^{(i_1)},z_1]\cdots[x^{(i_n)},z_n]$ is an differential of order less than $4(d_1+\ldots+d_n)$. In particular we can conclude as in loc.cit. that the term lies in a finite dimensional vector space which gives now easily the claim.

ii) In characteristic 0 we have the formula $x^n y^n.v = n! \cdot [x,y]^n v$, cf. [OS, Lemma 8.2 ii)]. We only have to divide two times by $n!$. □

[3]Note that this definition is stronger than the one in characteristic 0.

Proposition 2.13. *Let $p > 3$. Let $\mathfrak{p} = \mathfrak{p}_I$ for some $I \subset \Delta$. Suppose $M \in \dot{\mathcal{O}}^{\mathfrak{p}}$ is a highest weight module with highest weight λ and*

$$I = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}\ .$$

Then no non-zero element of a root space of $\mathfrak{u}_{\mathfrak{p}}^-$ acts locally finitely on M.

Proof. The proof is in principal the same as in the case of characteristic 0 [OS, Cor. 8.2]. However we have to modify some technical ingredients of the necessary lemmas due the different characteristic.

let $y \in (\mathfrak{u}_{\mathfrak{p}}^-)_\gamma$ for some root γ. Let v^+ be a weight vector with weight λ. Write $\gamma = \sum_{\alpha\in\Delta} c_\alpha \alpha$ (with non-negative integers c_α). We show by induction on the height $ht(\gamma)$ of γ (Recall that $ht(\gamma) = \sum_{\alpha\in\Delta} c_\alpha$) that y_γ can not act locally finite. For this it suffices by weight reasons to show that $y_\gamma^{[n]}.v^+ \neq 0$ for infinitely many positive integers n.

If $ht(\gamma) = 1$, then γ is an element of $\Delta \setminus I$. Rescaling y_γ we can choose $x_\gamma \in \mathfrak{g}_\gamma$ such that $[x_\gamma, y_\gamma] = h_\gamma$ and $[h_\gamma, x_\gamma] = 2x_\gamma$ and $[h_\gamma, y_\gamma] = -2y_\gamma$. Then by [Hab, 5.2] we get

$$x_\gamma^{[n]} y_\gamma^{[n]}.v^+ = \binom{\lambda(h_\gamma)}{n}.v^+ = \frac{1}{n!}\prod_{i=0}^{n-1}(\langle\lambda,\gamma^\vee\rangle - i).v^+\ . \tag{2.3}$$

As $I = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$, it follows that $\langle \lambda, \gamma^\vee\rangle \notin \mathbb{Z}_{\geq 0}$ and the term on the right of 2.3 does not vanish for infinitely many n. In particular, $y_\gamma^n.v^+ \neq 0$ for infinitely many $n \geq 0$.

Now suppose $ht(\gamma) > 1$. Then we can write $\gamma = \alpha + \beta$ with $\alpha \in \Delta$ and $\beta \in \Phi^+$. Clearly, not both α and β can be contained in Φ_I. We distinguish two cases.

(a) Let $\beta - \alpha \notin \Phi$. Then we get for $\alpha \notin I$ by Lemma 2.12:

$$x_\beta^{[n]} y_\gamma^{[n]}.v^+ = [x_\beta, y_\gamma]^{[n]}.v^+$$

where x_β is a non-zero element of $\mathfrak{g}_\beta$. We conclude by induction that $[x_\beta, y_\gamma]^{[n]}.v^+ \neq 0$ for infinitely many $n \geq 0$.

For $\alpha \in I$ we have by Lemma 2.12:

$$x_\alpha^{[n]} y_\gamma^{[n]}.v^+ = [x_\alpha, y_\gamma]^{[n]}.v^+\ .$$

where x_α be a non-zero element of $\mathfrak{g}_\alpha$. Again we conclude by induction the claim. And thus $y_\gamma^{[n]}.v^+ \neq 0$ for infinitely many $n \geq 0$.

(b) Let $\beta - \alpha$ is in Φ. Then we have $\gamma - k\alpha \in \Phi^+$ for $0 \leq k \leq k_0$ (with $k_0 \leq 3$, cf. [Hu, 0.2]), and $\gamma - k\alpha \notin \Phi \cup \{0\}$ for $k > k_0$. This implies $[x_\alpha^{(i)}, y_\gamma] = 0$ for $i > k_0$. By Lemma 2.12 we conclude as in loc.cit.

$$x_\alpha^{[nk_0]} y_\gamma^n .v^+ = \sum_{i_1+\ldots+i_n = nk_0} \frac{1}{i_1! \ldots i_n!} [x_\alpha^{(i_1)}, y_\gamma] \cdot \ldots \cdot [x_\alpha^{(i_n)}, y_\gamma].v^+$$

which can be rewritten as (the corresponding term vanishes if there is one $i_j > k_0$)

$$\frac{1}{(k_0!)^n} \, [x_\alpha^{(k_0)}, y_\gamma]^n .v^+.$$

Thus we get

$$x_\alpha^{[nk_0]} y_\gamma^{[n]} .v^+ = \frac{1}{(k_0!)^n} \, [x_\alpha^{(k_0)}, y_\gamma]^{[n]} .v^+.$$

If $\gamma - k_0\alpha$ is not in Φ_I we are done by induction. Otherwise we necessarily have $\alpha \notin I$. In this case, if we choose some $x_\beta \in \mathfrak{g}_\beta \setminus \{0\}$ and deduce as in loc.cit that

$$x_\beta^{[n]} y_\gamma^{[n]} .v^+ = [x_\beta, y_\gamma]^{[n]} .v^+ \,,$$

As we are now in the case of height one, we can thus conclude again. □

Remark 2.14. Unfortunately objects in the category $\dot{\mathcal{O}}$ do not have finite length in general. This holds in particular for the local cohomology modules $H^{d-i}_{\mathbb{P}^i}(\mathbb{P}^d, \mathcal{O})$ as discussed in [Ku]. However in loc.cit. it was pointed out that one can consider composition series of countable length in the sense of Birkhoff [Bi]. In this way one can use similar to the p-adic case [OS] the functors $\mathcal{F}^G_P$ for a description of the composition factors of the terms $\mathrm{Ind}^G_{P_{(j+1,d-j)}}(\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^n, \mathcal{E}) \otimes St_{d+1-j})$ appearing in the Theorem of Kuschkowitz.

3. Second approach

This section is inspired by the theory of $\mathcal{D}$-modules. Here we carry out the theory presented in the previous section for the rings of differential operators on the flag variety $X := \mathbf{B}_\mathbb{F} \backslash \mathbf{G}_\mathbb{F}$.

Let $D_{\mathbb{P}^d_\mathbb{F}}(\mathbb{P}^d_\mathbb{F})$ be the space of global sections of the $\mathcal{D}$-module sheaf $D_{\mathbb{P}^d_\mathbb{F}}$ on the projective variety $\mathbb{P}^d_\mathbb{F}$. For a homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}^d_\mathbb{F}$, set

$$D^{\mathcal{E}}_{\mathbb{P}^d_\mathbb{F}} = \mathcal{E}(\mathbb{P}^d_\mathbb{F}) \otimes D_{\mathbb{P}^d_\mathbb{F}}(\mathbb{P}^d_\mathbb{F}) \otimes \mathcal{E}^*(\mathbb{P}^d_\mathbb{F}).$$

Then $D^{\mathcal{E}}_{\mathbb{P}^d_{\mathbb{F}}}$ acts naturally on $\mathcal{E}(\mathcal{X})$ and the filtration appearing in Kuschkowitz's theorem. Instead we consider (which become clear later) the space of global sections $D = D_X(X)$ of the differential operators on X and

$$D^{\mathcal{E}} = \mathcal{E}(X) \otimes D \otimes \mathcal{E}(X)$$

for any homogeneous vector bundle $\mathcal{E}$ on $B\backslash G$. There is an action of $D^{\mathcal{E}}$ on all the above objects as well. We consider further the Beilinson-Bernstein homomorphism

$$\pi^{\mathcal{E}} : \dot{\mathcal{U}}(\mathfrak{g}) \to D^{\mathcal{E}}$$

which is not surjective (for $\mathcal{E} = \mathcal{O}_X$) in positive characteristic as shown by Smith in [Sm].

Consider the covering $X = \bigcup_{w \in W} B\backslash BU^- w$ by translates of the big open cell $B\backslash BU^-$. Let $D^1 = D(B\backslash BU^-)$. Thus D^1 is the crystalline Weyl algebra

$$D^1 = \mathbb{F}[T_\alpha \mid \alpha \in \Phi^-]\langle y_\alpha^{[n]} \mid \alpha \in \Phi^-, n \in \mathbb{N}\rangle.$$

By the sheaf property we see that D coincides with the set

$$\{\Theta \in D^1 \mid \Theta(\mathcal{O}(B\backslash BU^- w)) \subset \mathcal{O}(B\backslash BU^- w)\ \forall w\}. \tag{3.1}$$

For any prime power $q = p^n$ we let D^1_q be the differential operators which are $\mathbb{F}[T_\alpha^q \mid \alpha \in \Phi^-]$-linear. Then we have $D = \bigcup_n D_{p^n}$. The next statement is a generalization of [Sm, lemma 3.1]. We set for $\alpha > 0$, $T_\alpha := T_{-\alpha}^{-1}$.

Lemma 3.1. *Let $\Theta \in D^1_q$. Then $\Theta \in D$ iff*

i) $\Theta(1) \in \mathbb{F}$

and

ii) $\Theta(\prod_{\alpha \in \Phi^-} T_\alpha^{i_\alpha}) \in V := \bigoplus_{0 \leq j_\alpha \leq q} \prod_{\alpha \in \Phi^-} T_\alpha^{j_\alpha}$ *for all tuples* $(i_\alpha)_\alpha$ *with* $0 \leq i_\alpha \leq q-1$.

Proof. $\Rightarrow$: The first item follows from the sheaf property (3.1) since $\mathcal{O}(B\backslash G) = \mathbb{F}$. Now let $\Theta \in D \cap D^1_q$. Let $w_0 \in W$ be the longest element and $f = \prod_{\alpha<0} T_\alpha^{i_\alpha}$ as above. Then $g = f \cdot \prod_{\alpha>0} T_\alpha^q \in \mathcal{O}(B\backslash BU^- w_0)$. But then

$$\Theta(f) = (\prod_{\alpha<0} T_\alpha^q)\Theta(g) \in (\prod_\alpha T^q_{\alpha<0})\mathcal{O}(B\backslash BU^- w_0) \cap \mathcal{O}(B\backslash BU^-) \subset V.$$

$\Leftarrow$: We show that $\Theta(\mathcal{O}(B\backslash BU^-w)) \subset \mathcal{O}(B\backslash BU^-w)$ $\forall w \in W$. We consider the element $f = \prod_{\beta\in w(\Phi^-)} T_\beta^{i_\beta} \in \mathcal{O}(B\backslash BU^-w)$. Write

$$f = \prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{i_\beta} \prod_{\substack{\beta\in w(\Phi^-)\\ \beta>0}} T_\beta^{i_\beta} = \prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{i_\beta} \prod_{\substack{\beta\in w(\Phi^-)\\ \beta>0}} T_{-\beta}^{-i_\beta}.$$

For each $\beta > 0$ let m_β be the integer with $m_\beta q < i_\beta \leq (m_\beta+1)q$. On the other hand, for each $\beta < 0$ let m_β be the integer with $m_\beta q \leq i_\beta < (m_\beta+1)q$. Then $\prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{i_\beta} = \prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{m_\beta q} T_\beta^{i_\beta - m_\beta q}$. Putting this together we get by assumption (ii)

$$\Theta(\prod_{\substack{\beta\in w(\Phi^-)\\ \beta>0}} T_{-\beta}^{(m_\beta+1)q-i_\beta} \prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{i_\beta-m_\beta q}) \in V.$$

Thus $\Theta(f) \in \prod_{\substack{\beta\in w(\Phi^-)\\ \beta>0}} T_{-\beta}^{-(m_\beta+1)q} \prod_{\substack{\beta\in w(\Phi^-)\\ \beta<0}} T_\beta^{m_\beta q} V \subset \mathcal{O}(B\backslash BU^-w)$. □

We fix the same setup as in the previous section. I.e. $\mathbf{P} \subset \mathbf{G}$ is a parabolic subgroup, $\mathbf{U_P}$ its unipotent radical and $\mathbf{U_P^-}$ its opposite unipotent radical. Moreover we have fixed as before lifts $\mathbf{P}_\mathbb{Z}$ etc. inside $\mathbf{G}_\mathbb{Z}$. We consider the following subalgebras of D in terms of generators:

$$D(P) = \langle T_\alpha^m \cdot y_\alpha^{[n]} \in D \mid m \leq n \text{ for } y_\alpha \in \mathfrak{p}\cap\mathfrak{b}^-, m \geq n \text{ for } L_{-\alpha} \in \mathfrak{u}\rangle.$$

$$D(U_P) = \langle (T_\alpha)^m \cdot y_\alpha^{[n]} \in D \mid m > n, L_{-\alpha} \in \mathfrak{u}_P\rangle.$$

$$D(U_P^-) = \langle (T_\alpha)^m \cdot y_\alpha^{[n]} \in D \mid m < n, y_\alpha \in \mathfrak{u}_P^-\rangle.$$

$$D(L_P) = \langle (T_\alpha)^m \cdot y_\alpha^{[n]} \in D \mid m \leq n \text{ for } y_\alpha \in \mathfrak{l}_P\cap\mathfrak{b}^-, m > n \text{ for } L_{-\alpha} \in \mathfrak{l}_P\cap\mathfrak{u}\rangle.$$

$$D(T) = \langle (T_\alpha)^m \cdot y_\alpha^{[n]} \in D \mid m = n, \alpha \in \Delta\rangle.$$

Remark 3.2. i) Note that $D(T)$ is for $p \neq 2$ nothing else but $\pi^{\mathcal{O}_X}(\dot{\mathcal{U}}(\mathfrak{t}))$ as $T_\alpha y_\alpha = \pi(2h_\alpha)$ for all $\alpha \in \Delta$. Hence if $\lambda \in X^*(T)$, it induces a $D(T)$-module structure on $\mathbb{F}$ which we denote by $\mathbb{F}_\lambda$.

ii) By Lemma 3.1 one checks that $D(U_P) = \pi^{\mathcal{O}_X}(\dot{\mathcal{U}}(\mathfrak{u}_P))$ since $T_\alpha^2 y_\alpha = \pi(L_{-\alpha}) \forall \alpha \in \Phi^-$.

Lemma 3.3. *There is for all $n \in \mathbb{N}$ and $\alpha \in \Delta$ the identity $\binom{T_\alpha y_\alpha}{n} = T_\alpha^n y_\alpha^{[n]}$.*

Proof. This is left to the reader. □

We set $D^{\mathcal{E}}(P) = \mathcal{E}(X) \otimes D(P) \otimes \mathcal{E}^*(X)$ etc. Then there is a product decomposition $D^{\mathcal{E}} = D^{\mathcal{E}}(P) D^{\mathcal{E}}(U_P^-)$ (an almost PBW-decomposition).

Again we mimic the definition of the category $\mathcal{O}$ in the sense of BGG. Let $\mathcal{O}^P_{D^{\mathcal{E}}}$ be the category of $D^{\mathcal{E}}$-modules such that

i) M is finitely generated as a $D^{\mathcal{E}}$-module

ii) As a $D^{\mathcal{E}}(L_P)$-module it is a direct sum of finite-dimensional modules.

iii) $D^{\mathcal{E}}(U_P)$ acts locally finite-dimensional, i.e. for all $m \in M$ the subspace $D^{\mathcal{E}}(U_P) \cdot v$ is finite-dimensional.

Remark 3.4. For $\mathcal{E} = \mathcal{O}_X$ this category corresponds in analogy to the classical case to the principal block.

We define the algebraic part of $\mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}}$ as usual, i.e. we denote by $\mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}}$ the full subcategory of $\mathcal{O}^P_{D^{\mathcal{E}}}$ consisting of objects such that the action of $\dot{\mathcal{U}}(\mathfrak{t})$ on the weight spaces is given by algebraic characters $\lambda \in X^*(T)$. Note that axioms ii) and iii) induce together with the map $\pi^{\mathcal{E}} : \dot{\mathcal{U}}(\mathfrak{g}) \to D^{\mathcal{E}}$ an algebraic P-module structure on any object in $\mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}}$.

As in the classical case we see that the axioms imply the existence of a finite-dimensional $D^{\mathcal{E}}(P)$-module N which generates M as a $D^{\mathcal{E}}$-module. Further there are similar definitions. E.g. a vector in an $D^{\mathcal{E}}$-module $M \in \mathcal{O}_{D^{\mathcal{E}}}$ is called a maximal vector of weight $\lambda \in \mathfrak{t}^*$ if $v \in M_\lambda$ and $D^{\mathcal{E}}(U_P) \cdot v = 0$. A $D^{\mathcal{E}}$-module M is called a highest weight module of weight λ if there is a maximal vector $v \in M_\lambda$ such that $M = D^{\mathcal{E}} \cdot v$. By the very definition such a module satisfies $M = D^{\mathcal{E}}(U_B^-) \cdot v$. For a one-dimensional $\dot{\mathcal{U}}(\mathfrak{t})$-module λ we consider it as usual via the trivial $D^{\mathcal{E}}(U_B)$-action as a one-dimensional $D^{\mathcal{E}}(B)$-module $\mathbb{F}_\lambda$ and set $M(\lambda) = D^{\mathcal{E}} \otimes_{D^{\mathcal{E}}(B)} \mathbb{F}_\lambda$. More generally we may define for every finite-dimensional $D^{\mathcal{E}}(P)$-module W the generalized Verma module $M(W) = D^{\mathcal{E}} \otimes_{D(P)} W$. Note that we have surjections $D^{\mathcal{E}}(U_B^-) \otimes \bar{\mathbb{F}}_\lambda \to M(\lambda)$ and $D^{\mathcal{E}}(U_P^-) \otimes_{\mathbb{F}} W \to M(W)$. We see by the above surjections that [Hu, Thm. 1.3] holds true in our category, i.e. if $M(\lambda) \neq 0$ then it has a unique simple quotient $L(\lambda)$. Moreover these modules form a complete list of simple modules in the "union" of our categories $\mathcal{O}_{D^{\mathcal{E}}}$.

Consider the local cohomology module $\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{O})$. For $d-j\geq 2$ this coincides with the vector space of polynomials

$$\bigoplus_{\substack{n_0,\ldots,n_j\geq 0\\ n_{j+1}\cdots n_d<0\\ \sum_i n_i=0}} \mathbb{F}\cdot X_0^{n_0}\cdots X_j^{n_j}X_{j+1}^{n_{j+1}}\cdots X_d^{n_d}$$

cf. [O2]. In general there is some finite-dimensional $\mathbf{P_{(j+1,d-j)}}$-module V such that $\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{E})$ is a quotient of $\bigoplus_{\substack{n_0,\ldots,n_j\geq 0\\ n_{j+1}\cdots n_d\leq 0\\ \sum_i n_i=0}} \mathbb{F}\cdot X_0^{n_0}\cdots X_j^{n_j}X_{j+1}^{n_{j+1}}\cdots X_d^{n_d}\otimes V.$

Proposition 3.5. *Let $\mathcal{E}$ be a homogeneous vector bundle on $\mathbb{P}^d_{\mathbb{F}}$. Then $\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{E})$ is an object of $\mathcal{O}^{P_{(j+1,d-j)}}_{D^{\mathcal{E}}}$.*

Proof. The non-trivial aspect is to show that $\tilde{H}^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{E})$ is finitely generated. We will show this for $\mathcal{E}=\mathcal{O}$. We claim that

$$\bigoplus_{\substack{n_0,\ldots,n_j\geq 0\\ \sum_{i=0}^j n_i=d-j}} \mathbb{F}\cdot X_0^{n_0}\cdots X_j^{n_j}X_{j+1}^{-1}\cdots X_d^{-1}$$

is as in characteristic 0 a generating system of $H^{d-j}_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{O})$. Indeed, as in the latter case we can apply successively the differential operators $L_\alpha\in\mathfrak{u}^-_{P_{(j+1,d-j)}}$ to obtain all expressions $X_0^{n_0}\cdots X_j^{n_j}X_{j+1}^{n_{j+1}}\cdots X_d^{n_d}$ such that $|n_i|\leq p$ for all $i\geq j+1$. In order to obtain those where $n_i=-(p+1)$ for some $i\geq j+1$ we can apply $y^{[p]}_{(-,j+1)}$ to get the desired denominators. However, we do not get all possible nominators. But in our algebra D we have in contrast to $\dot{\mathcal{U}}(\mathfrak{g})$ the differential operator $T^{p-1}_{(a,b)}L^{[p]}_{(a,b)}$ with $j+1\leq a<b\leq d$ at our disposal. Applying these operators we can realize all nominators. For $|n_i|>p+1$ in particular for $|n_i|=rp+1, r\geq 2$ we use the same method as above etc.. □

Proposition 3.6. *The object $\tilde{H}^i_{\mathbb{P}^j}(\mathbb{P}^d,\mathcal{O})$ is a simple module isomorphic to $L(s_i\cdots s_1\cdot 0)$.*

Proof. In characteristic 0 we gave a proof in [OS, Prop. 7.5]. Here we can argue with the differential operators at our disposal in the same way. Note that for general $\lambda\in X^*(T)$ the simple module $L(\lambda)$ is an avatar of the characteristic 0 version. □

We let

$$\mathcal{A}^{\mathcal{E}}_G := \mathbb{F}[G]\# D^{\mathcal{E}}$$

be the smash product of the group algebra $\mathbb{F}[G]$ and $D^{\mathcal{E}}$.

Let M be an object of $\mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}}$ and let V be a finite-dimensional P-module. Then we set

$$\mathcal{F}^G_P(M,V) := \mathbb{F}[G] \otimes_{\mathbb{F}[P]} (M \otimes V).$$

Note that $\mathcal{F}^G_P(M,V) = \mathrm{Ind}^G_P(M \otimes V)$. This is a $\mathcal{A}^{\mathcal{E}}_G$-module. In this way we get a bi-functor

$$\mathcal{F}^G_P : \mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}} \times \mathrm{Rep}(P) \to Mod_{\mathcal{A}^{\mathcal{E}}_G}.$$

The proof of the next statement is the same as in Propositions 2.8 and 2.9.

Proposition 3.7. *a) The bi-functor $\mathcal{F}^G_P$ is exact in both arguments.*

b) If $Q \supset P$ is a parabolic subgroup, M an object of $\mathcal{O}^Q_{D^{\mathcal{E}},\mathrm{alg}}$, then

$$\mathcal{F}^G_P(M,V) = \mathcal{F}^G_Q(M, \mathrm{Ind}^Q_P(V)),$$

where $\mathrm{Ind}^Q_P(V)$ denotes the corresponding induced representation. □

Theorem 3.8. *Let M be an simple object of $\mathcal{O}^P_{D^{\mathcal{E}},\mathrm{alg}}$ such that P is maximal for M and let V be a simple P-representation. Then $\mathcal{F}^G_P(M,V)$ is simple as $\mathcal{A}^{\mathcal{E}}_G$-module.*

Proof. The proof follows the strategy of Theorems 2.10 and 2.11. Note that Proposition 2.13 does also hols true for our objects $L(\lambda)$ as avatars of their character zero versions. □

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