

THE COHOMOLOGY OF DELIGNE-LUSZTIG VARIETIES FOR THE GENERAL LINEAR GROUP

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ABSTRACT. We propose two inductive approaches for determining the cohomology of Deligne-Lusztig varieties in the case of $G = \mathrm{GL}_n$. The first one uses Demazure compactifications and analyses the corresponding Mayer-Vietoris spectral sequence. This allows us to give an inductive formula for the Tate twist -1 contribution of the cohomology of a DL-variety. The second approach relies on considering more generally DL-varieties attached to hypersquares in the Weyl group. Here we give explicit formulas for the cohomology of height one elements.

1. INTRODUCTION

In 1976 Deligne and Lusztig [DL] introduced certain locally closed subvarieties in flag varieties over finite fields which are of particular importance in the representation theory of finite groups of Lie type. They proved that their Euler-Poincaré characteristic considered as a virtual representation of the corresponding finite group detect all irreducible representations. However, a description of the individual cohomology groups of Deligne-Lusztig varieties has been determined since then only in a few special cases. In fact Lusztig [L2] considered the case of Coxeter elements, whereas Digne, Michel and Rouquier [DMR] gave explicit formulas for groups of type $A_2, {}^2A_2, B_2, {}^2B_2, G_2, {}^2G_2$. Further Dudas [Du] treated the cohomology associated to some short-length regular elements for split groups of type F_4 and E_n and [DM] one special case of type D_4 . On the other hand, the intersection cohomology groups of their Zariski closures are considered in [L3]. In this paper we propose two inductive approaches for determining all of them in the case of $G = \mathrm{GL}_n$ (resp. for reductive groups of Dynkin type A_{n-1}). Although the key ideas work for other (split) reductive groups as well, we have decided to treat here only the case of the general linear group since things are more concrete in this special situation.

For a split reductive group \mathbf{G} defined over $k = \mathbb{F}_q$, let X be the variety of all Borel subgroups of \mathbf{G} . Let $F : X \rightarrow X$ be the Frobenius map over \mathbb{F}_q . The Deligne-Lusztig

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variety associated to an element $w \in W$ of the Weyl group is the locally closed subset of X given by

$$X(w) = \{x \in X \mid \text{inv}(x, F(x)) = w\}.$$

Here $\text{inv} : X \times X \rightarrow W$ is the relative position map induced by the Bruhat lemma. Then $X(w)$ is a smooth quasi-projective variety defined over \mathbb{F}_q . It is naturally equipped with an action of $G = \mathbf{G}(k)$ and has dimension equal to the length of w . The ℓ -adic cohomology with compact support $H_c^*(X(w)) := H_c^*(X(w), \overline{\mathbb{Q}}_\ell)$ has therefore the structure of a $G \times \text{Gal}(\overline{k}/k)$ -module.

Let $\mathbf{G} = \text{GL}_n$. In this paper, we make heavily use of certain maps $\gamma : X_1 \rightarrow X(w')$ resp. $\delta : X_2 \rightarrow X(sw')$ introduced in [DL, Theorem 1.8] and implicitly further studied in [DMR]. Here $w, w' \in W$ and s is a simple reflection with $w = sw's$ and $\ell(w) = \ell(w') + 2$. Further X_1 is a closed subset of $X(w)$ and X_2 denotes its open complement. It is proved in loc.cit that γ is a \mathbb{A}^1 -bundle whereas δ is a \mathbb{G}_m -bundle. Here we consider instead of the map δ its look-alike $X_2 \rightarrow X(w's)$. The above maps extend to \mathbb{P}^1 -bundles $X_2 \cup X(sw') \cup X(w's) \rightarrow X(w's)$ and $X_1 \cup X(w') \rightarrow X(w')$ which glue in turn to a \mathbb{P}^1 -bundle

$$\gamma : X(Q) \rightarrow X(w's) \cup X(w')$$

where $X(Q) = X(w) \cup X(sw') \cup X(w's) \cup X(w')$. Here $\gamma|_{X(w's) \cup X(w')} = \text{id}$ whereas the restriction of γ to $X(w, sw') := X(w) \cup X(sw')$ is a \mathbb{A}^1 -bundle over the base $X(w's, w') := X(w's) \cup X(w')$. In particular, we deduce that

$$H_c^i(X(Q)) = H_c^i(X(w's, w')) \oplus H_c^{i-2}(X(w's, w'))(-1)$$

for all integers $i \geq 2$, which has been already known since [DMR].

The quadruple $Q = \{w', sw', w's, w\} \subset W$ is a square in the sense of [BGG]. The notion of a square appears in the theory of BGG-resolutions of finite-dimensional Lie algebra representations. It seems to be also useful in the study of the cohomology of Deligne-Lusztig varieties. We consider more generally hypersquares in W and even in the monoid F^+ which is freely generated by the subset S of simple reflections in W . In fact we work more generally with DL-varieties and their Demazure compactifications attached to elements in F^+ in the spirit of [DMR]. More precisely, let $w = s_{i_1} \cdots s_{i_r}$ be a fixed reduced decomposition of $w \in W$ and let $\overline{X}(w)$ be the associated Demazure compactification of $X(w)$. This variety is equipped with a compatible action of G . We consider the closed complement of $X(w)$ in $\overline{X}(w)$ which is - as already observed in [DL] - a union of smooth equivariant divisors. We analyse the resulting spectral sequence converging to the cohomology of $X(w)$. The crucial point is that the intersection of these divisors is again a compactification of a DL-variety

attached to some subexpression of $s_{i_1} \cdots s_{i_r} \in F^+$. Concretely the spectral sequence has the shape

$$E_1^{p,q} = \bigoplus_{v \preceq w, \ell(v) = \ell(w) - p} H^q(\overline{X}(v)) \implies H_c^{p+q}(X(w)).$$

Another feature is that if $w = sw's \in F^+$, then $\overline{X}(w)$ is a \mathbb{P}^1 -bundle over $\overline{X}(w's)$. This comes about from the fact that $\overline{X}(w)$ is paved by DL-varieties attached to squares of the special type as above. So by induction on the length of $w's$ we know the cohomology of the compactification $\overline{X}(w)$. Of course not every element w in F^+ has the shape $w = sw's$, but by using a result of [GKP], every element can be transformed into such an element by applying the usual Weyl group relations and a cyclic shift operator. We study henceforth the effect on the cohomology by these operations. The start of induction is given by elements of minimal length in their conjugacy classes, i.e. by Coxeter elements in Levi subgroups of G . This is one reason why we deal only with reductive groups of Dynkin type A_{n-1} . In this case the Demazure compactification of the standard Coxeter element can be considered as one of the Drinfeld halfspace $\Omega^n = \mathbb{P}^{n-1} \setminus \bigcup_{H/\mathbb{F}_q} H$ (complement of all \mathbb{F}_q -rational hyperplanes in the projective space of lines in $V = \mathbb{F}^n$), cf. [Dr], and may be realised as a sequence of blow ups as it comes up in the arithmetic theory of Ω^n over a local field [Ge, GK, I].

Theorem. *Let $G = \mathrm{GL}_n$.*

i) For all $w \in F^+$, we have $H^i(\overline{X}(w)) = 0$ for odd i .

ii) Let $w = sw's$ with $s \in S$ and $w' \in F^+$. Then there are decompositions

$$H^i(\overline{X}(w)) = H^i(\overline{X}(w's)) \oplus H^{i-2}(\overline{X}(w's))(-1) = H^i(\overline{X}(sw')) \oplus H^{i-2}(\overline{X}(sw'))(-1).$$

iii) The action of the Frobenius on $H^i(\overline{X}(w))$ and $H_c^i(X(w))$ is semi-simple for all $w \in F^+$ and for all $i \geq 0$.

iv) For all $i \geq 0$, the cycle map $A^i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^{2i}(\overline{X}(w))$ is an isomorphism (where $A^i(\overline{X}(w))$ is the Chow group of $\overline{X}(w)$ in degree i).

Whereas part ii) of this theorem is already contained more generally for reductive groups (and also for elements of the completed braid monoid) in [DMR, Prop. 3.2.3], it was pointed out to me that part i) and iii) and iv) can be deduced from [L4]. Our proofs differ from loc.cit.

It turns out that the cohomology of the varieties $\overline{X}(w)$ is similar to the classical situation of Schubert varieties. Indeed, let \prec be the Bruhat order on F^+ . For a parabolic subgroup $P \subset G$, let $i_P^G = \mathrm{Ind}_P^G(\overline{\mathbb{Q}}_\ell)$ be the induced representation of the trivial one. By part ii) of the above theorem and by studying the effect of the usual Weyl group relations on the

Demazure compactifications of DL-varieties we are able to deduce the next statement. For its proof and also the also the theorem hereafter, we use the identification of the cohomology with Chow groups and consider explicit cycle classes.

Theorem. *Let $w \in F^+$. Then the cohomology of $\overline{X}(w)$ can be written as*

$$H^*(\overline{X}(w)) = \bigoplus_{z \preceq w} i_{P_z^w}^G(-\ell(z))[-2\ell(z)]$$

for certain std parabolic subgroups $P_z^w \subset G$.

The gradings are not canonical as there are in general plenty of choices. However, they behave functorial for appropriate choices as the following theorem says.

Theorem. *Let $w, v \in F^+$ with $v \prec w$. Then there are gradings $H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(z)=i}} i_{P_z^w}^G$ and $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} i_{P_z^v}^G$ such that the natural homomorphism $H^{2i}(\overline{X}(w)) \rightarrow H^{2i}(\overline{X}(v))$ is graded. Moreover, the maps $i_{P_z^w}^G \rightarrow i_{P_z^v}^G$ (which are induced by the double cosets of 1 in $W_{P_z^w} \setminus W/W_{P_z^v}$ via Frobenius reciprocity) are either injective or surjective for all $z \preceq v$.*

In a next step we analyse the above spectral sequence attached to $\overline{X}(w)$ and its divisors. By weight reasons the spectral sequence degenerates in E_1 and we believe that it can be evaluated via the following approach.

Conjecture. *Let $w \in F^+$ and fix an integer $i \geq 0$. For $v \preceq w$, there are gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} i_{P_z^v}^G$ such that the complex*

$$E_1^{\bullet, 2i} : H^{2i}(\overline{X}(w)) \rightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \rightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-2}} H^{2i}(\overline{X}(v)) \rightarrow \dots \rightarrow H^{2i}(\overline{X}(e))$$

is quasi-isomorphic to a direct sum $\bigoplus_{\substack{z \preceq w \\ \ell(z)=i}} H(\cdot)_z$ of complexes of the shape

$$H(\cdot)_z : i_{P_z^w}^G \rightarrow \bigoplus_{\substack{z \preceq v \prec w \\ \ell(v)=\ell(w)-1}} i_{P_z^v}^G \rightarrow \bigoplus_{\substack{z \preceq v \prec w \\ \ell(v)=\ell(w)-2}} i_{P_z^v}^G \rightarrow \dots \rightarrow i_{P_z^e}^G$$

(Here the maps $i_{P_z^w}^G \rightarrow i_{P_z^v}^G$ in the complex are induced - up to sign - by the double cosets of 1 in $W_{P_z^w} \setminus W/W_{P_z^v}$ via Frobenius reciprocity. Further $i_{P_z^v}^G = (0)$ if $z \not\preceq v$).

As a consequence each element z which is smaller than w would induce by taking the cohomology of the complex $H(\cdot)_z$ a contribution with Tate twist $-\ell(z)$ to the cohomology of $X(w)$. A similar statement holds true for period domains over finite fields [DOR]. We are able to prove the conjecture in some cases. Additionally, we prove how to reduce the issue to the case where w is again of the form $w = sw's$.

Theorem. *i) The conjecture is true for Coxeter elements w .*

ii) If $w \in F^+$ is arbitrary, then the conjecture is true for $i \in \{0, 1, \ell(w) - 1, \ell(w)\}$.

Of course the cases $i = 0$ and $i = \ell(w)$ are trivial. As a consequence we derive an inductive formula for the Tate twist -1 -contribution $H_c^*(X(w))\langle -1 \rangle$ of the cohomology of a DL-variety $X(w)$. For a parabolic subgroup P of G , let v_P^G be the corresponding generalized Steinberg representation.

Corollary. *Let $w = sw's \in F^+$ with $\text{ht}(sw') \geq 1$. Then*

$$H_c^*(X(w))\langle -1 \rangle = \begin{cases} H_c^*(X(sw'))\langle -1 \rangle[-1] & \text{if } s \in \text{supp}(w') \\ H_c^*(X(sw'))\langle -1 \rangle[-1] \oplus v_{P(s)}^G(-1)[- \ell(w)] & \text{if } s \notin \text{supp}(w') \end{cases}.$$

Here $\text{supp}(w')$ denotes the set of simple reflections appearing in w' whereas $P(s)$ is the parabolic subgroup of G generated by B and s . Moreover, ht is the height function on F^+ (cf. Definition 5.3 for its definition). It has the property that $\text{ht}(w) = \text{ht}(w') + 1$ if $w = sw's$ as above. Further $\text{ht}(w) = 0$ if w is minimal length in its conjugacy class. The start of the inductive formula is hence given by height one elements. Here we are even able to determine all cohomology groups. For a partition λ of n , let j_λ be the corresponding irreducible G -representation.

Theorem. *Let $w = sw's \in W$ with $\text{ht}(w) = 1$ and assume that $\text{supp}(w) = S$.*

i) If $\text{ht}(sw') = 0$, then for $i \in \mathbb{N}$, with $\ell(w) < i < 2\ell(w) - 1$,

$$H_c^i(X(w')) = (H_c^{i-2}(X(w')) - j_{(i+1-n, 1, \dots, 1)}(n-i))(-1) - j_{(i+2-n, 1, \dots, 1)}(n-i-1).$$

Furthermore,

$$H_c^i(X(w)) = \begin{cases} v_B^G \oplus (v_{P(s)}^G - j_{(2, 1, \dots, 1)})(-1) & ; \quad i = \ell(w) \\ 0 & ; \quad i = 2\ell(w) - 1 \\ i_G^G(-\ell(w)) & ; \quad i = 2\ell(w) \end{cases}.$$

ii) If $\text{ht}(sw') = 1$, then

$$H_c^i(X(w)) = H_c^{i-2}(X(w's) \cup X(w'))(-1) \oplus H_c^{i-1}(X(sw'))$$

for all $i \neq 2\ell(w) - 1, 2\ell(w) - 2$ and $H_c^{2\ell(w)-1}(X(w)) = H_c^{2\ell(w)-2}(X(w)) = 0$.

Moreover, we give an inductive recipe for the cohomology in degree $2\ell(w) - 2$ of a DL-variety $X(w)$. Here the case of $\text{ht}(w') = 0$ is treated by the above results.

Corollary. *Let $w = sw's \in F^+$ with $\text{ht}(w') \geq 1$ and assume that $\text{supp}(w) = S$. Then*

$$H_c^{2\ell(w)-2}(X(w)) = H_c^{2\ell(w's)-2}(X(w's))(-1) \oplus (i_{P(w')}^G - i_G^G)(-\ell(w) + 1).$$

Here $P(w')$ is the parabolic subgroup of G which is generated by B and $\text{supp}(w') \subset S$.

The proof of the two last formulas is based on the second approach for determining the cohomology of DL-varieties. This alternative approach is pursued for arbitrary elements of the Weyl group in the appendix. Whereas the previous approach uses Demazure resolutions, i.e., DL-varieties attached to maximal hypersquares, this time the procedure goes the other way round in the sense that the considered hypersquare grows. In fact, for determining the cohomology of $X(w)$, we study first the map

$$H_c^i(X(w) \cup X(sw')) \longrightarrow H_c^i(X(sw'))$$

induced by the closed embedding $X(sw') \hookrightarrow X(w) \cup X(sw')$. By induction on the length the cohomology of the RHS is known. Further we give a conjecture on the structure of this map. Hence it suffices to know the cohomology of the edge $X(w) \cup X(sw')$ which is - as explained above - induced by the cohomology of the edge $X(w's) \cup X(w')$. Thus we have transferred the question of determining the cohomology of the vertex $X(w)$ to the knowledge of the cohomology of the edge $X(w's) \cup X(w')$, but which has smaller length, i.e. $\ell(w's) < \ell(w)$. In the next step one reduces similar to the case of an edge to the case of a square etc. This second approach is a little bit vague as it depends among other things on some more conjectures. Nevertheless, I have decided to include it into this paper for natural reasons.

In Section 2 we review some facts on unipotent representations of $\text{GL}_n(\mathbb{F}_q)$. In Section 3 we consider DL-varieties and study explicitly the case of a Coxeter element. In Section 4 we deal with squares and their associated DL-varieties. Here we treat in particular the case of the special square $Q = \{sw's, w's, sw', w'\}$ and prove that the map $X(Q) \longrightarrow X(w's, w')$ is a \mathbb{P}^1 -bundle. In section 5 we determine the cohomology of DL-varieties for $G = \text{GL}_4$ and in general for height 1 elements in W . In Section 6 we generalize the ideas of the foregoing section to hypersquares. Section 7 deals with the cohomology of Demazure Varieties. In Section 8 we reconsider the spectral sequence and discuss the conjecture mentioned above. Finally in Section 9 we illustrate the Conjecture resp. the Theorem in the case of $G = \text{GL}_4$.

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Notation:

- Let $k = \mathbb{F}_q$ be a finite field of cardinality q with fixed algebraic closure \bar{k} and absolute Galois group $\Gamma := \text{Gal}(\bar{k}/k)$.

- We denote for any Γ -module V and any integer i , by $V\langle i \rangle$ the eigenspace of the arithmetic Frobenius with eigenvalues of absolute value q^i .

- Let $\mathbf{G}_0 = GL_n$ be the general linear group over k . Denote by $\mathbf{G} = \mathbf{G}_0 \times_{\mathbb{F}_q} \mathbb{F}$ the base change to the algebraic closure. Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ be the diagonal torus resp. the Borel subgroup of upper triangular matrices. Let $W \cong S_n$ be the Weyl group of \mathbf{G} and S be the subset of simple reflections. For a subset $I \subset S$, we denote as usual by W_I be the subgroup of W generated by I .

- We also use the cyclic notation for elements in the symmetric group. Hence the expression $w = (i_1, i_2, \dots, i_r)$ denotes the permutation with $w(i_j) = i_{j+1}$ for $j = 1, \dots, r-1$, $w(i_r) = i_1$ and $w(i) = i$ for all $i \notin \{i_1, \dots, i_r\}$.

- For a vector space V of dimension n over k and any integer $1 \leq i \leq n$, we let $\text{Gr}_i(V)$ be the Grassmannian parametrizing subspaces of dimension i .

- We denote by 1 or e the identity in any group or monoid.

- If a representation V admits a decomposition $V = \bigoplus_{i \in I} V_i$ into subrepresentations, then we simply call it a grading instead of I -grading.

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2. UNIPOTENT REPRESENTATIONS OF GL_n

We start with a discussion on representations of $G = \mathbf{GL}_n(k)$ which are called unipotent. These kind of objects appear in the cohomology of Deligne-Lusztig varieties. In this section, all representations will be in vector spaces over a fixed algebraically closed field C of characteristic zero.

Recall that a standard parabolic subgroup (std psgp) $\mathbf{P} \subset \mathbf{G}$ is a parabolic subgroup of \mathbf{G} with $\mathbf{B} \subset \mathbf{P}$. The set of all std psgp is in bijection with the set

$$\mathcal{D} = \mathcal{D}(n) = \{(n_1, \dots, n_r) \in \mathbb{N}^r \mid n_1 + \dots + n_r = n, r \in \mathbb{N}\}$$

of decompositions of n . For a decomposition $d = (n_1, \dots, n_r) \in \mathcal{D}(n)$, we let \mathbf{P}_d be the corresponding std psgp with Levi subgroup $\mathbf{M}_{\mathbf{P}_d} = \prod_i \mathbf{GL}_{n_i}$. There is a partial order \leq on $\mathcal{D}(n)$ defined by

$$d' \leq d \text{ if and only if } \mathbf{P}_{d'} \subset \mathbf{P}_d.$$

If $\mathbf{P} = \mathbf{P}_d$ is a std psgp to a decomposition $d \in \mathcal{D}$, then any $d' \in \mathcal{D}$ induces a std psgp $\mathbf{Q}_{d'} = \mathbf{M}_{\mathbf{P}} \cap \mathbf{P}_{d'}$ of $\mathbf{M}_{\mathbf{P}}$ and this assignment gives a bijection between the sets

$$\{d' \in \mathcal{D} \mid d' \leq d\} \xrightarrow{\sim} \{\text{std psgps of } \mathbf{M}_{\mathbf{P}}\}.$$

In the sequel we call the finite group $P = \mathbf{P}(k)$ attached to a std psgp \mathbf{P} of \mathbf{G} standard parabolic of G , as well.

Recall that a parabolic subgroup of W is by definition the Weyl group of the Levi component $\mathbf{M}_{\mathbf{P}}$ of some std psgp \mathbf{P} . This defines a one-to-one correspondence between the std psgp \mathbf{P} of \mathbf{G} and the parabolic subgroups W_P of W . If $\mathbf{P} = \mathbf{P}_d$ with $d = (n_1, \dots, n_r) \in \mathcal{D}$, then $W_P \cong S_{n_1} \times \dots \times S_{n_r}$.

We denote by $\mathcal{P} = \mathcal{P}(n)$ the set of partitions of n . There is a map $\lambda : \mathcal{D} \rightarrow \mathcal{P}$ by ordering a decomposition (n_1, \dots, n_r) in decreasing size. Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ and $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_n)$ be two partitions of n . Consider the order \leq on \mathcal{P} defined by $\mu' \leq \mu$

if for all $r = 1, \dots, n$, we have

$$\sum_{i=1}^r \mu'_i \leq \sum_{i=1}^r \mu_i.$$

Then the map λ is compatible with both orders in the sense that if $d' \leq d$ then $\lambda(d') \leq \lambda(d)$. If $P = P_d$ is a std psgp, then we also write $\lambda(P)$ for $\lambda(d)$. On the other hand, two parabolic subgroups are called associate, if their Levi components are conjugate under G . If $P = P_{d_1}$ and $P' = P_{d_2}$ with associated decompositions $d_1 = (n_1, \dots, n_r)$ and $d_2 = (n'_1, \dots, n'_{r'})$ of n , then P and P' are associate if and only if $\lambda(d_1) = \lambda(d_2)$.

Let P be a std psgp of G . We denote by

$$i_P^G = \text{Ind}_P^G \mathbf{1}$$

the induced representation of the trivial representation $\mathbf{1}$ of P . It coincides with the set $C[G/P]$ of C -valued functions on G/P equipped with the natural action. Let $\hat{G}(i_B^G)$ be the set of isomorphism classes of irreducible subobjects of i_B^G . We remind the reader of the following properties of the representations i_P^G , cf. [DOR, Thm. 3.2.1].

Theorem 2.1. (i) i_P^G is equivalent to $i_{P'}^G$ if and only if P is associate to P' .

(ii) i_P^G contains a unique irreducible subrepresentation j_P^G which occurs with multiplicity one and such that

$$\text{Hom}_G(j_P^G, i_{P'}^G) \neq (0) \Leftrightarrow \lambda(P') \leq \lambda(P).$$

(iii) We set for every $\mu \in \mathcal{P}$,

$$j_\mu = j_{P_\mu}^G$$

where P_μ is any std psgp with $\lambda(P_\mu) = \mu$. Then $\{j_\mu \mid \mu \in \mathcal{P}\}$ is a set of representatives for $\hat{G}(i_B^G)$.

Remarks 2.2. i) The proof of the above theorem makes use of the representation theory of the symmetric group and is based on the following result of Howe [Ho]. Let \hat{W} be the set of isomorphism classes of irreducible representations of W . Analogously to the definition of i_P^G , we set $i_{W'}^W := \text{Ind}_{W'}^W \mathbf{1}$ for any subgroup W' of W . There exists a unique bijection

$$\alpha : \hat{G}(i_B^G) \longrightarrow \hat{W}$$

characterized by the following property. An irreducible representation $\sigma \in \hat{G}(i_B^G)$ occurs in i_P^G if and only if $\alpha(\sigma)$ occurs in $i_{W_P}^W$. Furthermore

$$\dim \text{Hom}_G(\sigma, i_P^G) = \dim \text{Hom}_W(\alpha(\sigma), i_{W_P}^W).$$

In particular, we get by Frobenius reciprocity

$$\text{Hom}_G(i_Q^G, i_P^G) \cong \text{Hom}_W(i_{W_Q}^W, i_{W_P}^W) = \text{Hom}_{W_Q}(\mathbf{1}, i_{W_P}^W) = C[W_Q \backslash W / W_P].$$

ii) Let $P \subset Q$ be two standard parabolic subgroups. Then there is a natural inclusion of G -representations $i_Q^G \subset i_P^G$ which corresponds just to the double coset of $e \in W$ in $W_Q \backslash W / W_P$. On the other hand, the map $i_P^G \rightarrow i_Q^G$ induced by e is given by $\delta_{gP} \mapsto \sum_{gP \subset gQ} \delta_{gQ}$ where $\delta_{gP} \in C[G/P]$ is the Kronecker function with respect to $gP \in G/P$. In general, for two arbitrary standard parabolic subgroups P, Q of G , the map $i_P^G \rightarrow i_Q^G$ induced by $e \in W$ is injective (surjective) if and only if $\lambda(P) \geq \lambda(Q)$ ($\lambda(P) \leq \lambda(Q)$). In fact, this property which is a refinement of Theorem 2.1 ii) was observed by Liebler and Vitale [LV] and reproved by Hazewinkel and Vorst [HV] for the symmetric group. In particular, one can speak of the maximum $\max\{i_P^G, i_Q^G\}$ of these representations in this sense.

Corollary 2.3. *Let V and W be two finite-dimensional isomorphic G -representations which have decompositions $V = \bigoplus_{i=1}^s i_{P_i}^G$, $W = \bigoplus_{i=1}^t i_{Q_i}^G$ into induced representations. Then $s = t$ and after a possible permutation the parabolic subgroups P_i and Q_i are associate for all $i = 1, \dots, s$.*

Proof. The trivial G -representation $i_G^G = \mathbf{1}$ appears with multiplicity one in each induced representation i_P^G . Hence $s = t$.

The remaining proof is by induction on s . Let $s = 1$. Then the claim follows by the theorem above. Let $s > 1$. Then again we use this theorem to deduce that there are associate parabolic subgroups P_i, Q_j in these decompositions. By dividing out the summand $i_{P_i}^G \cong i_{Q_j}^G$ on both sides, respectively, we get two decompositions of isomorphic representations of the type above. Now the claim follows by induction on s . \square

Definition 2.4. The *generalized Steinberg representation* associated to a std psgp $P = P_d$ is the quotient

$$v_P^G = i_P^G / \sum_{Q \supsetneq P} i_Q^G = i_{P_d}^G / \sum_{d' > d} i_{P_{d'}}^G.$$

For $P = G$, we have $v_G^G = \mathbf{1}$ whereas if $B = P$ then we get the ordinary Steinberg representation v_B^G which is irreducible. In general, the generalized Steinberg representations v_P^G are not irreducible. More precisely, we have the following criterion.

Proposition 2.5. *Let $d \in \mathcal{D}$. Then $v_{P_d}^G$ is irreducible if and only if $d = (k, 1, \dots, 1)$ for some k with $1 \leq k \leq n$.*

Proof. By Remark 2.2 it suffices to show that the corresponding claim is correct for the attached Weyl group representation $v_{W_d}^W := i_{W_d}^W / \sum_{d' > d} i_{W_{d'}}^W$.

First let $d = (k, 1, \dots, 1)$ for some k with $1 \leq k \leq n$. By induction hypothesis the $n - 1$ -tuple $\tilde{d} = (k, 1, \dots, 1) \in \mathbb{Z}^{n-1}$ which is induced by deleting the last entry gives rise

to an irreducible S_{n-1} -representation V . By Pieri's formula [FH] we have a decomposition $\text{Ind}_{S_{n-1}}^{S_n}(V) = \sum_{\lambda} V_{\lambda}$ where $\lambda \in \mathcal{P}$ ranges over all partitions obtained by adding the integer 1 to the $n-1$ -tuple \tilde{d} . Hence for $k \geq 2$, there are exactly 3 irreducible representations appearing in this sum. Otherwise there are 2 of them. In any case the irreducible representation j_d is one of them. On the other hand, we have $\text{Ind}_{S_{n-1}}^{S_n}(V) = i_{W_d}^W / \sum_{d' \neq (k, 1, \dots, 1, 2)}^{d' > d} i_{W_{d'}}^W$. The missing contribution $i_{W^{(k, 1, \dots, 1, 2)}}^W$ covers the remaining irreducible representations. Thus $v_{W_d}^W$ and therefore $v_{P_d}^G = j_d$ are irreducible.

Suppose now that d is not of the shape above. We shall see that the representation $v_{W_d}^W$ is reducible. In a first step we may assume that d is a partition since $v_{W_{\lambda(d)}}^W \subset v_{W_d}^W$. Then we consider the partition $(d_1 + 1, d_2 - 1, d_3, \dots, d_n)$. By Remark 2.2 the irreducible representation attached to $(d_1 + 1, d_2 - 1, d_3, \dots, d_n)$ appears in $v_{W_d}^W$. Hence the latter one is not irreducible. \square

Let us recall some further properties of the representations i_P^G . Let $P = P_d \subset G$ for some $d \in \mathcal{D}$. Let $d' \leq d$ and consider the std psgp $Q_{d'} \subset M_P$ of M_P . We consider the induced representation $i_{Q_{d'}}^{M_P}$ as a P -module via the trivial action of the unipotent radical of P . Then

$$\text{Ind}_P^G(i_{Q_{d'}}^{M_P}) = i_{P_{d'}}^G.$$

Since Ind_P^G is an exact functor we get for any $d'' \in \mathcal{D}$ with $d' \leq d'' \leq d$, the identity $\text{Ind}_P^G(i_{Q_{d'}}^{M_P} / i_{Q_{d''}}^{M_P}) = i_{P_{d'}}^G / i_{P_{d''}}^G$. In particular, we conclude that

$$\text{Ind}_P^G(v_{Q_{d'}}^{M_P}) = i_{P_{d'}}^G / \sum_{\{d'' \in \mathcal{D} | d' < d'' \leq d\}} i_{P_{d''}}^G.$$

Consider the special situation where $d = (n_1, n_2) \in \mathcal{D}$. Then $P = P_{(n_1, n_2)}$ and $M_P = M_1 \times M_2$ with $M_1 = \text{GL}_{n_1}$, $M_2 = \text{GL}_{n_2}$. Let for $i = 1, 2$, $d_i \in \mathcal{D}(n_i)$ be a decomposition of n_i and consider the corresponding std psgps P_{d_i} of M_i . Denote by $(d_1, d_2) \in \mathcal{D}(n)$ the glued decomposition of n .

Lemma 2.6. *We have the identity*

$$(2.1) \quad \text{Ind}_{P_{(n_1, n_2)}}^G(v_{P_{d_1}}^{M_1} \boxtimes v_{P_{d_2}}^{M_2}) = i_{P_{(d_1, d_2)}}^G / \sum_{d'_1 > d_1} i_{P_{(d'_1, d_2)}}^G + \sum_{d'_2 > d_2} i_{P_{(d_1, d'_2)}}^G.$$

Proof. Since

$$i_{P_{d_1}}^{M_1} \boxtimes i_{P_{d_2}}^{M_2} = i_{P_{d_1} \times P_{d_2}}^{M_1 \times M_2}$$

we get

$$\text{Ind}_{P_{(n_1, n_2)}}^G(i_{P_{d_1}}^{M_1} \boxtimes i_{P_{d_2}}^{M_2}) = i_{P_{(d_1, d_2)}}^G.$$

Then the identity above follows by applying the exact functor $\text{Ind}_{P_{(n_1, n_2)}}^G$ to the exact sequence

$$0 \longrightarrow \sum_{d'_1 > d_1} i_{P_{d'_1}}^{M_1} \boxtimes i_{P_{d_2}}^{M_2} + \sum_{d'_2 > d_2} i_{P_{d_1}}^{M_1} \boxtimes i_{P_{d'_2}}^{M_2} \longrightarrow i_{P_{d_1}}^{M_1} \boxtimes i_{P_{d_2}}^{M_2} \longrightarrow v_{P_{d_1}}^{M_1} \boxtimes v_{P_{d_2}}^{M_2} \longrightarrow 0.$$

□

For the next property of generalized Steinberg representations, we refer to [Le] (resp. to [DOR, Thm. 3.2.5] for a detailed discussion on this complex). Here we set for any decomposition $d = (n_1, \dots, n_r) \in \mathcal{D}$, $r(d) = r$.

Proposition 2.7. *Let $P = P_d$, where $d \in \mathcal{D}$. Then there is an acyclic resolution of v_P^G by G -modules,*

$$(2.2) \quad 0 \rightarrow i_G^G \rightarrow \bigoplus_{\substack{d' \geq d \\ r(d')=2}} i_{P_{d'}}^G \rightarrow \bigoplus_{\substack{d' \geq d \\ r(d')=3}} i_{P_{d'}}^G \rightarrow \cdots \rightarrow \bigoplus_{\substack{d' \geq d \\ r(d')=r(d)-1}} i_{P_{d'}}^G \rightarrow i_{P_d}^G \rightarrow v_P^G \rightarrow 0.$$

□

Remark 2.8. The prove of Proposition 2.7 relies on some simplicial arguments as follows, cf. loc.cit. Let $d, d' \in \mathcal{D}$. Then we have

$$(2.3) \quad i_{P_d}^G \cap i_{P_{d'}}^G = i_{P_{d \vee d'}}^G$$

for some $d \vee d' \in \mathcal{D}$. Further for all $d_1, \dots, d_r \in \mathcal{D}$, we have

$$(2.4) \quad i_{P_d}^G \cap (i_{P_{d_1}}^G + i_{P_{d_2}}^G + \cdots + i_{P_{d_r}}^G) = (i_{P_d}^G \cap i_{P_{d_1}}^G) + \cdots + (i_{P_d}^G \cap i_{P_{d_r}}^G).$$

We reinterpret the complex (2.2) as follows. Let F^+ be the monoid which is freely generated by the subset $S \subset W$ of simple reflections. Denote by

$$(2.5) \quad \gamma : F^+ \longrightarrow W$$

the natural map. For $w = s_{i_1} \cdots s_{i_r} \in F^+$, let $\ell(w) := r$ be the length and

$$\text{supp}(w) := \{s_{i_1}, \dots, s_{i_r}\}$$

its support. Any subword v which is induced by erasing factors in w gives by definition rise to an element which is shorter with respect to the Bruhat ordering \preceq on F^+ , cf. also [Hu]. Note that this ordering is not compatible with the usual one \leq on W via γ .

For $w \in F^+$, let $I(w) \subset S$ be a minimal subset such that w is contained in the submonoid generated by $I(w)$. Let

$$(2.6) \quad P(w) = P_{I(w)} \subset G$$

be the std parabolic subgroup generated by B and $I(w)$. Alternatively, let $d(w) \in \mathcal{D}$ be the decomposition which corresponds to the subset $\text{supp}(w) \subset S$ under the natural bijection $\mathcal{D} \xrightarrow{\sim} S$. Then $P(w) = P_{d(w)}$.

We may define for $w \in F^+$ the following complex where the differentials are defined similar as above:

$$(2.7) \quad C_w^\bullet : 0 \rightarrow i_{P(w)}^G \rightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v) = \ell(w) - 1}} i_{P(v)}^G \rightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v) = \ell(w) - 2}} i_{P(v)}^G \rightarrow \cdots \rightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v) = 1}} i_{P(v)}^G \rightarrow i_{P(e)}^G \rightarrow 0.$$

Example 2.9. Let $w = \text{Cox}_n = s_1 s_2 \dots s_{n-1} \in F^+$ be the standard Coxeter element. Then the complex C_w^\bullet coincides - up to augmentation in the Steinberg representation v_B^G - with the complex (2.2) where $d = d(e)$.

Definition 2.10. Let $w \in F^+$. Then we say that w has full support if $\text{supp}(w) = S$.

Proposition 2.11. *Let $w \in F^+$ have full support. Then the complex C_w^\bullet is quasi-isomorphic to C_{Cox}^\bullet .*

Proof. We may suppose that w is not a Coxeter element. Hence there exists a simple reflection $s \in S$ which appears at least twice in w . Write $w = w_1 s w' s w_2$ with $s \in S$ and $w_1, w_2, w' \in F^+$. Of course the subword $v = w_1 w' s w_2$ has full support, as well. Hence by induction on the length the complex C_v^\bullet is quasi-isomorphic to C_{Cox}^\bullet . On the other hand, for any subword $v_1 s v' s v_2$ of w , we have $P(v_1 s v' s v_2) = P(v_1 s v' s v_2)$. Hence the difference between the complexes C_w^\bullet and C_v^\bullet is a contractible complex. The result follows. \square

Let $w \in F^+$ and $s \in \text{supp}(w)$. Hence we may write $w = w_1 s w_2$ for some subwords $w_1, w_2 \in F^+$ of w . Then we also use the notation

$$w/s := w_1 w_2 \in F^+$$

for convenience. We may define analogously to (2.7) the complex

$$(2.8) \quad C_{w,s}^\bullet : 0 \rightarrow i_{P(w)}^G \rightarrow \bigoplus_{\substack{s \preceq v \preceq w \\ \ell(v) = \ell(w) - 1}} i_{P(v)}^G \rightarrow \bigoplus_{\substack{s \preceq v \preceq w \\ \ell(v) = \ell(w) - 2}} i_{P(v)}^G \rightarrow \cdots \rightarrow i_{P(s)}^G \rightarrow 0.$$

Example 2.12. Let $w = \text{Cox}_n \in F^+$ be the standard Coxeter element and $s \in S$. Then the complex $C_{w,s}^\bullet$ coincides - up to augmentation with respect to the generalized Steinberg representation - with the complex (2.2) where $d = d(s)$, i.e., $P = P(s)$.

With the same arguments as in Proposition 2.11 one proves the next statement.

Proposition 2.13. *Let $w \in F^+$ have full support and let $s \in \text{supp}(w)$. Then the complex $C_{w,s}^\bullet$ is quasi-isomorphic to $C_{\text{Cox},s}^\bullet$ if $s \notin \text{supp}(w/s)$. Otherwise, it is acyclic. \square*

More generally, let $w \in F^+$ and fix a subword $u \prec w$. For any $v \in F^+$ with $u \preceq v \preceq w$, let $P_v \subset G$ be a std psgp chosen inductively in the following way. Start with an arbitrary std psgp $P_u \subset G$. Let $u \prec v \preceq w$ and suppose that for all $u \preceq z \prec v$, the std psgp P_z are already defined. Set $\{z_1, \dots, z_r\} = \{u \preceq z \prec v \mid \ell(z) = \ell(v) - 1\}$. Then let $P_v \subset G$ be a std psgp such that $i_{P_v}^G = A \oplus B$ where $A \subset \bigcap_i i_{P_{z_i}}^G$ and B maps to zero under all the maps $i_{P_v}^G \rightarrow i_{P_{z_i}}^G$ induced by the double cosets of $e \in W$ in $W_{P_v} \setminus W/W_{P_{z_i}}$ via Frobenius reciprocity. Hence we get a sequence of G -representations

$$(2.9) \quad 0 \rightarrow i_{P_w}^G \rightarrow \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v) = \ell(w) - 1}} i_{P_v}^G \rightarrow \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v) = \ell(w) - 2}} i_{P_v}^G \rightarrow \cdots \rightarrow \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v) = \ell(u) + 1}} i_{P_v}^G \rightarrow i_{P_u}^G \rightarrow 0$$

which we equip analogously with the same signs as above. By the very construction of the sequence we derive the following fact.

Lemma 2.14. *The sequence (2.9) is a complex. \square*

Example 2.15. Let $\mathbf{G} = \text{GL}_4$ and $w, u \in F^+$ with $\ell(w) = \ell(u) + 2$. The sequence

$$i_{P_{(3,1)}}^G \longrightarrow i_{P_{(3,1)}}^G \oplus i_{P_{(2,2)}}^G \longrightarrow i_{P_{(3,1)}}^G$$

(with differentials as explained above) is a complex in the sense above, whereas

$$i_{P_{(2,2)}}^G \longrightarrow i_{P_{(2,2)}}^G \oplus i_{P_{(3,1)}}^G \longrightarrow i_{P_{(2,2)}}^G$$

is not.

For later use, we introduce the next definition.

Definition 2.16. Let V be a finite-dimensional G -representation. We denote by $\text{supp}(V)$ the set of isomorphism classes of irreducible subrepresentations which appear in V .

Let $f : V \rightarrow W$ be a homomorphism of G -representations. We get for each irreducible G -representation Z an induced map

$$f^Z : V^Z \rightarrow W^Z$$

of the Z -isotypic parts.

Definition 2.17. Let $f : V \rightarrow W$ be a homomorphism of G -representations.

i) We call f si-surjective resp. si-injective resp. si-bijective if the map f^Z is surjective resp. injective resp. bijective for all $Z \in \text{supp}(V) \cap \text{supp}(W)$.

ii) We say that f has si-full rang if the map f^Z has full rang for all $Z \in \text{supp}(V) \cap \text{supp}(W)$.

Remarks 2.18. i) The above definition makes of course sense for arbitrary groups. We will apply it in the upcoming sections in the case of $H := G \times \Gamma$. We shall see later on that the action of H on the considered geometric representations is semi-simple.

ii) Obviously, the homomorphism f has si-full rang if and only if the map f^Z is injective or surjective for all $Z \in \text{supp}(V) \cap \text{supp}(W)$.

We close this section with the following observation.

Lemma 2.19. Let $V^1 \xrightarrow{f_1} V^2 \xrightarrow{f_2} V^3 \xrightarrow{f_3} V^4$ be an exact sequence of H -modules with $\text{supp}(V^1) \cap \text{supp}(V^4) = \emptyset$.

a) If f_1 is si-surjective then f_2 is si-injective.

b) If f_3 is si-injective then f_2 is si-surjective,

c) f_2 has si-full rang.

Proof. a) Let $Z \in \text{supp}(V^2) \cap \text{supp}(V^3)$. If the map f_2^Z has a kernel it follows that $V \in \text{supp}(V^1)$. Since f_1 is si-surjective we deduce that f_2^Z is the zero map. Hence V_3^Z maps injectively into V^4 which implies $Z \in \text{supp}(V^4)$ which is a contradiction to the assumption.

b) Let $Z \in \text{supp}(V^2) \cap \text{supp}(V^3)$. If the map f_2^Z is not surjective it follows that $V \in \text{supp}(V^4)$. Since f_3 is si-injective we deduce that f_2^Z is the zero map which implies $Z \in \text{supp}(V^1)$ which is again a contradiction to the assumption.

c) This is obvious. □

3. DELIGNE-LUSZTIG VARIETIES

Let $X = X_{\mathbf{G}}$ be the set of all Borel subgroups of \mathbf{G} . This is a smooth projective algebraic variety homogeneous under \mathbf{G} . By the Bruhat lemma the set of orbits of \mathbf{G} on $X \times X$ can be identified with W . We denote by $\mathcal{O}(w)$ the orbit of $(B, wBw^{-1}) \subset X \times X$ and by $\overline{\mathcal{O}(w)} \subset X \times X$ its Zariski closure.

Let $F : X \rightarrow X$ be the Frobenius map over \mathbb{F}_q . The *Deligne-Lusztig variety associated to $w \in W$* is the locally closed subset of X given by

$$X(w) = X_{\mathbf{G}}(w) = \{x \in X \mid \text{inv}(x, F(x)) = w\}$$

where by definition $\text{inv}(x, F(x)) = w \Leftrightarrow (x, F(x)) \in \mathcal{O}(w)$. Denote by \leq the Bruhat order and by ℓ the length function on W . Then $X(w)$ is a smooth quasi-projective variety of dimension $\ell(w)$ defined over \mathbb{F}_q and which is equipped with an action of G , cf. [DL, 1.4]. We denote by $\overline{X(w)} \subset X$ its Zariski closure.

Before we proceed, let us recall some properties of the varieties $\mathcal{O}(w)$ we need in the sequel, cf. loc.cit. If $w = w_1 w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$ then

- (1) a) $(B, B') \in \mathcal{O}(w_1)$ and $(B', B'') \in \mathcal{O}(w_2)$ implies $(B, B'') \in \mathcal{O}(w)$
 b) If $(B, B'') \in \mathcal{O}(w)$, then there is a unique $B' \in X$ with $(B, B') \in \mathcal{O}(w_1)$ and $(B', B'') \in \mathcal{O}(w_2)$.

In other words, there is an isomorphism of schemes $\mathcal{O}(w_1) \times_X \mathcal{O}(w_2) \cong \mathcal{O}(w)$.

- (2) Let $w, w' \in W$. Then $\mathcal{O}(w') \subset \overline{\mathcal{O}(w)} \Leftrightarrow w' \leq w$ for the Bruhat order \leq on W .

As in the case of usual Schubert cells, there is by item (2) the following relation concerning the closures of DL-varieties. Let $w', w \in W$. Then

$$X(w') \subset \overline{X(w)} \Leftrightarrow w' \leq w.$$

It follows that we have a Schubert type stratification

$$(3.1) \quad \overline{X(w)} = \bigcup_{v \leq w} X(v).$$

In particular if $w' \leq w$ with $\ell(w) = \ell(w') + 1$, then $X(w) \cup X(w')$ is a locally closed subvariety of X which is moreover smooth since the dimensions of $X(w)$ and $X(w')$ differs by one.

Example 3.1. Let $G = \text{GL}_3$ and identify X with the full flag variety of $V = \mathbb{F}^3$. Then

$$\begin{aligned} X(1) &= X(\mathbb{F}_q) \\ X((1, 2)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^2 \text{ is } k\text{-rational}, F(V^1) \neq V^1\} \\ X((2, 3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^1 \text{ is } k\text{-rational}, F(V^2) \neq V^2\} \\ X((1, 2, 3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid F(V^1) \subset V^2, F(V^i) \neq V^i, i = 1, 2\} \\ X((1, 3, 2)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^1 \subset F(V^2), F(V^i) \neq V^i, i = 1, 2\} \\ X((1, 3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid F(V^1) \not\subset V^2, V^1 \not\subset F(V^2)\}. \end{aligned}$$

Let $S = \{s_1, \dots, s_{n-1}\} \subset W$ be the set of simple reflections. Recall that a Coxeter element $w \in W$ is given by any product of all $s \in S$ (with multiplicity one). In the sequel we denote by

$$\text{Cox}_n := s_1 \cdot s_2 \cdots s_{n-1} = (1, 2, \dots, n) \in W$$

the standard Coxeter element.

Example 3.2. Let $w = \text{Cox}_n$. Then $X(w)$ can be identified via the projection map $X \rightarrow \mathbb{P}^{n-1}$ with the Drinfeld space

$$\Omega(V) = \Omega^n = \mathbb{P}^{n-1} \setminus \bigcup_{H/\mathbb{F}_q} H$$

(complement of all \mathbb{F}_q -rational hyperplanes in the projective space of lines in $V = \mathbb{F}^n$), cf. [DL], §2. Its inverse is given by the map

$$\begin{aligned} \Omega(V) &\longrightarrow X(w) \\ x &\mapsto x \subset x + F(x) \subset x + F(x) + F^2(x) \subset \cdots \subset V \end{aligned}$$

For any Coxeter element w for GL_n , the corresponding DL-variety $X(w)$ is universally homeomorphic to Ω^n , cf. [L2], Prop. 1.10.

In the sequel we denote for any variety X defined over k , by $H_c^i(X) = H_c^i(X, \overline{\mathbb{Q}}_\ell)$ (resp. $H^i(X) = H^i(X, \overline{\mathbb{Q}}_\ell)$) the ℓ -adic cohomology with compact support (resp. the ℓ -adic cohomology) in degree i . For a Deligne-Lusztig variety $X(w)$, there is by functoriality an action of $H = G \times \Gamma$ on these cohomology groups.

Proposition 3.3. *Let $w = \text{Cox}_n$ be the standard Coxeter element. Then*

$$H_c^*(X(w)) = \bigoplus_{k=1}^n j_{(k,1,\dots,1)}(-k-1)[-n-1-(k-1)].$$

(Here for an integer $m \in \mathbb{Z}$, we denote as usual by (m) the Tate twist of degree m .)

Proof. See [L2, O] resp. [SS] in the case of a local field. □

The cohomology of the Zariski closure of $X(\text{Cox}_n)$ has the following description.

Proposition 3.4. *Let $w = \text{Cox}_n \in W$ be the standard Coxeter element. Then*

$$H^*(\overline{X(w)}) = \bigoplus_{v \leq w} J(v)(-\ell(v))[-2\ell(v)],$$

where $J(v) = J^G(v)$ is defined inductively as follows. Write v in the shape

$$v = v' \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}$$

where $1 \leq j \leq n-2$, $l \geq 0$ and where $v' \leq \text{Cox}_j = s_1 \cdots s_{j-1}$. Then

$$J(v) = \text{Ind}_{\mathbb{P}_{(j,n-j)}^G}^G (J^{\text{GL}_j}(v') \boxtimes \mathbf{1}).$$

Proof. In terms of flags the DL-variety $X(w)$ has the description

$$X(w) = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^j) \neq V^j, \forall 1 \leq j \leq n-1\}.$$

The Zariski closure $\overline{X(w)}$ of $X(w)$ in X is then given by the subset

$$\overline{X(w)} = \{V^\bullet \mid F(V^j) \subset V^{j+1}, \forall 1 \leq j \leq n-1\}$$

which can be identified with a sequence of blow-ups, cf. [Ge, GK, I]. Start with $Y_0 = \mathbb{P}^{n-1} = \mathbb{P}(V)$ where $V = \mathbb{F}^n$ and consider the blow up B_1 in the set of rational points $Z_0 = \mathbb{P}^{n-1}(k) \subset Y_0$. Then we may identify B_1 with the variety $\{V^1 \subset V^2 \subset V \mid F(V^1) \subset V^2\}$. We set $Y_1 = \bigcup_{W \in \text{Gr}_2(V)(k)} \mathbb{P}(W) \subset B_1$ which is the strict transform of the finite set of k -rational planes and blow up B_1 in Y_1 . The resulting variety B_2 can be identified with $\{V^1 \subset V^2 \subset V^3 \subset V \mid F(V^i) \subset V^{i+1}, i = 1, 2\}$. Now we repeat this construction successively until we get $B_{n-2} = \overline{X(w)}$. Hence the cohomology of $\overline{X(w)}$ can be deduced from the usual formula for blow ups [SGA5]. More precisely, each time we blow up, we have to add the cohomology of the variety

$$\coprod_{W \in \text{Gr}_j(V)(k)} (\overline{X_{\text{GL}(W)}(\text{Cox}_j)} \times \mathbb{P}(V/W)) \setminus (\overline{X_{\text{GL}(W)}(\text{Cox}_j)} \times \mathbb{P}^0)$$

which is

$$\bigoplus_{v' \leq \text{Cox}_j} \bigoplus_{l=1}^{n-j-1} \text{Ind}_{\mathbb{P}_{(j,n-j)}^G}^G (J^{\text{GL}_j}(v') \boxtimes \mathbf{1})(-\ell(v' \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}))[-2(\ell(v') + l)].$$

The start of this procedure is given by the cohomology of the projective space $\mathbb{P}(V)$, which we initialize by $H^{2i}(\mathbb{P}(V)) = J(s_1 s_2 \cdots s_i)(-i) = i_G^G(-i), i = 1, \dots, n-1$. \square

Remarks 3.5. i) Since all DL-varieties for Coxeter elements are homeomorphic [L2], they have all the same cohomology. By considering stratifications (3.1) for different Coxeter elements, the same is true for their Zariski closures. Alternatively, one might argue that the morphism σ, τ of the upcoming Proposition 4.3 for different Coxeter elements extend to their Zariski closures, thus inducing an isomorphism on their cohomology.

ii) It follows by the description of $\overline{X(\text{Cox}_n)}$ in terms of blow ups together with Remark 3.5 i) that the cycle map

$$A^i(\overline{X(w)})_{\mathbb{Q}_\ell} \longrightarrow H^{2i}(\overline{X(w)})$$

is an isomorphism for every Coxeter element w . In fact, in the Chow group the same formulas concerning blow ups are valid [SGA5]. The first Chow group of $\overline{X(\text{Cox}_n)}$ is also considered in [RTW].

iii) Let $w = s_{n-1}s_{n-2}\cdots s_2s_1$. Then by symmetry, there is by considering the dual projective space $(\mathbb{P}^{n-1})^\vee$ a procedure for realising

$$\overline{X(w)} = \{V^\bullet \mid V^i \subset F(V^{i+1}), i = 1, \dots, n-1\}$$

as a sequence of blow ups.

iv) Since $\overline{X(\text{Cox}_n)}$ is smooth and projective the above formula can be also deduced by considering its stratification into DL-varieties together with Propositions 3.3 and 3.6.

On the other hand, for elements $w \in W$ having not full support, the cohomology of $X(w)$ can be deduced by an induction process. Here recall that we say that w has full support if it is not contained in any proper parabolic subgroup W_P of W .

Proposition 3.6. *Let $w \in W$ have not full support. Let $\mathbf{P} = \mathbf{P}_{(i_1, \dots, i_r)}$ be a minimal parabolic subgroup such that $w \in W_P$. Then*

$$H_c^*(X(w)) = \text{Ind}_P^G (H_c^*(X_{\mathbf{M}_P}(w)))$$

and

$$H^*(\overline{X(w)}) = \text{Ind}_P^G (H^*(\overline{X_{\mathbf{M}_P}(w)}))$$

where $X_{\mathbf{M}_P}(w) = \prod_{j=1}^r X_{GL_{i_j}}(w_j)$ and $w = w_1 \cdots w_r$ with $w_j \in S_{i_j}, j = 1, \dots, r$.

Proof. See [DL, Prop. 8.2]. □

Remark 3.7. The proof in loc.cit. shows that there is an identification of varieties $X(w) = \text{Ind}_P^G (X_{\mathbf{M}_P}(w))$. Hence we have the same formulas for Chow groups. In particular, by using Remark 3.5 we see that the cycle map is an isomorphism for all Coxeter elements in Levi-subgroups, as well.

In the case when we erase some simple reflection in a reduced expression of a Coxeter element, we may deduce from the above results the following consequence. Here we abbreviate the partition or decomposition $(k, 1, \dots, 1) \in \mathcal{P}$ by $(k, 1^{(n-k)})$. For any triple of integers $i, k, l \in \mathbb{Z}$ with $1 \leq i \leq n$ and $1 \leq k \leq i, 1 \leq l \leq n-i$, we set

$$A_{k,l} = i_P^G_{(k, 1^{(i-k)}, l, 1^{(n-i-l)})} / \sum_{d_1 > (k, 1^{(i-k)})} i_P^G_{(d_1, l, 1^{(n-i-l)})} + \sum_{d_2 > (l, 1^{(n-i-l)})} i_P^G_{(k, 1^{(k-i)}, d_2)}.$$

Corollary 3.8. *Let $w = \text{Cox}_n = s_1 \cdots s_{n-1}$ be the standard Coxeter element and let¹ $w' = s_1 \cdots \hat{s}_i \cdots s_{n-1} \in W$. Then*

$$H_c^*(X(w')) = \bigoplus_{m=2}^n \bigoplus_{k+l=m} A_{k,l}(-(m-2))[-(n-2) - (m-2)].$$

Proof. The Weyl group element w' is contained in the parabolic subgroup W_P with $P = P_{(i,n-i)}$. Now the expressions $s_1 \cdots s_{i-1}$ and $s_{i+1} \cdots s_{n-1}$ are both Coxeter elements in the Weyl groups attached to $M_1 = \text{GL}_i$ resp. $M_2 = \text{GL}_{n-i}$. The cohomology of $H_c^*(X_{M_1}(w_1))$ resp. $H_c^*(X_{M_2}(w_2))$ are given by Proposition 3.3:

$$H_c^*(X_{M_1}(w_1)) = \bigoplus_{k=1, \dots, i} j_{(k,1, \dots, 1)}^{M_1}(-(k-1))[-(i-1) - (k-1)]$$

resp.

$$H_c^*(X_{M_2}(w_2)) = \bigoplus_{l=1, \dots, n-i} j_{(l,1, \dots, 1)}^{M_2}(-(l-1))[-((n-i)-1) - (l-1)].$$

Thus we get by Proposition 3.6 and identity (2.1)

$$\begin{aligned} H_c^*(X(w')) &= \text{Ind}_{P_{(i,n-i)}}^G \left(H_c^*(X_{M_1}(w_1)) \boxtimes H_c^*(X_{M_2}(w_2)) \right) \\ &= \bigoplus_{m=2}^n \bigoplus_{k+l=m} A_{k,l}(-(k+l-2))[-(n-2) - (k+l-2)]. \end{aligned}$$

□

Let $w = \text{Cox}_n \in W$. As we see from Proposition 3.4 the cohomology of $\overline{X}(w)$ vanishes in odd degree. For any integer $i \geq 0$, let

$$(3.2) \quad C^\bullet : H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v < w \\ \ell(v) = \ell(w) - 1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v < w \\ \ell(v) = 1}} H^{2i}(\overline{X}(v)) \longrightarrow H^{2i}(\overline{X}(e))$$

be the natural complex (with $C^0 = H^{2i}(\overline{X}(w))$) induced by the closed complement $\bigcup_{v < w} \overline{X}(v)$ of $X(w)$ in $\overline{X}(w)$. This complex determines by Propositions 3.4 and 3.6 the contribution with Tate twist $-i$ to the cohomology of $X(w)$ (compare also the lines following Lemma 7.1 for a more precise explanation of this issue), i.e.

$$H_c^*(X(w))\langle -i \rangle = \bigoplus_{j \geq 0} H^j(C^\bullet)[-2i - j].$$

On the other hand, we may consider the grading

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \leq w \\ \ell(z) = i}} G(w)_z$$

¹Here the symbol \hat{s}_i means as usual that s_i is deleted from the above expression.

described in Proposition 3.4, i.e. $G(w)_z = J(z)(-\ell(z))$ for $z < w$. By Proposition 3.6 we also have such a grading for all subexpressions $v < w$, i.e. $H^{2i}(\overline{X(v)}) = \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} G(v)_z$.

Lemma 3.9. *Let $v_1 < v_2 \leq w$ with $\ell(v_2) = \ell(v_1) + 1$. Then $G(v_2)_z \subset G(v_1)_z$ for all $z \leq v_1$.*

Proof. By Proposition 3.6 we may suppose that $w = v_2$ and $v = v_1 = s_1 \cdots \hat{s}_i \cdots s_{n-1}$. Let $z = z' \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}$ with $z' \leq \text{Cox}_j$ and $l \geq 1$. Then $G(w)_z = \text{Ind}_{P_{(j,n-j)}}^G (J^{\text{GL}_j}(z') \boxtimes \mathbf{1})$.

If $i < j$ then $z' = z'_1 z'_2$ with $z'_1 < \text{Cox}_i$ and $z'_2 | s_{i+1} \cdots s_{j-1}$. Thus we see that $G(v)_z = \text{Ind}_{P_{(i,n-i)}}^G (J^{\text{GL}_i \times \text{GL}_{n-i}}(z))$ with

$$\begin{aligned} J^{\text{GL}_i \times \text{GL}_{n-i}}(z) &= J^{\text{GL}_i}(z'_1) \boxtimes J^{\text{GL}_{n-i}}(z'_2 \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}) \\ &= J^{\text{GL}_i}(z'_1) \boxtimes \text{Ind}_{P_{(j-i,n-j)}}^{\text{GL}_{n-i}} (J^{\text{GL}_{j-i}}(z'_2) \boxtimes \mathbf{1}). \end{aligned}$$

Hence $G(v)_z = \text{Ind}_{P_{(i,j-i,n-j)}}^G (J^{\text{GL}_i}(z'_1) \boxtimes J^{\text{GL}_{j-i}}(z'_2) \boxtimes \mathbf{1})$. Since we have by induction on j the inclusion $J^{\text{GL}_j}(z') \subset \text{Ind}_{P_{(i,j-i)}}^{\text{GL}_j} (J^{\text{GL}_i}(z'_1) \boxtimes J^{\text{GL}_{j-i}}(z'_2))$ the claim follows in this case.

If $i = j$ then one deduces moreover by arguing in the same way that the inclusion is an identity, i.e., $G(w)_z = G(v)_z$ as $J^{\text{GL}_{n-i}}(s_{i+1} s_{i+2} \cdots s_{i+l})$ is the trivial representation.

If $i > j$ then we necessarily have $i > j + l$. Then $z = z_1 z_2$ with $z_1 \leq \text{Cox}_i$ and $z_2 = e$. Thus

$$\begin{aligned} G(v)_z &= \text{Ind}_{P_{(i,n-i)}}^G (J^{\text{GL}_i \times \text{GL}_{n-i}}(z)) \\ &= \text{Ind}_{P_{(i,n-i)}}^G (J^{\text{GL}_i}(z_1) \boxtimes J^{\text{GL}_{n-i}}(z_2)) \\ &= \text{Ind}_{P_{(i,n-i)}}^G (\text{Ind}_{P_{(j,i-j)}}^{\text{GL}_i} (J^{\text{GL}_j}(z') \boxtimes \mathbf{1}) \boxtimes i_{B \cap \text{GL}_{n-i}}^{\text{GL}_{n-i}}) \\ &= \text{Ind}_{P_{(j,i-j,n-i)}}^G (J^{\text{GL}_j}(z') \boxtimes \mathbf{1} \boxtimes i_{B \cap \text{GL}_{n-i}}^{\text{GL}_{n-i}}). \end{aligned}$$

Hence it contains $G(w)_z$. □

By Lemma 2.14 we get for each subword $z \leq w$, a complex

$$G(\cdot)_z : G(w)_z \longrightarrow \bigoplus_{\substack{z \leq v \leq w \\ \ell(v)=\ell(w)-1}} G(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{z \leq v \leq w \\ \ell(v)=1}} G(v)_z \longrightarrow \bigoplus_{\substack{z \leq e \\ \ell(z)=i}} G(e)_z.$$

Lemma 3.10. *If $z \notin \{s_1, s_1 s_2, \dots, \text{Cox}_n\}$ then this complex is acyclic. Otherwise, it coincides with the complex (2.2) with respect to the generalized Steinberg representation $v_{P(z)}^G$.*

Proof. Let $z = s_1 s_2 \cdots s_i$. Then the complex $G(\cdot)_z$ is easily verified by the above description just the complex (2.2) with respect to $v_{P(z)}^G$. So let $z \notin \{s_1, s_1 s_2, \dots, \text{Cox}_n\}$ and write $z = z' \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}$ with $z' \leq \text{Cox}_j$ and $l \geq 1$ as above. In particular, we have $j \geq 1$. By the proof of the foregoing lemma, we know that for $z \leq v \leq w$ with $s_j | v$ we

have $G(v)_z = G(v/s_j)_z$. By standard simplicial arguments it follows that the complex is contractible. \square

Hence by summing up we get a graded complex

$$(3.3) \quad \bigoplus_{\substack{z \leq w \\ \ell(z)=i}} G(w)_z \longrightarrow \bigoplus_{\substack{v \leq w \\ \ell(v)=\ell(w)-1}} \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} G(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v \leq w \\ \ell(v)=1}} \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} G(v)_z \longrightarrow \bigoplus_{\substack{z \leq e \\ \ell(z)=i}} G(e)_z$$

$$= \bigoplus_{\substack{z \leq w \\ \ell(z)=i}} \left(G(w)_z \longrightarrow \bigoplus_{\substack{z \leq v \leq w \\ \ell(v)=\ell(w)-1}} G(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{z \leq v \leq w \\ \ell(v)=1}} G(v)_z \longrightarrow \bigoplus_{\substack{z \leq e \\ \ell(z)=i}} G(e)_z \right).$$

Proposition 3.11. *The complexes (3.2) and (3.3) are quasi-isomorphic.*

Proof. We have a natural morphism of complexes (3.3) \rightarrow (3.2). The claim follows now from the foregoing lemma and Proposition 3.3. \square

More generally, we believe that the natural generalization holds true.

Conjecture 3.12. *Let $u < w$ and $i \in \mathbb{N}$. Then for all $v \in W$ with $u \leq v \leq w$, there are gradings*

$$H^{2i}(\overline{X(v)}) = \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} H(v)_z$$

into induced representations $H(v)_z = i_{P_z^v}^G$ for certain parabolic subgroups P_z^v (here the $H(v)_z$ do not necessarily coincide with the expressions $G(v)_z$ in (3.3)) such that the complex

$$(3.4) \quad H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=\ell(u)+1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(u)})$$

is quasi-isomorphic to a graded complex

$$(3.5) \quad \bigoplus_{\substack{z \leq w \\ \ell(z)=i}} H(w)_z \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=\ell(w)-1}} \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=1}} \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} H(v)_z \longrightarrow \bigoplus_{\substack{z \leq u \\ \ell(z)=i}} H(u)_z$$

$$= \bigoplus_{\substack{z \leq w \\ \ell(z)=i}} \left(H(w)_z \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=\ell(w)-1}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u \leq v \leq w \\ \ell(v)=1}} H(v)_z \longrightarrow \bigoplus_{\substack{z \leq u \\ \ell(z)=i}} H(u)_z \right).$$

We shall prove a more concrete version of the conjecture in the case where $\ell(u) = \ell(w) - 1$.

Proposition 3.13. *Let w be a Coxeter element and let $w' \in W$ with $w' < w$ and $\ell(w') = \ell(w) - 1$. There are gradings $H^{2j}(\overline{X(w)}) = \bigoplus_{\substack{z \leq w \\ \ell(z)=j}} H(w)_z$ and $H^{2j}(\overline{X(w')}) = \bigoplus_{\substack{z \leq w' \\ \ell(z)=j}} H(w')_z$ into induced representations such that the homomorphism $H^{2j}(\overline{X(w)}) \rightarrow H^{2j}(\overline{X(w')})$ coincides with the graded one. Moreover, the maps $H(w)_z \rightarrow H(w')_z$ are injective for all $z \leq w'$.*

As for the proof we note that by Remark 3.5 the cycle map $A^j(\overline{X(w)})_{\mathbb{Q}_\ell} \rightarrow H^{2j}(\overline{X(w)})$ is an isomorphism. The same holds true for the subvariety $\overline{X(w')}$. Moreover for the Chow groups, we have by the constructive proof an integral version of Proposition 3.4, i.e. the induced representations i_P^G appearing in loc.cit. are replaced by their integral models $i_P^G(\mathbb{Z}) = \{f : G \rightarrow \mathbb{Z} \mid f(gp) = f(g) \forall p \in P\}$. Thus by Poincaré duality, the above result follows from the following statement.

Proposition 3.14. *Let w be a Coxeter element and let $w' \in W$ with $w' < w$ and $\ell(w') = \ell(w) - 1$. There are gradings $A_j(\overline{X(w)}) = \bigoplus_{\substack{z \leq w \\ \ell(z)=j}} A(w)_z$ and $A_j(\overline{X(w')}) = \bigoplus_{\substack{z \leq w' \\ \ell(z)=j}} A(w')_z$ into induced representations $i_P^G(\mathbb{Z})$ such that the homomorphism $A_j(\overline{X(w')}) \rightarrow A_j(\overline{X(w)})$ coincides with the graded one. Moreover, the maps $A(w')_z \rightarrow A(w)_z$ are surjective for all $z \leq w'$.*

Proof. We may assume that $w = \text{Cox}_n$ is the standard Coxeter element. If $w' = s_2 \cdots s_{n-1} \in W$, the claim is a result of the proof of Proposition 3.4 following inductively the process of blow ups. Here we may consider on the cohomology groups $A_j(\overline{X(w)})$ and $A_j(\overline{X(w')}) = \text{Ind}_{P_{(1,n-1)}}^G (A_j(\overline{X_{\text{GL}_1 \times \text{GL}_{n-1}}(w')}))$ the natural gradings given by loc.cit.

If on the other extreme $w' = s_1 \cdots s_{n-2}$, then we identify $\overline{X(\text{Cox}_n)}$ with the variety $\overline{X(s_{n-1}s_{n-2} \cdots s_2s_1)}$ (cf. Remark 3.5 i)) and argue in the same way as above using the variant of Proposition 3.4 (cf. Remark 3.5 iii)) describing the latter space as a sequence of blow ups using hyperplanes. Here the gradings are induced by mirroring the Dynkin diagram and the induced representations.

In general, let $w' = s_1 \cdots \hat{s}_i \cdots s_{n-1}$. Here the reasoning is a kind of mixture of the previous cases. We consider the exact sequences [Fu, Prop. 6.7 e)]

$$0 \rightarrow A_i(Y_{n-3}) \rightarrow A_i(\tilde{Y}_{n-3}) \oplus A_i(B_{n-3}) \rightarrow A_i(\overline{X(w)}) \rightarrow 0$$

and

$$0 \rightarrow A_i(Y'_{n-3}) \rightarrow A_i(\tilde{Y}'_{n-3}) \oplus A_i(B'_{n-3}) \rightarrow A_i(\overline{X(w')}) \rightarrow 0$$

where $B'_{n-3} = B_{n-3} \cap \overline{X(w')}$, $Y'_{n-3} = Y_{n-3} \cap \overline{X(w')}$, $\tilde{Y}_{n-3} = \overline{X(s_1s_2 \cdots s_{n-3}s_{n-1})}$ is the preimage of Y_{n-3} in $\overline{X(w)}$ and $\tilde{Y}'_{n-3} = \tilde{Y}'_{n-3} \cap \overline{X(w')}$. By induction on n and the steps in the blow up process it suffices to show that the claim of the statement is true for the maps $i_* : A_j(Y_{m-1}) \rightarrow A_j(B_{m-1})$ with $m \neq 1, n-1$.

Recall that

$$B_{m-1} = \{(0) \subset V^1 \subset V^2 \subset \cdots \subset V^{m-1} \subset V^m \subset V \mid F(V^j) \subset V^{j+1}, j = 1, \dots, m-1\}$$

and

$$Y_{m-1} = \{V^\bullet \in B_{m-1} \mid F(V^m) = V^m\}.$$

We can consider the equivalent situation given by

$$B'_{m-1} = \{(0) \subset V^1 \subset V^2 \subset \dots \subset V^{m-1} \subset V^m \subset V \mid V^j \subset F(V^{j+1}), j = 1, \dots, m-1\}$$

and

$$Y'_{m-1} = \{V^\bullet \in B'_{m-1} \mid F(V^m) = V^m\}.$$

Similarly as above this situation is induced by successive blow ups starting with $B = \{(0) \subset V^{m-1} \subset V^m \subset V \mid V^{m-1} \subset F(V^m)\}$ and $Y = \{V^\bullet \in B \mid F(V^m) = V^m\}$, respectively. Hence the claim follows from the next lemma. \square

Lemma 3.15. *Let $m \geq 1$ and let $B = B_{(m)} = \{(0) \subset V^{m-1} \subset V^m \subset V \mid F(V^{m-1}) \subset V^m\}$.*

i) The cycle map induces for all $j \geq 0$, an isomorphism $A^j(B)_{\mathbb{Q}_\ell} \cong H^{2j}(B)$. Set $I = \{z \leq \text{Cox}_n \mid z = s_{k+1}s_{k+2}\dots s_{k+l} \text{ for } l \geq 1 \text{ and } k \leq m-1\}$. Then for $m \leq \frac{n}{2}$, we further have $A_j(B) = \bigoplus_{\substack{z \in I \\ \ell(z)=j}} A_B(z)$ with $A_B(z) \cong i_{P_{(k,n-k)}}^G(\mathbb{Z})$ for $z = s_{k+1}s_{k+2}\dots s_{k+l}$.

ii) Let $Y = \{V^\bullet \in B \mid F(V^m) = V^m\} \subset B$. Then there are gradings on $A_j(B)$ and $A_j(Y)$ such the induced homomorphism $A_j(Y) \rightarrow A_j(B)$ is in diagonal form.

Proof. The proof is by induction on m . By symmetry we may assume that $m \leq \frac{n}{2}$. If $m = 1$, then the first claim follows from the proof of Proposition 3.4. In general, we consider the diagram

$$\begin{array}{ccc} & \{(0) \subset V^{m-1} \subset V^m \subset V^{m+1} \subset V \mid F(V^k) \subset V^{k+1}, k = m-1, m\} & \\ \swarrow & & \searrow \\ B_{(m)} & & B_{(m+1)} \end{array}$$

where the maps are the projections. By induction we may suppose that the statement is true for $B_{(m)}$. All the appearing varieties are smooth and projective. Thus we get the desired formula in i) by first blowing up and then blowing down and using [Fu, Prop. 6.7].

Concerning the second statement, we have $Y = \coprod_{W \in \text{Gr}_m(V)(k)} \check{\mathbb{P}}(W)$. Hence $A_j(Y) = \text{Ind}_{P_{(m,n-m)}}^G(A_j(\check{\mathbb{P}}(W)))$ which is labeled by the (single) element $s_{m-j} \cdots s_{m-1} \leq \text{Cox}_n$ (which is e for $j = 0$) and which is identified with the $\ell(z)$ -dimensional cycle class

$$\bigoplus_{H \in \text{Gr}_m(V)(\mathbb{F}_q)} \{(0) \subset V^{m-1} \subset H \subset V \mid V^{m-1} \text{ contains an } m-j-1\text{-dim. rational subspace}\}.$$

A priori we identify (by the very construction) for $z = s_m \cdots s_{m+l}$ the $\ell(z)$ -dimensional cycle class $A_B(z)$ with

$$\bigoplus_{W \in \text{Gr}_{m-1}(V)(\mathbb{F}_q)} \{(0) \subset W \subset V^m \subset V \mid V^m \text{ is contained in an } m+l+1\text{-dim. rational subspace}\}.$$

We replace $A_B(z)$ by the cycle class

$$\bigoplus_{W \in \text{Gr}_{m-1}(V)(\mathbb{F}_q)} \sum_{\substack{H \in \text{Gr}_m(V)(\mathbb{F}_q) \\ W \subset H}} \{(0) \subset V^{m-1} \subset H \subset V \mid V^{m-1} \subset H \subset V \mid V^{m-1} \\ \text{contains an } m-j-1\text{-dim. rational subspace}\}$$

and relabel it with $s_{m-l-1} \dots s_{m-1}$. On the other hand, we replace the cycle class $A_B(z)$ where $z = s_{m-l-1} \dots s_{m-1}$ with

$$\bigoplus_{L \in \text{Gr}_{m-l-2}(V)(\mathbb{F}_q)} \sum_{\substack{W \in \text{Gr}_{m-1}(V)(\mathbb{F}_q) \\ L \subset W}} \{(0) \subset W \subset V^m \subset V \mid V^m \text{ is contained} \\ \text{in a } m+l+1\text{-dim. rational subspace}\}$$

and relabel it by $s_m \dots s_{m+l}$. Hence we get a new grading $A_j(B) = \bigoplus_{\substack{z \in I \\ \ell(z)=j}} A'_B(z)$. Then it is clear that the map i_* is in diagonal form and maps surjectively $A_Y(z)$ onto $A'_B(z)$ (since $i \leq \frac{n}{2}$) for all $z \leq \text{Cox}_{m-1}$. The claim follows. \square

Examples 3.16. i) Let $j = 1$. If $w' = s_2 \dots s_{n-1}$, then we have the natural gradings given by loc.cit., i.e.

$$H(w)_{s_1} = i_G^G(-1) \text{ and } H(w)_{s_{i+1}} = i_{P_{(i,n-i)}}^G(-1), \quad i \geq 1,$$

resp.

$$H(w')_{s_2} = i_{P_{(1,n-1)}}^G(-1) \text{ and } H(w')_{s_{i+1}} = i_{P_{(1,i-1,n-i)}}^G(-1), \quad i \geq 2.$$

If on the other extreme $w' = s_1 \dots s_{n-2}$, then we identify as explained above $\overline{X(\text{Cox}_n)}$ with $\overline{X(s_{n-1}s_{n-2} \dots s_2s_1)}$ and argue in the same way as above using the variant of Proposition 3.4 We set

$$H(w)_{s_{n-1}} = i_G^G(-1) \text{ and } H(w)_{s_i} = i_{P_{(i+1,n-i-1)}}^G(-1), \quad i < n-1,$$

resp.

$$H(w')_{s_{n-2}} = i_{P_{(n-1,1)}}^G(-1) \text{ and } H(w')_{s_i} = i_{P_{(i+1,n-i-2,1)}}^G(-1), \quad i < n-2.$$

Now let $w' = s_1 \dots \hat{s}_i \dots s_{n-1} \in W$ for some $1 < i < n-2$. On $H^2(\overline{X(w')})$ we consider the natural grading induced by Proposition 3.4, i.e., we set

$$H(w')_{s_j} = \begin{cases} i_{P_{(j-1,i-(j-1),n-i)}}^G(-1) & ; 1 < j < i \\ i_{P_{(i,n-i)}}^G(-1) & ; j = 1, j = i+1 \\ i_{P_{(i,j-(i+1),n-j+1)}}^G(-1) & ; i+1 < j < n \end{cases} .$$

As for w we set,

$$H(w)_{s_j} = \begin{cases} i_{P_{(j-1, n-j+1)}}^G(-1) & ; j \neq i, \neq 1 \\ i_G^G(-1) & ; j = i \\ i_{P_{(i-1, n-i+1)}}^G(-1) & ; j = 1 \end{cases}$$

Here the contribution $i_{P_{(k, n-k)}}^G(-1)$ is induced by the cycle classes $\{V^\bullet \in \overline{X}(w) \mid V^k \text{ is rational}\}$.

ii) Let $j = n - 2$ and $w' = s_1 \cdots \hat{s}_i \cdots s_{n-1} \in W$ for some $1 \leq i \leq n - 2$. Using Poincaré duality we treat the equivalent situation by considering the homomorphism $i_* : A_{n-2}(\overline{X}(w')) \rightarrow A_{n-2}(\overline{X}(w))$. We supply the latter object with the basis given by the cycle classes $\{V^\bullet \in \overline{X}(w) \mid V^k \text{ is rational}\}$ labeled by $s_1 s_2 \cdots \hat{s}_k \cdots s_{n-1}$ for $1 \leq k \leq n - 2$. We realize the missing trivial representation by the cycle class $f^*(\mathbb{P}(H))$ labeled as usual by $s_1 s_2 \cdots s_{n-2}$. Here $f : \overline{X}(w) \rightarrow \mathbb{P}(V)$ is composite of all blow up maps, i.e., the projection map onto the first filtration step and $H \subset V$ is a (rational) hyperplane. Hence we have a grading $A_{n-2}(\overline{X}(w)) = \bigoplus_k A(w)_k$.

If $i < n - 2$, then the map $A_{n-2}(\overline{X}(w')) \rightarrow A_{n-2}(\overline{X}(w))$ is graded by the choice of bases. If $i = n - 1$ then $i_* = \bigoplus_k i_k$ where $i_k : A(w') \rightarrow A(w)_k$ is induced by Frobenius reciprocity by the double coset of $e \in W$ multiplied by -1 . The reason is that $f^*(\mathbb{P}(H))$ coincides with the cycle class

$$\overline{X}(w)_H + \sum_{i=1}^{n-2} \sum_{\substack{W \in \text{Gr}_i(V)^{(k)} \\ W \subset H}} \overline{X}(w)_W,$$

where $\overline{X}(w)_W := \{V^\bullet \in \overline{X}(w) \mid V^i = W\}$ when $\dim W = i$.

If we want to have a graded morphism, then we simply use the bases on $A_{n-2}(\overline{X}(w))$ by using the approach via the dual projective space, cf. Remark 3.5 iii).

To the end of this section we recall the definition of Deligne-Lusztig varieties attached to elements of the Braid monoid B^+ of W and the description of smooth compactifications of them, cf. [DL, DMR, L4]. The Braid monoid B^+ is the quotient of F^+ given by the relations $(st)^{m_{s,t}} = 1$ where $s, t \in S$ with $s \neq t$. Here $m_{s,t} \in \mathbb{Z}$ is the order of the element $st \in W$. Thus we have surjections

$$(3.6) \quad F^+ \xrightarrow{\alpha} B^+ \xrightarrow{\beta} W$$

with $\gamma = \beta \circ \alpha$. There is a section $W \hookrightarrow B^+$ of β which identifies W with the subset

$$B_{\text{red}}^+ = \{w \in B^+ \mid \ell(w) = \ell(\beta(w))\}$$

of reduced elements in B^+ , cf. [GKP]. In the sequel we will identify W with B_{red}^+ .

For any element $w = s_{i_1} \cdots s_{i_r} \in F^+$, set

$$\begin{aligned} X(w) &:= X(s_{i_1}, \dots, s_{i_r}) \\ &:= \{x = (x_0, \dots, x_r) \in X^{r+1} \mid \text{inv}(x_{j-1}, x_j) = s_{i_j}, j = 1, \dots, r, x_r = F(x_0)\}. \end{aligned}$$

This is a smooth variety over k equipped with an action of G . If $w \in W$ and $w = s_{i_1} \cdots s_{i_r}$ is a fixed reduced decomposition, then we also simply write $X^{F^+}(w)$ for $X(s_{i_1}, \dots, s_{i_r})$. For any $w \in W$, the map

$$(3.7) \quad \begin{array}{ccc} X(s_{i_1}, \dots, s_{i_r}) & \longrightarrow & X(w) \\ (x_0, \dots, x_r) & \mapsto & x_0 \end{array}$$

defines a G -equivariant isomorphism of varieties over k . Moreover by Broué, Michel [BM] and Deligne [De] the variety $X^{F^+}(w)$ depends up to a unique isomorphism only on the image of $s_{i_1} \cdots s_{i_r}$ in B^+ .

Finally, we consider compactifications of the varieties $X(w)$. More generally, we associate to certain elements of the completed braid monoid \underline{B}^+ a DL-variety. Here we do not treat the general machinery as developed in [DMR], not to mention the definition of \underline{B}^+ , as we need these varieties only for some elements of \underline{B}^+ . For this let \hat{F}^+ be the free monoid generated by the set \hat{S} consisting of S and of all reflections in W of the shape sts with $s, t \in S, st \neq ts \in W$. In order to distinguish the generator $sts \in \hat{S}$ where $s, t \in S$ and $st \neq ts \in W$ from the product $sts \in F^+ \subset \hat{F}^+$, we also write \widehat{sts} for this element.

For any product $t = t_1 \dots t_r \in \hat{F}^+$ with $t_i \in \hat{S}$, we set

$$\begin{aligned} \overline{X}(t) &:= \overline{X}(t_1, \dots, t_r) \\ &:= \{x = (x_0, \dots, x_r) \in X^{r+1} \mid \text{inv}(x_{j-1}, x_j) \leq t_j, j = 1, \dots, r, x_r = F(x_0)\}. \end{aligned}$$

Again this is a k -variety with an action of G .

Proposition 3.17. *The variety $\overline{X}(t)$ is smooth and projective.*

Proof. See [DMR, Prop. 2.3.5, 2.3.6]. □

Let $w = s_{i_1}, \dots, s_{i_r} \in F^+$. Then the variety $\overline{X}(w)$ includes $X(w)$ as an open subset so that we get a compactification of $X(w)$. More precisely, for all $v \preceq w$ we can identify $X(v)$ with a locally closed subvariety of $\overline{X}(w)$ and we get in this way a stratification $\overline{X}(w) = \bigcup_{v \preceq w} X(v)$.

Hence if $w \in W$ and $w = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition, then the variety

$$\overline{X}^{F^+}(w) := \overline{X}(s_{i_1}, \dots, s_{i_r})$$

is a smooth compactification of $X(w)$. The map (3.7) extends to a surjective proper birational morphism

$$\pi : \overline{X}^{F^+}(w) \longrightarrow \overline{X(w)}$$

which we call the Bott-Samelson or Demazure resolution of $X(w)$ with respect to the reduced decomposition.

Remark 3.18. If w is a Coxeter element, then the map π is an isomorphism. In fact, this follows easily by considering the natural stratifications on both sides.

Remark 3.19. When $w \in F^+$ has not full support, then the obvious variant of Proposition 3.6 does also hold true for the DL-varieties $X(w)$ and their compactifications $\overline{X}(w)$, cf. [DMR, Cor. 3.1.3].

4. SQUARES

We consider the action of W on itself by conjugation. The set of conjugacy classes C in W is in bijection with the set of partitions \mathcal{P} of n . For a partition $\mu \in \mathcal{P}$, let C^μ be the corresponding conjugacy class. Let C_{\min} be the set of minimal elements in a given conjugacy class C .

Corollary 4.1. *Let C be a conjugacy class and let $v, w \in C_{\min}$. Then $H_c^*(X(v)) \cong H_c^*(X(w))$.*

Proof. We start with the remark that Coxeter elements are minimal within their conjugacy class. In this case the result is due to Lusztig [L2], cf. also Remark 3.5 i). Hence by Proposition 3.6 the same is true for all Coxeter elements in a fixed Levi subgroup. On the other hand, the Levi subgroups to the minimal elements v and $w \in C_{\min}$ are conjugated by an element in G which induces the isomorphism. \square

In order to deal with non-minimal elements we recall the following result of Geck, Kim and Pfeiffer. Let $w, w' \in W$ and $s \in S$. Set $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leq \ell(w)$. We write $w \rightarrow w'$ if $w = w'$ or if there are elements $s_1, \dots, s_r \in S$ and $w = w_1, \dots, w_r = w' \in W$ with $w_i \xrightarrow{s_i} w_{i+1}$, $i = 1, \dots, r - 1$.

Theorem 4.2. ([GKP], Thm. 2.6) *Let C be a conjugacy class of W . For any $w \in C$, there exists some $w' \in C_{\min}$ such that $w \rightarrow w'$.* \square

As a consequence, for any $w \in W$, which is not minimal in its conjugacy class there exists a finite set of cyclic shifts (i.e. elementary conjugations $w \rightarrow sws$, $s \in S$) such that the resulting element has the shape $sw's$ with $\ell(w) = \ell(w') + 2$.

Let $s \in S$ and let $w, w' \in W$ with $w = sw's$. Suppose that $\ell(w) = \ell(w') + 2$. We put

$$X(w, sw') = X(w) \cup X(sw') \text{ and } X(w's, w') = X(w's) \cup X(w')$$

and

$$X(w, w's) = X(w) \cup X(w's) \text{ and } X(sw', w') = X(sw') \cup X(w').$$

Proposition 4.3. (*Deligne-Lusztig*) *i) The varieties $X(w's)$ and $X(sw')$ are universally homeomorphic by maps $\sigma : X(sw') \rightarrow X(w's)$ and $\tau : X(w's) \rightarrow X(sw')$ with $\tau \circ \sigma = F$ and $\sigma \circ \tau = F$. Hence $H_c^*(X(w's)) \cong H_c^*(X(sw'))$.*

ii) The above maps extend to morphisms $\tau : X(w's, w') \rightarrow X(sw', w')$ and $\sigma : X(sw', w') \rightarrow X(w's, w')$ with $\sigma|_{X(w')} = id$ and $\tau|_{X(w')} = F$.

Proof. i) For later use we just recall the construction of the maps and refer for the proofs to [DL]. Let $B \in X(sw')$. Then there is by property (1) b) of the varieties $O(-)$ a unique Borel subgroup $\sigma(B) \in X$ with $(B, \sigma(B)) \in \mathcal{O}(s)$ and $(\sigma B, F(B)) \in \mathcal{O}(w')$. An immediate computation shows that $(\sigma(B), F(\sigma B)) \in \mathcal{O}(w's)$, cf. [DL, Thm 1.6]. The map τ is defined analogously.

ii) This follows easily from the definitions of the maps σ, τ . □

The following statement is based on a observation made in Theorem 1.6 in [DL].

Proposition 4.4. *There exists a \mathbb{A}^1 -bundle $\gamma : X(w, sw') \rightarrow X(w's, w')$.*

Proof. As in the proof of loc.cit., we may write $X(w)$ as a (set-theoretical) disjoint union

$$X(w) = X_1 \cup X_2$$

where X_1 is closed in $X(w)$ and X_2 is its open complement (Note that we have interchanged the role of w and w' compared to [DL]). Recall the definition of $X_i, i = 1, 2$. For $B \in X(w)$ there are unique Borel subgroups $\delta(B), \gamma(B) \in X$ with $(B, \gamma(B)) \in \mathcal{O}(s)$, $(\gamma(B), \delta(B)) \in \mathcal{O}(w')$ and $(\delta(B), F(B)) \in \mathcal{O}(s)$. Then

$$X_1 = \{B \in X(w) \mid \delta(B) = F(\gamma(B))\} \text{ resp. } X_2 = \{B \in X(w) \mid \delta(B) \neq F(\gamma(B))\}.$$

In [DL] it is shown that the map $\gamma : X_1 \rightarrow X(w')$ is a \mathbb{A}^1 -bundle whereas $\delta : X_2 \rightarrow X(w's)$ is a \mathbb{G}_m -bundle.

On the other hand, if $B \in X_2$, then $(\delta(B), F(B)) \in \mathcal{O}(s)$, $(F(B), F(\gamma B)) \in \mathcal{O}(s)$, $\delta(B) \neq F(\gamma B)$, hence $(\delta(B), F(\gamma(B))) \in \mathcal{O}(s)$. This was already shown in [DL]. It follows that

$(\gamma(B), F(\gamma(B))) \in \mathcal{O}(w's)$, hence $\gamma(B) \in X(w's)$. Thus we have a morphism $\gamma : X(w) \rightarrow X(w's, w')$ of varieties compatible with the action of G and F .

By Proposition 4.3 there is a homeomorphism $\sigma : X(sw') \rightarrow X(w's)$. This map is compatible with $\gamma : X(w) \rightarrow X(w's, w')$ in the sense that both maps glue to a morphism $\gamma : X(w, sw') \rightarrow X(w's, w')$. This map is clearly an \mathbb{A}^1 -bundle. \square

Corollary 4.5. *There is an isomorphism $H_c^i(X(w, sw')) = H_c^{i-2}(X(w's, w'))(-1)$ for all $i \geq 2$.* \square

Remark 4.6. This corollary and the upcoming results Corollary 4.5, Corollary 4.12, Proposition 6.5 and Remark 6.6 were already proved in [DMR, Prop. 3.2.10]). Indeed, they consider the situation where a cyclic shift is applied to $w = sw's$.

The map γ even extends to a larger locally closed subvariety as follows. We set $X(w, sw', w's, w') := X(w, sw') \cup X(w's, w')$.

Lemma 4.7. *The set $X(w, sw', w's, w')$ is locally closed in X .*

Proof. For proving the assertion, it suffices to show (by considering topological closures of DL-varieties) that there is no element $v \in W$ different from sw' resp. $w's$ with $w' \leq v \leq w$. This is a consequence of the following result. \square

Lemma 4.8. *Let $w_1, w_2 \in W$ with $w_1 \leq w_2$ and $\ell(w_2) = \ell(w_1) + 2$. Then there are uniquely determined elements $v_1, v_2 \in W$ with $w_1 \leq v_i \leq w_2$ and $\ell(v_i) = \ell(w_1) + 1$.*

Proof. See [BGG, Lemma 10.3]. \square

In the above situation, Bernstein, Gelfand, Gelfand call the quadruple $Q = \{w_1, v_1, v_2, w_2\}$ a square in W . Here we use sometimes the graphical illustration of [Ku] (resp. for technical reasons sometimes without arrows) to indicate this kind of object:

$$Q : \begin{array}{ccc} & w_2 & \\ \nearrow & & \nwarrow \\ v_1 & & v_2 \\ \nwarrow & & \nearrow \\ & w_1 & \end{array} \quad (\text{ resp. } \begin{array}{ccc} & w_2 & \\ v_1 & & v_2 \\ & w_1 & \end{array}).$$

Lemma 4.7 generalizes as follows.

Lemma 4.9. *For any square $Q = \{w_1, v_1, v_2, w_2\}$ in W , the subset*

$$X(Q) := X(w_2) \cup X(v_2) \cup X(v_1) \cup X(w_1)$$

is locally closed in X . □

For later use, we also mention the following property.

Lemma 4.10. *For any square $Q = \{w_1, v_1, v_2, w_2\} \subset W$, let*

$$\delta_{w_1, v_1 \cup v_2}^{i-1} : H_c^{i-1}(X(w_1)) \longrightarrow H_c^i(X(v_1) \cup X(v_2))$$

and

$$\delta_{v_1 \cup v_2, w_2}^i : H_c^i(X(v_1) \cup X(v_2)) \longrightarrow H_c^{i+1}(X(w_2))$$

be the corresponding boundary homomorphisms. Then $\delta_{v_1 \cup v_2, w_2}^i \circ \delta_{w_1, v_1 \cup v_2}^{i-1} = 0$.

Proof. This is clear as the map $\delta_{v_1 \cup v_2, w_2}^i \circ \delta_{w_1, v_1 \cup v_2}^{i-1}$ is just the composite of the corresponding differentials in the E_1 -term associated to the stratification

$$X(Q) = X(w_2) \bigcup (X(v_1) \cup X(v_2)) \bigcup X(w_1).$$

□

Now we come back to the locally closed subvariety $X(w, sw', w's, w') \subset X$.

Proposition 4.11. *The map γ extends to a \mathbb{P}^1 -bundle $X(w, sw', w's, w') \longrightarrow X(w's, w')$ with $\gamma|_{X(w's, w')} = \text{id}_{X(w's, w')}$.*

Proof. This is a direct consequence of the definitions of γ and the variety $X(w's, w')$ realising the latter space as the set $\{(B_0, B_1, B_2, B_3) \in X^4 \mid (B_0, B_1) \in \mathcal{O}(e), (B_1, B_2) \in \mathcal{O}(w'), (B_2, B_3) \in \overline{\mathcal{O}(s)}, B_3 = F(B_0)\}$. □

Recall from Remark 2.18 that $H = G \times \Gamma$.

Corollary 4.12. *There is an isomorphism of H -modules*

$$H_c^i(X(w, sw', w's, w')) = H_c^i(X(w's, w')) \oplus H_c^{i-2}(X(w's, w'))(-1)$$

for all $i \geq 0$. □

For the next statement, we consider the open subset $Y := X(w) \cup X(sw') \cup X(w's)$ of $X(w, sw', w's, w')$.

Corollary 4.13. *There is a geometrically induced splitting $H_c^i(Y) = H_c^i(X(w, sw')) \oplus H_c^i(X(w's))$ as H -modules for all $i \geq 0$.*

Proof. The existence of a splitting is easily verified by considering the diagram of long exact cohomology sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_c^i(X(w, sw')) & \longrightarrow & H_c^i(X(w, sw', w's, w')) & \longrightarrow & H_c^i(X(w's, w')) & \xrightarrow{\delta^i} & H_c^{i+1}(X(w, sw')) & \longrightarrow \cdots \\
& & \parallel & & \uparrow & & \uparrow & & \parallel & \\
\cdots & \longrightarrow & H_c^i(X(w, sw')) & \longrightarrow & H_c^i(Y) & \longrightarrow & H_c^i(X(w's)) & \longrightarrow & H_c^{i+1}(X(w, sw')) & \longrightarrow \cdots
\end{array}$$

together with the fact that the differential map δ^i vanishes. That it is natural comes about from the fact that the subset $U := X_2(w) \cup X(sw') \cup X(w's)$ is open in Y and there is a \mathbb{P}^1 -bundle $\gamma : U \rightarrow X(w's)$ with $\gamma \circ i = \text{id}$ where $i : X(w's) \hookrightarrow U$ is the inclusion. \square

In the sequel, we denote by

$$r_{w,sw'}^i : H_c^i(X(w, sw')) \longrightarrow H_c^i(X(sw'))$$

the map which is induced by the closed immersion $X(sw') \hookrightarrow X(w, sw')$. We consider the corresponding long exact cohomology sequence

$$\cdots \longrightarrow H_c^{i-1}(X(sw')) \longrightarrow H_c^i(X(w)) \longrightarrow H_c^i(X(w, sw')) \longrightarrow H_c^i(X(sw')) \longrightarrow \cdots$$

which by Corollary 4.5 identifies with the sequence

$$(4.1) \quad \begin{array}{l} \cdots \longrightarrow H_c^{i-1}(X(sw')) \longrightarrow H_c^i(X(w)) \longrightarrow H_c^{i-2}(X(w's) \cup X(w'))(-1) \\ \longrightarrow H_c^i(X(sw')) \longrightarrow \cdots \end{array}$$

Remark 4.14. In [DMR] it is proved that there is a long exact cohomology sequence

$$\begin{array}{l} \cdots \longrightarrow H_c^i(X(w)) \longrightarrow H_c^{i-2}(X(w'))(-1) \longrightarrow H_c^{i-1}(X(sw'))(-1) \oplus H_c^i(X(sw')) \\ \longrightarrow H_c^{i+1}(X(w)) \longrightarrow \cdots \end{array}$$

which relies on the fact that the natural maps $\delta^i : H_c^{i-2}(X(sw'))(-1) \rightarrow H_c^i(X(sw'))$ induced by the \mathbb{G}_m -bundle X_2 over $X(sw')$ are trivial (as already stated in [DL, Thm. 1.6]). In particular, it follows that the cokernel of the boundary map $H_c^{i-3}(X(w')) \rightarrow H_c^{i-2}(X(w's))$ always contributes to $H_c^i(X(w))$.

Remark 4.15. The same statements presented here (Prop. 4.4. - Cor. 4.12) are true if we work with elements in F^+ instead in W , cf. also [DMR]. More precisely, if $w = sw's$ for $w, w' \in F^+$ and $s \in S$, then we can define subsets $X_1, X_2 \subset X(w)$ such that X_1 is a \mathbb{A}^1 -bundle over $X(w')$ and such that X_2 is an \mathbb{G}_m -bundle over $X(w's)$. With the same reasoning, the subset $X(w) \cup X(sw')$ is an \mathbb{A}^1 -bundle over $X(w's) \cup X(w')$, etc.

Example 4.16. We reconsider Example 3.1 (which is also discussed in [DMR, ch. 4]). So let $w = sw's = (1, 3)$ with $s = s_2 = (2, 3)$, $w' = s_1 = (1, 2)$. We are going to determine the cohomology of the DL-variety $X(w)$. The cohomology of $X(s_1s_2)$ resp. $X(s_2s_1)$ is given in Proposition 3.3 by

$$H_c^*(X(s_2s_1)) = H_c^*(X(s_1s_2)) = v_B^G[-2] \oplus v_{P_{(2,1)}}^G(-1)[-3] \oplus i_G^G(-2)[-4].$$

Further we have $H_c^*(X(w')) = i_B^G/i_{P_{(2,1)}}^G[-1] \oplus i_{P_{(2,1)}}^G(-1)[-2]$. Now the variety $X(w's, w') = X(w's) \cup X(w')$ coincides with the set $\{V^\bullet \mid F(V^1) \subset V^2, V^1 \neq F(V^1)\}$ which we may identify with the open subset $\mathbb{P}(V) \setminus \mathbb{P}(V)(k)$ of $\mathbb{P}(V)$. Hence we obtain (which follows also by applying Proposition 5.7)

$$H_c^*(X(w's, w')) = v_{P_{(2,1)}}^G[-1] \oplus i_G^G(-1)[-2] \oplus i_G^G(-2)[-4]$$

and therefore

$$H_c^*(X(w, sw')) = v_{P_{(2,1)}}^G(-1)[-3] \oplus i_G^G(-2)[-4] \oplus i_G^G(-3)[-6]$$

by Corollary 4.5. We claim that the maps $r_{w,sw'}^3, r_{w,sw'}^4$ are surjective. Indeed, for $i = 4$ this is clear since $H_c^4(X(sw'))$ is the top cohomology group of $X(sw')$. As for $i = 3$, we consider the boundary map $H_c^2(X(s)) \rightarrow H_c^3(X(sw'))$ which is surjective since $X(s) \cup X(sw')$ has the same cohomology as $X(w's, s)$. Let $\tilde{Z} = X(s^2) \cup X(s)$. Then the map $H_c^2(\tilde{Z}) \rightarrow H_c^2(X(s))$ is surjective, as well, since the RHS is the top cohomology degree and both varieties have the same number of connected components. We consider the resulting commutative diagram

$$\begin{array}{ccc} H_c^3(X(w, sw')) & \longrightarrow & H_c^3(X(sw')) \\ \uparrow & & \uparrow \\ H_c^2(\tilde{Z}) & \longrightarrow & H_c^2(X(s)). \end{array}$$

It follows that the map $r_{w,sw'}^3 : H_c^3(X(w, sw')) \rightarrow H_c^3(X(sw'))$ is surjective. Hence we get

$$H_c^*(X(w)) = v_B^G[-3] \oplus i_G^G(-3)[-6].$$

5. COHOMOLOGY OF DL-VARIETIES OF HEIGHT ONE

In this section we determine the cohomology of DL-varieties attached to Weyl group elements which are slightly larger than Coxeter elements, i.e. to elements which are of height one. For the definition of the height function we recall that by Theorem 4.2 there is for any non-minimal $w \in W$ some element $w' \in W$ with $\ell(w) = \ell(w') + 2$ and $w \rightarrow w'$.

Definition 5.1. We define the height of w inductively by $\text{ht}(w) = \text{ht}(w') + 1$. Here we set $\text{ht}(w) = 0$ if w is minimal in its conjugacy class.

In order to extend the definition of the height function to F^+ , we use the following statement.

Lemma 5.2. *Let $w \in B^+$ such that $\ell(\beta(w)) < \ell(w)$, i.e. such that $w \in B^+ \setminus W$. Then w has the shape $w = w_1 s s w_2$ for some $s \in S$ and $w_1, w_2 \in B^+$.*

Proof. See [GP, Exercise 4.1]. □

Definition 5.3. i) Let $w \in B^+$. We define the height inductively by

$$\text{ht}(w) := \begin{cases} \text{ht}(w) & \text{if } w \in W \\ h(w_1 w_2) + 1 & \text{if } w = w_1 s s w_2 \text{ is as above.} \end{cases}$$

ii) For $w \in F^+$, we set $\text{ht}(w) := \text{ht}(\alpha(w))$.

Apart from minimal elements induced by W we may write by Lemma 5.2 resp. Theorem 4.2 the elements in B^+ modulo cyclic shift in the shape $w = s w'$ for some $s \in S$ and $w' \in B^+$. The proof of the next statement is immediate.

Lemma 5.4. *Let $w \in F^+$ and let $w_{\min} \in W$ be a minimal element lying in the conjugacy class of $\gamma(w)$. Then $\ell(w) = \ell(w_{\min}) + 2 \text{ht}(w)$.* □

For any irreducible H -representation $V = j_\mu(i)$, $\mu \in \mathcal{P}$, $i \in \mathbb{Z}$, we set $t(V) = i$.

Proposition 5.5. *Let $v, w \in F^+$, $i, j, m \in \mathbb{Z}_{\geq 0}$ and suppose that $\text{ht}(v) = 0$. Let $V \subset H_c^i(X(w))$ be a subrepresentation such that $V(m) \subset H_c^j(X(v))$. Then*

$$\ell(w) - \ell(v) + m \geq i - j \geq \ell(w) - \ell(v) + m - \text{ht}(w).$$

Proof. As $\text{ht}(v) = 0$ we deduce by Proposition 3.3 and Proposition 3.6 that $j = \ell(v) - t(V(m))$. As the maps $\bigoplus_{\substack{v' < v \\ \ell(v') = \ell(v) - 1}} H_c^{j-1}(X(v')) \longrightarrow H_c^j(X(v))$ are surjective for $j < 2\ell(v)$, we may suppose by considering the above inequalities that $V(m)$ sits in the top cohomology degree of $X(v)$. Then $\ell(v) = -t(V(m))$. As any unipotent representation is realized in $H^0(X(e))$ we may assume that $v = 1$ and therefore $j = 0$.

We start with the case where $\text{ht}(w) = 0$. In this case one has even - by looking again at Proposition 3.3 - the stronger identity

$$i = \ell(w) + m.$$

Now let $\text{ht}(w) \geq 1$ and suppose that $w = sw's$. Consider the long exact cohomology sequence (4.1). We distinguish the following cases:

Case a) Let $V \subset H_c^{i-1}(X(sw'))$. By induction on the length we deduce that $\ell(sw') + m \geq i - 1 \geq \ell(sw') + m - \text{ht}(sw')$. As $\ell(sw') = \ell(w) - 1$ and $\text{ht}(w) \geq \text{ht}(sw')$, we see that $\ell(w) + m \geq i \geq \ell(w) + m - h(w)$.

Case b) Let $V \subset H_c^i(X(w, sw'))$. Then $m \geq 1$.

Subcase i) Let $V(1) \subset H_c^{i-2}(X(sw'))$. By induction on the length we deduce that $\ell(sw') + m - 1 \geq i - 2 \geq \ell(sw') + m - 1 - h(sw')$. As $\ell(sw') = \ell(w) - 1$ and $\text{ht}(w) \geq \text{ht}(sw')$, we see that $\ell(w) + m \geq i \geq \ell(w) + m - \text{ht}(w)$.

Subcase ii) Let $V(1) \subset H_c^{i-2}(X(w'))$. By induction on the length we deduce that $\ell(w') + m - 1 \geq i - 2 \geq \ell(w') + m - 1 - \text{ht}(w')$. As $\ell(w') = \ell(w) - 2$ and $h(w) = h(w') + 1$, we see that even the stronger identity $\ell(w) + m - 1 \geq i \geq \ell(w) + m - h(w)$ holds true. \square

Corollary 5.6. *Let $v, w \in W$ and $i, j, m \in \mathbb{Z}_{\geq 0}$. Let $V \subset H_c^i(X(w))$ be a subrepresentation such that $V(m) \subset H_c^j(X(v))$. Then*

$$\ell(w) - \ell(v) + m + \text{ht}(v) \geq i - j \geq \ell(w) - \ell(v) + m - \text{ht}(w).$$

Proof. We apply the foregoing proposition where w is replaced by v and v by 1. Then $\ell(v) + t(V(m)) \geq j \geq \ell(v) + t(V(m)) - \text{ht}(v)$. Multiplying this term with -1 and adding the result to the sequence of inequalities $\ell(w) + t(V) \geq i \geq \ell(w) + t(V) - \text{ht}(w)$ gives the statement. \square

For $w' \in W$ with $w' \leq w$ and $\ell(w') = \ell(w) - 1$, the set $X(w, w') = X(w) \cup X(w')$ is a subvariety of X as already observed above. Here $X(w')$ is closed and $X(w)$ is open in $X(w, w')$. We denote by

$$\delta_{w',w}^* : H_c^*(X(w')) \longrightarrow H_c^{*+1}(X(w))$$

the associated boundary map.

Proposition 5.7. *Let w be a Coxeter element and let $w' \in W$ with $w' \leq w$ and $\ell(w') = \ell(w) - 1$. Then the boundary homomorphism $\delta_{w',w}^j : H_c^j(X(w')) \longrightarrow H_c^{j+1}(X(w))$ is surjective for all $j \leq 2\ell(w') = 2(n-2)$. (In particular, $\delta_{w',w}^j$ is si-surjective for all $j = 0, \dots, 2\ell(w) = 2(n-1)$.)*

Proof. We may suppose that $w = \text{Cox}_n$. Since all the representations $H_c^i(X(w)) \neq (0)$ are irreducible, it suffices to show that the boundary maps $\delta_{w',w}^i$ for $i < 2\ell(w) - 1$, are non-trivial. Let $w' = s_1 \cdots \hat{s}_i \cdots s_n$ be as above. In terms of flags the DL-varieties in question

have the following description

$$X(w) = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^j) \neq V^j, 1 \leq j \leq n-1\},$$

$$X(w') = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^i) = V^i, F(V^j) \neq V^j, 1 \leq j \neq i \leq n-1\}.$$

Their Zariski closures are given by

$$\overline{X(w)} = \{V^\bullet \mid F(V^j) \subset V^{j+1}, 1 \leq j \leq n-1\},$$

$$\overline{X(w')} = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^i) = V^i, 1 \leq j \neq i \leq n-1\}.$$

The complement of $X(w)$ in $\overline{X(w)}$ is a divisor $D = \bigcup_W D_W$ where the union is over all k -rational subspaces W of V . For any rational flag $W^\bullet = (0) \subsetneq W^{i_1} \subsetneq W^{i_2} \subsetneq \dots \subsetneq W^{i_k} \subsetneq V$ of V , we set $D_{W^\bullet} = D_{W^{i_1}} \cap D_{W^{i_2}} \cap \dots \cap D_{W^{i_k}}$ and $\text{lg}(W^\bullet) = k$. This construction gives rise for any constant sheaf A on $\overline{X(w)}$ to a resolution

$$A \longrightarrow \bigoplus_W A_{D_W} \longrightarrow \bigoplus_{W^\bullet, \text{lg}(W^\bullet)=2} A_{D_{W^\bullet}} \longrightarrow \dots \longrightarrow \bigoplus_{W^\bullet, \text{lg}(W^\bullet)=n-1} A_{D_{W^\bullet}}$$

of $A_{X(w)}$. On the other hand, we have $\overline{X(w')} = \bigcup_{W \in \text{Gr}_i(V)(k)} D_W$. Similarly as above, we get a resolution

$$A_{\overline{X(w')}} \longrightarrow \bigoplus_{\substack{W^\bullet, \text{lg}(W^\bullet)=2 \\ W^i \in W^\bullet}} A_{D_{W^\bullet}} \longrightarrow \bigoplus_{\substack{W^\bullet, \text{lg}(W^\bullet)=3 \\ W^i \in W^\bullet}} A_{D_{W^\bullet}} \longrightarrow \dots \longrightarrow \bigoplus_{\substack{W^\bullet, \text{lg}(W^\bullet)=n-1 \\ W^i \in W^\bullet}} A_{D_{W^\bullet}}$$

of $A_{X(w')}$. The second complex is a subcomplex of the first one and this inclusion induces just the boundary map. In other terms, applying $H^{2i}(-)$ to both resolutions (strictly speaking to injective resolutions of $A = \mathbb{Z}/l^n\mathbb{Z}$, $n \in \mathbb{N}$), we just get the complexes

$$H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{v < w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}) \longrightarrow \dots \longrightarrow \bigoplus_{\substack{v < w \\ \ell(v)=1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(e)})$$

and

$$H^{2i}(\overline{X(w')}) \longrightarrow \bigoplus_{\substack{v < w' \\ \ell(v)=\ell(w')-1}} H^{2i}(\overline{X(v)}) \longrightarrow \dots \longrightarrow \bigoplus_{\substack{v < w' \\ \ell(v)=1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(e)}).$$

If $w' = s_1 s_2 \dots s_{n-2}$, then $X(w') \cong \coprod_H \Omega(H)$ with H running through all rational hyperplanes in $V = \mathbb{F}^n$. Further we may identify $X(w)$ with $\Omega(V) \subset \mathbb{P}(V)$. Here the result is well-known in the setting of period domains. In fact, by considering also the varieties $\Omega(E)$ with E a rational subspace of V , we get a stratification of the projective space $\mathbb{P}(V)$. Then the result follows by weight reasons and the cohomology formula in Proposition 3.3

with respect to the varieties $\Omega(E)$. Alternatively, one might use the fundamental complex in [O]. By symmetry the same reasoning applies to $w' = s_2 s_3 \cdots s_{n-1}$.

In general we distinguish the cases whether $j = 2\ell(w')$ or $j < 2\ell(w')$. Suppose first that $j = 2\ell(w')$. Here the claim follows by Example 3.16 ii) since the contribution $H^j(\overline{X}(w'))$ does not lie in the image of the map $H^j(\overline{X}(w)) \rightarrow \bigoplus_{\substack{v < w \\ \ell(v) = \ell(w) - 1}} H^j(\overline{X}(v))$.

If $j < 2\ell(w')$ then we argue as follows. Let $v = s_1 s_2 \cdots s_{n-2}$ and $v' = \gcd(w', v) = s_1 \cdots \hat{s}_i \cdots s_{n-2}$. By induction on n the map $H_c^{j-1}(X(v')) \rightarrow H_c^j(X(v))$ is surjective. On the other hand, by what we have observed above the map $H_c^j(X(v)) \rightarrow H_c^{j+1}(X(w))$ is surjective, as well. Using Lemma 4.10 we deduce that the map $H_c^j(X(w')) \rightarrow H_c^{j+1}(X(w))$ is non-trivial. \square

The next two statements give the cohomology of all Weyl group elements having full support and which are of height 1. Arbitrary elements of height one are handled by Proposition 3.6.

Proposition 5.8. *Let $w = sw'$ where $w' \in W$ is a Coxeter element in some Levi subgroup of a proper maximal parabolic subgroup in G . Then the maps $r_{w,sw'}^j : H_c^j(X(w, sw')) \rightarrow H_c^j(X(sw'))$ are all surjective for $j > \ell(sw') = n - 1$. (In particular they are si-surjective for all $j \geq 0$.)*

Proof. We start with the observation that sw' and $w's$ are both Coxeter elements in W . We may suppose that $w' = s_1 \cdots \hat{s}_i \cdots s_{n-1}$ and $s = s_i$. It is clear that $v_B^G \notin \text{supp } H_c^{n-1}(X(w, sw'))$. So let $j > n - 1$ and suppose that $r_{w,sw'}^j$ is not surjective. Then the irreducible module $H_c^j(X(sw'))$ maps injectively into $H_c^{j+1}(X(w))$ via the boundary homomorphism $\delta_{sw',w}^j$. First let $i < n - 1$. Set

$$w'' := s_i s_1 s_2 \cdots \hat{s}_i \cdots s_{n-2} = s_i w' s_{n-1}.$$

This is a Coxeter element in the parabolic subgroup $W_{(n-1,1)}$ of W with $w'' \leq s_i w'$ and $\ell(w'') = \ell(w')$. Consider the square

$$\begin{array}{ccc} & w & \\ s_i w' & \nearrow & \nwarrow \\ & w'' & \nearrow \\ & & w'' s_i. \end{array}$$

The boundary homomorphism $\delta_{w'',s_i w'}^{j-1} : H_c^{j-1}(X(w'')) \rightarrow H_c^j(X(s_i w'))$ is si-surjective by Proposition 5.7. On the other hand, the boundary map $\delta_{w'',w'' s_i}^{j-1} : H_c^{j-1}(X(w'')) \rightarrow H_c^j(X(w'' s_i))$ vanishes as the map $r_{w'' s_i, w''}^j : H_c^{j-1}(X(w'' s_i) \cup X(w'')) \rightarrow H_c^j(X(w''))$ is

(si-)surjective by induction on n . Indeed, both elements $w'', w''s_i$ are of the shape above and in the Weyl group of GL_{n-1} . The start of induction is given by Example 4.16. By Lemma 4.10 the composite $\delta_{s_i w', w}^j \circ \delta_{w'', s_i w'}^{j-1}$ vanishes, a contradiction. The result follows in this special case.

If $i = n - 1$, then we set $w'' := s_1 w' s_{n-1}$ and consider $w' s_{n-1}$ instead of $s_{n-1} w'$ and $s_{n-1} w''$ instead of $w'' s_{n-1}$. Then the same argument goes through. \square

By Proposition 5.7 we are able to give a formula for the cohomology of these height 1 elements. Here we could give the description of the induced representation $H_c^*(X(w')) = \mathrm{Ind}_{P_{(i, n-i)}}^G (H_c^*(X_M(w')))$ by using Littlewood-Richardson coefficients, cf. [FH, §A] (Note that the structure or combinatoric of unipotent G - and W -representations is the same, cf. Remark 2.2). Instead we prefer to use the notation which is common in the Grothendieck group of G -representations. Hence if we write $V - W$ for two G -representations V, W , then we mean implicitly that W is a subrepresentation of V .

Corollary 5.9. *In the situation of the foregoing proposition, we have for $j \in \mathbb{N}$, with $\ell(w) < j < 2\ell(w) - 1$,*

$$H_c^j(X(w)) = (H_c^{j-2}(X(w')) - j_{(j+1-n, 1, \dots, 1)}(n-j))(-1) - j_{(j+2-n, 1, \dots, 1)}(n-j-1).$$

Moreover, we have $H_c^{\ell(w)}(X(w)) = v_B^G \oplus (v_{P(s)}^G - j_{(2, 1, \dots, 1)}(-1))$, $H_c^{2\ell(w)-1}(X(w)) = 0$ and $H_c^{2\ell(w)}(X(w)) = i_G^G(-\ell(w))$.

Proof. By Proposition 5.8, we deduce that $H_c^j(X(w)) = \ker (H_c^j(X(w, sw')) \rightarrow H_c^j(X(sw')))$ for all $j > \ell(w) = n$. By Proposition 3.3 we have $H_c^j(X(sw')) = j_{(j+2-n, 1, \dots, 1)}(n-j-1)$. Further $H_c^j(X(w, sw')) = H_c^{j-2}(X(w's) \cup X(w'))(-1)$ and the boundary map

$$H_c^{j-2}(X(w')) \rightarrow H_c^{j-1}(X(w's)) = j_{(j+1-n, 1, \dots, 1)}(n-j)$$

is surjective for $j - 2 \leq 2\ell(w') = 2\ell(w) - 4$ by Proposition 5.7. Hence we get the first identity in the statement.

If $j = \ell(w) = n$ then one verifies easily that

$$H_c^{j-2}(X(w')) - j_{(j+1-n, 1, \dots, 1)}(n-j) = H_c^{\ell(w)}(X(w')) - v_B^G = v_{P(s)}^G.$$

In addition the Steinberg representation appears as a summand in $H_c^n(X(w))$. It is induced via the boundary map $H_c^{\ell(w's)}(X(w's)) \rightarrow H_c^{\ell(w)}(X(w))$ from $H_c^{\ell(w)-1}(X(w's)) = H_c^{\ell(w's)}(X(w's)) = v_B^G$.

The remaining identities for $j = 2\ell(w) - 1, 2\ell(w)$ are easily verified in the same way. \square

Remark 5.10. Let $w \in W$ have full support. Then we always have $H_c^{\ell(w)}(X(w)) \supset v_B^G = j_{(1,\dots,1)}$ and $H_c^{2\ell(w)}(X(w)) = v_G^G(-\ell(w)) = j_{(n)}(-\ell(w))$, cf. [L2, Prop. 1.22], [DMR, Prop. 3.3.14, 3.3.15]. More precisely, these are the only cohomology degrees where these extreme unipotent representations appear. Further $H_c^i(X(w)) = 0$ for all $i < \ell(w)$.

Example 5.11. Let $n = 4$, $w = (1, 2)(2, 3)(3, 4)(1, 2) \in W$. Then

$$H_c^*(X(w)) = v_B^G[-4] \oplus j_{(2,2)}(-2)[-5] \oplus i_G^G(-4)[-8].$$

Example 5.12. Let $n = 4$, $w = (2, 3)(1, 2)(3, 4)(2, 3) \in W$. Then

$$\begin{aligned} H_c^*(X(w)) &= v_B^G[-4] \oplus j_{(2,2)}(-1)[-4] \oplus j_{(2,1,1)}(-2)[-5] \\ &\quad \oplus j_{(3,1)}(-2)[-5] \oplus j_{(2,2)}(-3)[-6] \oplus i_G^G(-4)[-8]. \end{aligned}$$

The remaining elements with full support and which are of height 1 are treated by the next result.

Corollary 5.13. Let $w = sw's \in W$ with $\ell(w) = \ell(w') + 2$ for some Coxeter element $w' \in W$ and $s \in S$. Then the map $r_{w,sw'}^j : H_c^j(X(w, sw')) \rightarrow H_c^j(X(sw'))$ vanishes for all $j \neq 2\ell(w) - 2$ and is an isomorphism for $j = 2\ell(w) - 2$. Hence we have

$$H_c^j(X(w)) = H_c^{j-2}(X(w's) \cup X(w'))(-1) \oplus H_c^{j-1}(X(sw'))$$

for all $j \neq 2\ell(w) - 1, 2\ell(w) - 2$ and $H_c^{2\ell(w)-1}(X(w)) = H_c^{2\ell(w)-2}(X(w)) = 0$.

Proof. By Prop. 5.8 the boundary map $H_c^{j-2}(X(w')) \rightarrow H_c^{j-1}(X(w's))$ vanishes for all $j \in \mathbb{N}$ with $j - 2 \neq \ell(w') = n - 1$. If $j - 2 = \ell(w')$, then it is an injection, since on the LHS we have the Steinberg representation v_B^G . Hence

$$\begin{aligned} H_c^j(X(sw's) \cup X(sw')) &\cong H_c^{j-2}(X(w's) \cup X(w'))(-1) \\ &= H_c^{j-2}(X(w's))(-1) \oplus H_c^{j-2}(X(w'))(-1) \end{aligned}$$

for all $j > \ell(w') + 2 = n + 1$. By Remark 4.14 we know that $r_{w,sw'}^j$ applied to a contribution of $H_c^{j-2}(X(w's))(-1)$ vanishes. On the other hand, by comparing weights we see that the Tate twist of a contribution in $H_c^j(X(sw's) \cup X(sw'))$ induced by $H_c^{j-2}(X(w'))(-1)$ is different from the Tate twist of $H_c^j(X(sw'))$, except for $j = 2\ell(w) - 2$. Here we have the trivial representation on both sides. The result follows. \square

Example 5.14. Let $n = 4$, $w = (2, 3)(1, 2)(2, 3)(3, 4)(2, 3) \in W$. Then one verifies that

$$\begin{aligned} H_c^*(X(w)) &= v_B^G[-5] \oplus j_{(2,2)}(-2)[-6] \oplus j_{(2,1,1)}(-2)[-6] \\ &\quad \oplus j_{(3,1)}(-3)[-7] \oplus j_{(2,2)}(-3)[-7] \oplus i_G^G(-5)[-10]. \end{aligned}$$

Although the previous results do not apply directly to the element $w = (1, 4) \in W$, we are able to compute the cohomology of $X((1, 4))$. Indeed, we write $w = sw's$ with $w' = (1, 3)$ and $s = (3, 4)$. The cohomology groups of $X((1, 3))$ and $X((1, 3)(3, 4))$ behave disjointly, cf. Examples 4.16, 5.11. We deduce that

$$\begin{aligned} H_c^*(X(w') \cup X(w's)) &= j_{(2,1,1)}[-3] \oplus j_{(2,2)}(-2)[-5] \oplus j_{(3,1)}(-3)[-6] \\ &\oplus i_G^G(-3)[-6] \oplus i_G^G(-4)[-8] \end{aligned}$$

and so

$$\begin{aligned} H_c^*(X(w) \cup X(sw')) &= j_{(2,1,1)}(-1)[-5] \oplus j_{(2,2)}(-3)[-7] \oplus j_{(3,1)}(-4)[-8] \\ &\oplus i_G^G(-4)[-8] \oplus i_G^G(-5)[-10]. \end{aligned}$$

But these groups behave again disjointly (apart from the top cohomology) from those of $X(sw')$. Hence we get

$$\begin{aligned} H_c^*(X(w)) &= v_B^G[-5] \oplus j_{(2,1,1)}(-1)[-5] \oplus j_{(2,2)}(-2)[-6] \oplus \\ &\oplus j_{(2,2)}(-3)[-7] \oplus j_{(3,1)}(-4)[-8] \oplus i_G^G(-5)[-10]. \end{aligned}$$

For determining the cohomology of the DL-variety attached to the longest element in the Weyl group of GL_4 we refer to the appendix.

6. HYPERSQUARES

Here we generalize some of the results of the previous section to hypersquares.

For elements $v, w \in W$ with $v \leq w$, we let $I(v, w) = \{z \in W \mid v \leq z \leq w\} \subset W$ be the interval between v and w . Analogously we define $I^{F^+}(v, w) = I(v, w)$ for $v, w \in F^+$. Note that if we have fixed reduced decompositions $\dot{v}, \dot{w} \in F^+$ of $v, w \in W$, then $\gamma(I^{F^+}(\dot{v}, \dot{w})) \neq I(v, w)$ in general. Further we set for any interval $I = I(v, w)$,

$$\mathrm{head}(I) = w \text{ and } \mathrm{tail}(I) = v.$$

Definition 6.1. Let $v \leq w \in W$ with $\ell(w) - \ell(v) = d$. We say that $I(v, w)$ is a hypersquare of dimension d in W if

$$\#\{z \in I(v, w) \mid \ell(z) = \ell(w) - i\} = \binom{d}{i}$$

for all $1 \leq i \leq d$.

If $I(v, w)$ is a hypersquare, then $\#I(v, w) = 2^d$ (the converse is also true). In this case we also write $Q(v, w) = I(v, w)$.

The definition of a hypersquare in F^+ is similar but easier in the sense that for all $v, w \in F^+$ with $v \preceq w$ the cardinality of $I(v, w)$ is always $2^{\ell(w)-\ell(v)}$.

Definition 6.2. Let $v \preceq w \in F^+$ with $\ell(w) - \ell(v) = d$. The associated hypersquare of dimension d in F^+ is given by the set $Q(v, w) = I(v, w)$.

If we consider $v, w \in W$ with $v \leq w$ and with fixed reduced decompositions $w = s_{i_1} \cdots s_{i_r}$ and $v = s_{j_1} \cdots s_{j_s}$, then we also write $Q^{F^+}(v, w)$ for $Q(s_{j_1} \cdots s_{j_s}, s_{i_1} \cdots s_{i_r})$. For a hypersquare resp. interval $I = I(v, w)$ of W (resp. F^+), let

$$X(v, w) := X(I) := \bigcup_{w \in I} X(w)$$

be the induced locally closed subvariety of X (resp. of $X^{\ell(w)+1}$ where $w = \text{head}(I)$). In particular, for $w \in F^+$ the compactification $\overline{X}(w)$ of $X(w)$ can be rewritten as

$$\overline{X}(w) = X(Q(1, w)).$$

Lemma 6.3. *Let $Q \subset W$ (resp. $Q \subset F^+$) be a hypersquare. Then the variety $X(Q)$ is smooth.*

Proof. If $Q \subset F^+$, then the claim follows from Proposition 3.17 since $X(Q)$ is an open subset of $\overline{X}(\text{head}(Q))$. If $Q \subset W$, then the claim follows from [BL, Theorem 6.2.10]. Indeed, let $d := \dim Q$. By the rigidity of Q it has to coincide with the Bruhat graph $B(\text{tail}(Q), \text{head}(Q))$. Then loc.cit. says that the Schubert type analogue

$$X^{\text{Sch}}(Q) := \bigcup_{w \in Q} X^{\text{Sch}}(w)$$

(where $X^{\text{Sch}}(w)$ is the Schubert cell to w) is (rationally) smooth if each vertex in the Bruhat graph has exactly d edges. But each vertex in Q has already by definition d edges. Now the argumentation is completely analogous as for ordinary DL-varieties. In fact, next we deduce that $\mathcal{O}(Q) := \bigcup_{w \in Q} \mathcal{O}(w) \subset X \times X$ is smooth. Since $\mathcal{O}(Q)$ is transversal to the graph of the Frobenius, we see that $X(Q)$ is smooth. \square

Definition 6.4. A square $Q \subset W$ (resp. $Q \subset F^+$) is called special if it has the shape $Q = \{sw's, sw', w's, w'\}$ for some $w, w' \in W$, $s \in S$ (resp. F^+) with $\ell(w) = \ell(w') + 2$. In this case we also write $Q_w = Q_{w,s} = Q$.

The generalization of Proposition 4.5 is given by the next result.

Proposition 6.5. *Let $Q' = Q(v', w') \subset W$ be a hypersquare of dimension d . Suppose that for $s \in S$, the sets $sQ', Q's$ and $Q := sQ's$ are hypersquares, as well, and that $\ell(sw's) =$*

$\ell(w') + 2$, $\ell(sv's) = \ell(v') + 2$. Then $X(Q) \cup X(sQ')$ is an \mathbb{A}^1 -bundle over $X(Q's) \cup X(Q')$. Consequently,

$$H_c^i(X(Q) \cup X(sQ')) \cong H_c^{i-2}(X(Q's) \cup X(Q'))(-1).$$

Moreover $X(Q) \cup X(sQ') \cup X(Q's) \cup X(Q') = X(Q(v', sw's))$ is a \mathbb{P}^1 -bundle over $X(Q's) \cup X(Q')$.

Proof. The claim follows easily from the fact the hypersquare $Q \cup sQ' \cup Q's \cup Q'$ is paved by special squares together with Corollary 4.12. \square

Remark 6.6. The same statement is true if we work in F^+ where the assumptions are automatically satisfied. In particular, if $w = sw's \in F^+$, then

$$H^i(\overline{X}(w)) = H^i(\overline{X}(w's)) \oplus H^{i-2}(\overline{X}(w's))(-1)$$

for all $i \geq 2$ and $H^0(\overline{X}(w)) = H^0(\overline{X}(w's))$. Analogously, we have

$$H^i(\overline{X}(w)) = H^i(\overline{X}(sw')) \oplus H^{i-2}(\overline{X}(sw'))(-1)$$

for all $i \geq 2$ and $H^0(\overline{X}(w)) = H^0(\overline{X}(sw'))$.

This statement is already proved more generally in [DMR, Prop. 3.2.3]. In fact, the notion of a hypersquare can be expressed by using elements of the completed braid monoid \underline{B}^+ .

For later use we mention the following statement. Recall from the notation fixed in the introduction that we denote for any Γ -module V and any integer i , by $V\langle i \rangle$ the eigenspace of the arithmetic Frobenius with eigenvalues of absolute value q^i .

Lemma 6.7. *Let $w = sw's \in F^+$ with $\text{ht}(sw') = 0$. Then $H_c^{2i+1}(X(s^2, w))\langle -i \rangle = 0$.*

Proof. For proving the assertion we may assume that w is full. If $\ell(w) > \ell(\gamma(w))$ then s commutes with every simple reflection in w' . Hence the variety $X(s^2, w)$ is homeomorphic to $X(s^2) \times \overline{X}(w')$. One computes easily that

$$H_c^*(X(s^2)) = i_B^G/i_{P(s)}^G[-2] \bigoplus i_{P(s)}^G(-2)[-4].$$

Further the cohomology of $\overline{X}(w')$ vanishes in odd degree by Proposition 3.4, thus we get $H_c^{2i+1}(X(s^2, w)) = (0)$ by the Künneth formula.

So let $\ell(w) = \ell(\gamma(w))$ and suppose that $V = j_\lambda(-i) \subset H_c^{2i+1}(X(s^2, w)) \neq 0$. Suppose first that V is induced by $H_c^{2i+1}(X(w))$. By Corollary 5.9 and by Proposition 5.5 it follows that $i = \ell(w) - 2$.

1. *Case:* $w = s_1 s_2 s_3 \cdots s_{n-1} s_1$ (or $w = s_{n-1} s_1 s_2 \cdots s_{n-2} s_{n-1}$ etc.) i.e., $s = s_1$ or $s = s_{n-1}$.

By Corollary 5.9 we conclude that $j_\lambda = j_{(n-2,2)}$. We consider the square

$$Q = \{w, sv'_1s, sv'_2s, su's\} \subset F^+$$

with

$$\begin{aligned} v'_1 &= s_2s_3 \cdots s_{n-3}s_{n-2}, \\ v'_2 &= s_2s_3 \cdots s_{n-3}s_{n-1}, \\ u' &= s_2s_3 \cdots s_{n-3}. \end{aligned}$$

Now $H_c^{2i}(X(su's)) = H_c^{2\ell(su's)}(X(su's)) = i_{P(su's)}^G(-i) = i_{P_d}^G(-i)$ with $d = (n-2, 1, 1) \in \mathcal{D}$. For proving our claim in this special situation, it is enough to see that the $j_{(n-2,2)}$ -isotypic part in $H_c^{2i+1}(X(Q))$ vanishes. For this we consider the boundary map $H_c^{2i}(X(sv'_2s) \cup X(su's)) \rightarrow H_c^{2i+1}(X(sv'_1s) \cup X(w))$ and moreover the extended boundary map

$$H_c^{2i}(X(sv'_2s) \cup X(su's) \cup X(sv'_2s) \cup X(su's)) \rightarrow H_c^{2i+1}(X(w) \cup X(sv'_1s) \cup X(sv'_1s) \cup X(su's))$$

which identifies by Proposition 6.5 with

$$H_c^{2i-2}(X(v'_2s) \cup X(u's) \cup X(v'_2s) \cup X(u's))(-1) \rightarrow H_c^{2i-1}(X(w's) \cup X(v'_1s) \cup X(v'_1s) \cup X(w's))(-1).$$

By weight reasons we deduce that

$$H_c^{2i}(X(sv'_2s) \cup X(su's)) \subset H_c^{2i}(X(sv'_2s) \cup X(su's) \cup X(sv'_2s) \cup X(su's)).$$

On the other hand, we have $V \not\subset H_c^{2i}(X(sv'_1s) \cup X(su's))$ as $H_c^{2i}(X(sv'_1s)) = i_{P_{(n-1,1)}}^G(-i)$ and $H_c^{2i}(X(su's)) = j_{(n-2,1,1)}(-i)$ by Proposition 3.3. Hence it suffices to see that V does not appear in the cokernel of the extended boundary map. Now

$$V(-1) \subset H_c^{2i-2}(X(u's)) = i_{P(u's)}^G(-i+1) = i_{P_{(n-2,1,1)}}^G(-i+1)$$

resp.

$$V(-1) \subset H_c^{2i-2}(X(v'_2s)) = i_{P(v'_2s)}^G(-i+1) = i_{P_{(1,n-3,2)}}^G(-i+1).$$

On the other hand,

$$V(-1) \subset H_c^{2i-1}(X(w')) = i_{P_{(1,n-2,1)}}^G / i_{P_{(1,n-1)}}^G(-i+1),$$

$$V(-1) \subset H_c^{2i-1}(X(v'_1s)) = i_{P_{(n-2,1,1)}}^G / i_{P_{(n-1,1)}}^G(-i+1)$$

and

$$V(-1) \subset H_c^{2i-2}(X(v'_1s)) = i_{P_{(1,n-2,1)}}^G(-i+1).$$

The result follows now easily by intertwining arguments as the contribution $V(-1) \subset H_c^{2i-2}(X(v'_1s))$ maps diagonally to $H_c^{2i-1}(X(v'_1s)) \oplus H_c^{2i-1}(X(w'))$ and $H_c^{2i-2}(X(v'_2s))$ maps surjectively onto $H_c^{2i-1}(X(w'))$ by Proposition 5.7.

2. Case: $w = s_i s_1 s_2 \cdots \widehat{s_i} \cdots s_{n-1} s_i$ with $2 \leq i \leq n-2$.

By Corollary 3.8 we see that $H_c^{2i+1}(X(w))$ is a direct sum of quotients of induced representations $i_P^G(-i)$ where P is not a proper maximal subgroup. But $H_c^{2i}(X(sw'))$ does not kill $H_c^{2i+1}(X(w))$ by Prop. 5.8. Further $H_c^{2i}(X(sw')) = H_c^{2\ell(w)-4}(X(sw')) = H_c^{2\ell(sw')}(X(sw')) = i_{P(sw')}^G(-i)$ for all $v' \prec w'$ with $\ell(v') = \ell(w') - 1$, where $P(sw') \subset G$ is a proper maximal subgroup. Hence the representations of the first kind have to be killed by weight reasons by some $H_c^{2i}(X(su's))$ with $\ell(u') = \ell(w') - 2$. The claim follows.

If V is not necessarily induced by $H_c^{2i+1}(X(w))$, then we argue as follows. In the first case above we consider the subsquares $X(s_1^2, s_1s_3s_4 \dots s_{n-1}s_1)$, $X(s_1s_2s_1, s_1s_2s_4s_5 \dots s_{n-1}s_1)$, $X(s_1s_2s_3s_1, s_1s_2s_3s_5s_6 \dots s_{n-1}s_1)$ within $X(s_1^2, w)$. These are homeomorphic to $X(s_1^2) \times \overline{X(s_3s_4 \dots s_{n-1})}$, $X(s_1s_2s_1) \times \overline{X(s_4s_5 \dots s_{n-1})}$, $X(s_1s_2s_3s_1) \times \overline{X(s_5s_6 \dots s_{n-1})}$ etc. As for the first two subvarieties U we have $H_c^{2i+1}(U)\langle -i \rangle = 0$. On the other hand, for the other varieties U , it follows by induction that we have $H_c^{2i+1}(U)\langle -i \rangle = 0$. Hence we get a contradiction since there has to be one out of these varieties which induce V .

In the second case one proceeds similarly. □

7. SOME FURTHER RESULTS ON THE COHOMOLOGY OF DL-VARIETIES

We shall prove some further statements given in the introduction.

Recall we may write all “non-minimal” elements in B^+ modulo cyclic shift in the shape $w = sw's$ for some $s \in S$ and $w' \in B^+$. In particular, for $w \in F^+$ such that $\gamma(w)$ is not minimal in its conjugacy class, there is always an element $sw's \in F^+$ with $\ell(w) = \ell(sw's)$ and with $H_c^*(X(w)) = H_c^*(X(sw's))$. We shall use this property to prove the next statement.

Lemma 7.1. *Let $w \in F^+$. Then $H^i(\overline{X}(w)) = 0$ for i odd.*

Proof. By Propositions 3.4 and 3.6 we may assume that $\gamma(w)$ is not minimal in its conjugacy class. Since $\overline{X}(w)$ is smooth and projective it suffices to show that all eigenvalues of the Frobenius on the cohomology groups $H^*(\overline{X}(w))$ are integral powers of q . By considering the spectral sequence to the stratification $\overline{X}(w) = \bigcup_{v \preceq w} X(v)$ it suffices to show that this property is valid for the cohomology groups $H_c^*(X(v))$. By what we have said above, we may suppose that $v = sv's$ for some $s \in S$ and $v' \in F^+$. By induction on the length we know that the assertion is true for $H_c^*(X(v's))$ and $H_c^*(X(v'))$, hence for $H_c^*(X(v's) \cup X(v'))$. But $X(v) \cup X(sv')$ is an \mathbb{A}^1 -bundle over $X(v's) \cup X(v')$ by Remark 4.15. Thus the assertion is true for $H_c^*(X(v) \cup X(sv'))$. Finally, by considering again the corresponding long exact cohomology sequence the claim follows. □

Let $X(w)$ be a DL-variety attached to an element $w \in W$ and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of w . In order to compute the cohomology of $X(w) \cong X^{F^+}(w)$ we consider the stratification $\overline{X}(w) = \bigcup_{v \preceq w} X^{F^+}(v)$ in which $X(w)$ appears as an open stratum. Write

$$\overline{X}(w) = X^{F^+}(w) \dot{\cup} Y$$

where $Y = \bigcup_{\substack{v \prec w \\ \ell(v) = \ell(w) - 1}} \overline{X}(v)$. We consider the induced spectral sequence

$$E_1^{p,q} \implies H_c^{p+q}(X^{F^+}(w))$$

with

$$E_1^{p,q} = \bigoplus_{\substack{\{v_1, \dots, v_p\} \\ v_i \prec w, \ell(v_i) = \ell(w) - 1}} H_c^q \left(\bigcap_{i=1}^p \overline{X}(v_i) \right)$$

for $p \geq 1$ and $E_1^{0,q} = H^q(\overline{X}(w))$. Note that the intersection $\bigcap_1^p \overline{X}(v_i)$ is nothing else but $\overline{X}(v)$ where $v \in F^+$ is the unique element of length $\ell(w) - p$ with $v \preceq v_i$, $i = 1, \dots, p$.

Remark 7.2. The element v could be considered as the greatest common divisor or the meet of the elements v_1, \dots, v_p . Note that the set $Q(1, w)$ is a bounded distributive lattice.

Hence the i th row of E_1 is given by the complex

$$(7.1) \quad 0 \longrightarrow H^i(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v) = \ell(w) - 1}} H^i(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v) = \ell(w) - 2}} H^i(\overline{X}(v)) \longrightarrow \dots$$

We shall analyse this spectral sequence. As all varieties $\overline{X}(v)$ are smooth and projective their cohomology is pure. We conclude that $E_2 = E_\infty$ and hence by weight reasons that

$$H_c^i(X(w)) = \bigoplus_{p+q=i} E_2^{p,q}.$$

Proposition 7.3. *The representations $H^*(\overline{X}(w))$ and $H_c^*(X(w))$ are Frobenius semisimple for all $w \in F^+$.*

Proof. Again the proof is by induction on $\ell(w)$. The start of induction is given by Proposition 3.4. As the weights of $H^{i-1}(Y)$ are different from $H^i(\overline{X}(w))$, it is enough to prove that both of these objects are Frobenius semisimple. But by considering the E_2 -term of the obvious spectral sequence converging to the cohomology of Y and by induction hypothesis it suffices to show that $H^i(\overline{X}(w))$ is Frobenius semisimple.

Since $\overline{X}(w)$ is smooth and projective we get by Poincaré duality the identity $H^i(\overline{X}(w)) = H^{2\ell(w)-i}(\overline{X}(w))(-\ell(w)+i)$ for $i \leq \ell(w)$. So it suffices to consider the case $i \leq \ell(w)$. Further we know that $H_c^i(X(w)) = (0)$ for all $i < \ell(w)$, cf. Remark 5.10. Hence we deduce that

$H^i(\overline{X}(w)) \subset H^i(Y)$ for all $i < \ell(w)$. In the latter case the claim follows by induction considering again the spectral sequence to Y .

If $i = \ell(w)$ (is even and positive), then we consider the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^{\ell(w)-1}(Y) &\longrightarrow H_c^{\ell(w)}(X(w)) \longrightarrow H^{\ell(w)}(\overline{X}(w)) \\ &\longrightarrow H^{\ell(w)}(Y) \longrightarrow H_c^{\ell(w)+1}(X(w)) \longrightarrow 0. \end{aligned}$$

We claim that if there is some irreducible subrepresentation $V = j_\lambda(-j) \subset H_c^{\ell(w)}(X(w))$, then $j < \frac{i}{2}$. Here we may suppose that $w = sw's$. If $V \subset H^{\ell(w)-1}(Y)$, then the claim follows by weight reasons. If $V \subset H^{\ell(w)}(\overline{X}(w))$, i.e. $j = \frac{i}{2}$, then it is in the kernel of the map

$$H^{\ell(w)}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v) = \ell(w) - 1}} H^{\ell(w)}(\overline{X}(v)).$$

Since $w's$ appears as index in this direct sum, the kernel is by Remark 6.6 the same as the kernel of the map

$$H^{\ell(w)-2}(\overline{X}(w's))(-1) \longrightarrow H^{\ell(w)}(\overline{X}(sw')) \bigoplus \bigoplus_{\substack{v' \prec w's \\ \ell(v') = \ell(w') - 1}} H^{\ell(w)-2}(\overline{X}(v's))(-1).$$

In particular, it is contained in the kernel of the map

$$H^{\ell(w)-2}(\overline{X}(w's))(-1) \longrightarrow \bigoplus_{\substack{v' \prec w's \\ \ell(v') = \ell(w') - 1}} H^{\ell(w)-2}(\overline{X}(v's))(-1).$$

Since the contribution of w' is missing on the RHS, we deduce that

$$V(1) = j_\lambda(-j+1) \subset H_c^{\ell(w)-2}(X(w's) \cup X(w')).$$

But $H_c^{\ell(w)-2}(X(w's)) = (0)$, as $\ell(w') = \ell(w) - 2 < \ell(w's)$. Hence $V(1) \subset H_c^{\ell(w')}(X(w'))$. Again by induction we know that $j - 1 < \frac{\ell(w')}{2}$. But $\frac{\ell(w')}{2} = \frac{\ell(w)-2}{2} = \frac{\ell(w)}{2} - 1$. Hence we get a contradiction. \square

Corollary 7.4. *(of the proof) Let $w \in F^+ \setminus \{e\}$ and let $V = j_\lambda(-i) \subset H_c^{\ell(w)}(X(w))$ for some $\lambda \in \mathcal{P}$. Then $2i < \ell(w)$.*

Remarks 7.5. i) The latter result is proved in [DMR, Prop. 3.3.31 (iv)] for arbitrary reductive groups.

ii) It was pointed out to me by O. Dudas, that the semi-simplicity of $H^*(\overline{X}(w))$, Lemma 7.1 and the upcoming Proposition 7.15 can be deduced from [L4].

We further have the following vanishing result.

Proposition 7.6. *Let $w \in F^+$ with $\text{ht}(w) \geq 1$. Then $H_c^{2\ell(w)-1}(X(w)) = 0$.*

Proof. By Corollary 5.9 we may suppose that $\text{ht}(sw') \geq 1$. We consider the long exact cohomology sequence

$$\begin{aligned} \dots &\longrightarrow H_c^{2\ell(w)-2}(X(w) \cup X(sw')) \xrightarrow{r} H_c^{2\ell(w)-2}(X(sw')) \longrightarrow H_c^{2\ell(w)-1}(X(w)) \\ &\longrightarrow H_c^{2\ell(w)-1}(X(w) \cup X(sw')) \longrightarrow \dots \end{aligned}$$

The map $r = r_{w,sw'}^{2\ell(w)-2}$ has to be surjective since $H_c^{2\ell(w)-2}(X(sw')) = H_c^{2\ell(sw')}(X(sw')) = i_G^G(-\ell(sw'))$ is the top cohomology group. On the other hand, by Proposition 4.5 we know that $H_c^{2\ell(w)-1}(X(w) \cup X(sw')) = H_c^{2\ell(w)-3}(X(w's) \cup X(w'))(-1)$. But

$$H_c^{2\ell(w)-3}(X(w's)) = H_c^{2\ell(w's)-1}(X(sw')) = 0$$

by induction. Further $H_c^{2\ell(w)-3}(X(w')) = H_c^{2\ell(w')+1}(X(w')) = 0$. Hence we conclude the claim. \square

This vanishing result has the following consequences.

Corollary 7.7. *Let $w \in F^+$ with $\text{ht}(w) \geq 1$. Then*

$$\begin{aligned} H_c^{2\ell(w)-2}(\overline{X}(w)) &= H_c^{2\ell(w)-2}(X(w)) \bigoplus \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H_c^{2\ell(w)-2}(X(v)) \\ &= H_c^{2\ell(w)-2}(X(w)) \bigoplus \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} i_{P(v)}^G(-(\ell(w)-1)). \end{aligned}$$

Proof. We consider the long exact cohomology sequence

$$\dots \longrightarrow H^{i-1}(Y) \longrightarrow H_c^i(X(w)) \longrightarrow H^i(\overline{X}(w)) \longrightarrow H^i(Y) \longrightarrow \dots$$

which coincides in degree $i = \ell(w) - 2$ with

$$\begin{aligned} \dots &\longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H_c^{2\ell(w)-3}(X(v)) \xrightarrow{\delta} H_c^{2\ell(w)-2}(X(w)) \longrightarrow H_c^{2\ell(w)-2}(\overline{X}(w)) \\ &\longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H_c^{2\ell(w)-2}(X(v)) \longrightarrow H_c^{2\ell(w)-1}(X(w)) \longrightarrow \dots \end{aligned}$$

Now $H_c^{2\ell(w)-1}(X(w)) = 0$. If $\text{ht}(v) \geq 1$ then we conclude by induction that $H_c^{2\ell(w)-3}(X(v)) = H_c^{2\ell(v)-1}(X(v)) = 0$. Thus if this latter condition is satisfied, we are done. Otherwise w is of height one and there is some $v \prec w$ with $\text{ht}(v) = 0$. Then we deduce from Corollary 5.9 and weight reasons that the map δ vanishes. \square

Corollary 7.8. *Let $w = sw's \in F^+$ with $\text{ht}(w') \geq 1$ and $\text{supp}(w) = S$. Then*

$$H_c^{2\ell(w)-2}(X(w)) = H_c^{2\ell(w's)-2}(X(w's))(-1) \bigoplus (i_{P(w')}^G - i_G^G)(-\ell(w) + 1).$$

Proof. Since $\text{ht}(sw') \geq 1$ we have an exact sequence

$$0 \longrightarrow H_c^{2\ell(w)-2}(X(w)) \longrightarrow H_c^{2\ell(w)-2}(X(w) \cup X(sw')) \longrightarrow H_c^{2\ell(w)-2}(X(sw')) \longrightarrow 0.$$

But the assumption $\text{supp}(w) = S$ also implies that $\text{supp}(sw') = S$. Hence we get

$$H_c^{2\ell(w)-2}(X(sw')) = H_c^{2\ell(sw')}(X(sw')) = i_G^G(-\ell(w) + 1).$$

Further the identity $H_c^{2\ell(w)-2}(X(w) \cup X(sw')) = H_c^{2\ell(w)-4}(X(w's) \cup X(w'))$ is satisfied by Corollary 4.5. But since $\text{ht}(w') \geq 1$ we deduce that

$$H_c^{2\ell(w)-5}(X(w')) = H_c^{2\ell(w')-1}(X(w')) = 0.$$

Now the result follows easily. \square

We reconsider the spectral sequence

$$E_1^{p,q} \Rightarrow H^{p+q}(H^*(\overline{X}(w)))$$

which is induced by the stratification $\overline{X}(w) = \bigcup_{v \leq w} X(v)$. The q th line in the E_1 -term is the complex

$$\bigoplus_{\substack{v' \leq w \\ \ell(v')=q}} H_c^{2q}(X(v')) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v' \leq w \\ \ell(v')=q+j}} H_c^{2q+j}(X(v')) \longrightarrow \bigoplus_{\substack{v \leq w \\ \ell(v)=q+j+1}} H_c^{2q+j+1}(X(v)) \longrightarrow \cdots$$

where the homomorphisms are induced by the boundary maps $\delta_{v',v}$.

Picture:

$$\begin{array}{cccccccc}
 & & & & & & \vdots & \vdots \\
 & & & & & & H^{i+2} & H^{i+3} \dots \\
 & & H^4 & H^5 & H^6 & \dots & & \\
 & H^2 & H^3 & H^4 & H^5 & \dots & H^{i+1} & H^{i+2} \dots \\
 H^0 & H^1 & H^2 & H^3 & H^4 & \dots & H^i & H^{i+1} \dots \\
 \hline
 e & \ell = 1 & \ell = 2 & \ell = 3 & \ell = 4 & \dots & \ell = i & \ell = i + 1
 \end{array}$$

Of course this spectral sequence degenerates and we may write by weight reasons and by Proposition 7.3 for all $0 \leq i \leq \ell(w)$,

$$H^{2i}(\overline{X}(w)) = \bigoplus_{j=i}^{\ell(w)} H_c^{2i}(X(w)(j))'$$

where

$$X(w)(j) := \bigcup_{\substack{v \preceq w \\ \ell(v)=j}} X(v)$$

and where $H_c^{2i}(X(w)(j))' \subset H_c^{2i}(X(w)(j)) = \bigoplus_{\substack{v \preceq w \\ \ell(v)=j}} H_c^{2i}(X(v))$.

For $v \preceq w$ with $\ell(v) = i$, we have $H_c^{2i}(X(v)) = H^{2i}(\overline{X}(v)) = i_{P(v)}^G(-i)$. Here $P(v) \subset G$ is the std psgp attached to v , cf. (2.6.) By Remark 4.10 the trivial representation does appear in the top cohomology degree of a DL-variety. Hence the subrepresentation $i_G^G(-i) \subset i_{P(v)}^G(-i)$ survives the spectral sequence. Thus there is grading

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(v)=i}} H(w)_z$$

with $H(w)_z \supset i_G^G(-i)$ for certain representations $H(w)_z$.

In the sequel we shall see that we may suppose that the objects $H(w)_z$ are induced representations from parabolic subgroups. Recall that the following three operations C, K, R on elements in F^+ allows us to transform an arbitrary element $w \in F^+$ into the shape $w = sw's$ with $s \in S$ and $w' \in F^+$.

(I) (Cyclic shift) If $w = sw'$ with $s \in S$, then we set $C(w) = w's$.

(II) (Commuting relation). If $w = w_1stw_2$ with $s, t \in S$ and $st = ts$. Then we set $K(w) = w_1tsw_2$.

(III) (Replace sts by tst) If $w = w_1stsw_2$ with $s, t \in S$ and $sts = tst$. Then we set $R(w) = w_1tstw_2$.

We shall analyse the induced behaviour on the cohomology of Demazure varieties. In [DMR] the following generalization of Proposition 4.3 is proved.

Proposition 7.9. *Let $w = s_{i_1} \cdots s_{i_r} \in \hat{F}^+$. Then for all $i \geq 0$, there are isomorphisms of H -modules*

$$H_c^i(X(w)) \longrightarrow H_c^i(X(C(w)))$$

and

$$H^i(\overline{X}(w)) \longrightarrow H^i(\overline{X}(C(w)))$$

Proof. These are special cases of [DMR, Proposition 3.1.6]. □

Corollary 7.10. *Let $sw's \in F^+$. Then $H^i(\overline{X}(sw's)) = H^i(\overline{X}(w's))$ for all $i \geq 0$.* □

Remark 7.11. Consider the equivalence relation \sim on F^+ leading to cyclic shift classes in the sense of [GP], i.e., $v, w \in F^+$ are equivalent if there is some integer $i \geq 0$ with $C^i(w) = v$.

Thus we can associate to any element in $C^+ = F^+ / \sim$ its cohomology. Moreover, the height function ht on F^+ depends only on the cycle shift class. Sometimes it is useful to interpret in what follows the image of an element $w = s_{i_1} s_{i_2} \cdots s_{i_{r-1}} s_{i_r}$ in C^+ as a circle, i.e.,

$$\begin{array}{ccc} & s_{i_1} & \\ & \cdot & \\ s_{i_r} & & s_2 \\ \vdots & & \vdots \\ s_{i_{j+1}} & & s_{i_{j-1}} \\ & s_{i_j} & \end{array}$$

As we have observed above the Cyclic shift operator does not affect the cohomology of a Demazure variety. The same holds true for operation (II).

Proposition 7.12. *Let $w = w_1 s t w_2 \in \hat{F}^+$ with $s, t \in S$ (or $s \in S, t \in \hat{S}$) such that $st = ts$. Set $K(w) = w_1 t s w_2$. Then $H^i(\overline{X}(w)) = H^i(\overline{X}(K(w)))$ for all $i \geq 0$.*

Proof. This property is implicitly contained in the definition of a generalized Deligne-Lusztig variety attached to elements of the completed Braid monoid [DMR]. The reason is that the stratifications of the varieties $\overline{X}(w)$ and $\overline{X}(K(w))$ are essentially the same in the obvious sense. \square

Let $w = w_1 s t s w_2 \in F^+$. In the sequel we also write $w = w_1 s_l t s_r w_2$ (l for left, r for right) in order to distinguish the reflection s in its appearance in w . Further we write $w_1 s^2 w_2$ for the subword $w_1 s_l s_r w_2$.

Proposition 7.13. *Let $w = w_1 s t s w_2 \in \hat{F}^+$ with $s, t \in S$ and $st \neq ts$ in W . Then there is a canonical decomposition $H^{2i}(\overline{X}(w)) = H^{2i}(X(w_1 \widehat{st} s w_2)) \oplus H^{2i-2}(X(w_1 s w_2))(-1)$.*

Proof. This is a special situation of [DMR, Prop. 3.2.9]. The argument is that the proper morphism $\pi : \overline{X}(w) \rightarrow \overline{X}(w_1 \widehat{st} s w_2)$ by forgetting appropriate entries is an isomorphism over the open subset $\overline{X}(w_1 \widehat{st} s w_2) \setminus \overline{X}(w_1 s w_2)$ and induces over $\overline{X}(w_1 s w_2)$ a \mathbb{P}^1 -bundle. As both varieties are smooth and projective the claim follows from considering long exact cohomology sequences. \square

Corollary 7.14. *Let $s, t \in S$, $w_1, w_2 \in \hat{F}^+$ such that $st \neq ts$ in W . Set $w = w_1 s t s w_2$ and $v = R(w) = w_1 t s t w_2$. Then for all $i \geq 0$, there is an identity*

$$\begin{aligned} H^i(\overline{X}(v)) &= H^i(\overline{X}(w)) - H_c^i(X(s_l, w_1 s^2 w_2)) + H_c^i(X(t_r, w_1 t^2 w_2)) \\ &= H^i(\overline{X}(w)) - H^{i-2}(\overline{X}(w_1 s w_2))(-1) + H^{i-2}(\overline{X}(w_1 t w_2))(-1). \end{aligned}$$

(Here we mean as before when writing $-H^{i-2}(\overline{X}(w_1sw_2))(-1)$ that $H^{i-2}(\overline{X}(w_1sw_2))(-1)$ appears canonically as a submodule in $H^i(\overline{X}(w))$.)

Proposition 7.15. *Let $w \in F^+$. Then for all $i \geq 0$, the cycle map $A^i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} \rightarrow H^{2i}(\overline{X}(w))$ is an isomorphism (where $A^i(\overline{X}(w))$ is the Chow group of $\overline{X}(w)$ in degree i).*

Proof. We may assume that $\text{supp}(w) = S$. If w is a Coxeter element, then the claim follows from Remark 3.5. If $w = sw's$ then $A^i(\overline{X}(w)) = A^i(\overline{X}(w's)) \oplus A^{i-1}(\overline{X}(w's))$ and the claim follows by induction on $\ell(w)$.

In general we have to consider the induced behaviour of the operations K, C, R on the Chow groups. First we observe that $A^i(\overline{X}(w)) = A^i(\overline{X}(C(w)))$ since the generating cycle classes are rational. Further $A^i(\overline{X}(w)) = A^i(\overline{X}(K(w)))$. Let $w = w_1stsw_2$, $R(w) = w_1tstw_2$, $y = w_1\widehat{st}sw_2$ and suppose that the claim is true for w . Then we consider as in Proposition 7.13 the map $\pi : \overline{X}(w) \rightarrow \overline{X}(y)$ which induces short exact sequences

$$0 \rightarrow H^{2i-4}(\overline{X}(w_1sw_2))(-2) \rightarrow H^{2i}(\overline{X}(y)) \oplus H^{2i-2}(\overline{X}(w_1s^2w_2))(-1) \rightarrow H^{2i}(\overline{X}(w)) \rightarrow 0$$

and

$$0 \rightarrow A^{i-2}(\overline{X}(w_1sw_2)) \rightarrow A^i(\overline{X}(y)) \oplus A^{i-1}(\overline{X}(w_1s^2w_2)) \rightarrow A^i(\overline{X}(w)) \rightarrow 0.$$

It follows by induction on the length that the cycle map for $\overline{X}(y)$ is an isomorphism, as well. Considering both exact sequences for $R(w)$ and using again induction the claim is true for $R(w)$. \square

Remark 7.16. For $m \geq 0$, the Tate twist $-i$ contribution of $H_c^{2i+m}(X(w))$ is Bloch's higher Chow group $CH^i(X(w), m)_{\overline{\mathbb{Q}}_\ell}$, cf. [Bl]. Indeed, this follows from the spectral sequence (7.1).

For $w \in F^+$ and $z \preceq w$, we denote by $i_{z,w} : A_{\ell(z)}(\overline{X}(z)) \rightarrow A_{\ell(z)}(\overline{X}(w))$ the map induced by the inclusion $\overline{X}(z) \subset \overline{X}(w)$. Moreover, we let $[\overline{X}(z)] \in A_{\ell(z)}(\overline{X}(z))$ be the sum of all the irreducible components appearing in $\overline{X}(z)$.

Definition 7.17. Let $w \in \widehat{F}^+$. A grading $A_i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} = \bigoplus_{\substack{z \preceq w \\ \ell(z)=i}} A(w)_z$ is called geometrical if there is an order z_1, \dots, z_r on the set $\{z \preceq w \mid \ell(z) = i\}$ such that

$$\bigoplus_{j=1, \dots, k} A(w)_{z_j} \supset \sum_{j=1, \dots, k} (i_{z_j, w})_*([\overline{X}(z_j)])$$

for all $k \leq r$.

By using the cycle map we may speak of geometrical gradings on $H^{2i}(\overline{X}(w))$.

Let $w = sw's \in F^+$ as before. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & H_c^{2i}(X(s^2, w)) & \longrightarrow & H_c^{2i}(X(s, w)) & \longrightarrow & H_c^{2i}(X(s, w's)) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (7.2) \quad \cdots & \longrightarrow & H_c^{2i}(X(s, w)) & \longrightarrow & H^{2i}(\overline{X}(w)) & \longrightarrow & H^{2i}(\overline{X}(w's)) & \longrightarrow \cdots \\
 & & \downarrow g & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & H_c^{2i}(X(s, sw')) & \longrightarrow & H^{2i}(\overline{X}(sw')) & \xrightarrow{f} & H^{2i}(\overline{X}(w')) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

where the maps in this diagram are the natural ones. We slightly generalize this setup as follows. Let $v = v_1 r v_2$ with $r \in S$ and $u = v_1 v_2$. Set $\check{v} = v_1 r^2 v_2 = v_1 r_l r_r v_2$ and $\bar{v} = v_1 r_r v_2$. Then $v = v_1 r_l v_2$ and we may form the following diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & H_c^{2i}(X(r^2, \check{v})) & \longrightarrow & H_c^{2i}(X(r_r, \check{v})) & \longrightarrow & H_c^{2i}(X(r_r, \bar{v})) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (7.3) \quad \cdots & \longrightarrow & H_c^{2i}(X(r_l, \check{v})) & \longrightarrow & H^{2i}(\overline{X}(\check{v})) & \longrightarrow & H^{2i}(\overline{X}(\bar{v})) & \longrightarrow \cdots \\
 & & \downarrow g & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & H_c^{2i}(X(r_l, v)) & \longrightarrow & H^{2i}(\overline{X}(v)) & \xrightarrow{f} & H^{2i}(\overline{X}(u)) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

In the setting of Chow groups and Poincaré duality the above diagram reads as

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \longrightarrow & A_i(X(r_r, \bar{v})) & \longrightarrow & A_i(X(r_r, \check{v})) & \longrightarrow & A_i(X(r^2, \check{v})) & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 (7.4) \quad \cdots & \longrightarrow & A_i(\overline{X}(\bar{v})) & \longrightarrow & A_i(\overline{X}(\check{v})) & \longrightarrow & A_i(X(r_l, \check{v})) & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow g' \\
 \cdots & \longrightarrow & A_i(\overline{X}(u)) & \xrightarrow{f'} & A_i(\overline{X}(v)) & \longrightarrow & A_i(X(r_l, v)) & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

The next result generalizes Proposition 3.13 to arbitrary pairs of elements $v, u \in F^+$ with $u \prec v$ and $\ell(v) = \ell(u) + 1$.

Theorem 7.18. *a) The cohomology of $\overline{X}(v)$ in degree $2i$ can be written as*

$$H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} H(v)_z$$

with $H(v)_z = i_{P_z^v}^G(-i)$ for certain standard parabolic subgroups $P_z^v \subset G$.

b) There are geometric gradings

$$H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} H(v)_z \text{ and } H^{2i}(\overline{X}(u)) = \bigoplus_{\substack{z \preceq u \\ \ell(z)=i}} H(u)_z$$

by induced representations as in part a), such that the map $f : H^{2i}(\overline{X}(v)) \rightarrow H^{2i}(\overline{X}(u))$ is in diagonal form, i.e. it coincides with the graded one. Further the induced homomorphisms $H(v)_z \rightarrow H(u)_z$ are injective or surjective for all $z \preceq u$.

c) There is a geometric grading as in part a)

$$H_c^{2i}(X(r_l, \check{v})) = H^{2i-2}(\overline{X}(\check{v}))(-1) = \bigoplus_{\substack{z' \preceq \check{v} \\ \ell(z')=i-1}} H(\check{v})_{z'}$$

such that $H(\check{v})_{z'}(-1) = H(v)_{z_1 r z_2}$ for all elements $z' = z_1 z_2 \preceq u$.

We leave the translation of this theorem for covariant Chow groups to the reader. We start with the following lemma.

Lemma 7.19. *Suppose that there are gradings as in Theorem 7.18 b). Then if we start with an arbitrary geometric grading on $H^{2i}(\overline{X}(u))$, then we can define a geometric grading on $H^{2i}(\overline{X}(v))$ such that f is graded with respect to these new gradings, as well.*

Proof. We consider the viewpoint of Chow groups. Let $A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} = \bigoplus_{\substack{z \preceq u \\ \ell(z)=i}} A(u)'_z$ be another grading. Then by Corollary 2.3 for any $z \preceq u$ there is an element $z' = \phi(z)$ with $z' \preceq u$, $\ell(z) = \ell(z') = i$ and with $A(u)_z \cong A(u)'_{z'}$. The map $\pi_{z, \phi(z)}^u : A(u)_z \xrightarrow{\sim} A(u)'_{\phi(z)}$ which is induced by the neutral element, cf. Remark 2.2, and the choice of P_z^u -invariant generating cycle classes for the induced representations $i_{P_z^u}^G$ gives rise to such an isomorphism.

Suppose first that the maps $A(u)_z \rightarrow A(v)_z$ are surjective for all $z \preceq u$ with $\ell(z) = i$. As G acts semi-simple we may consider $A(v)_z$ as a subrepresentation of $A(u)_z$. We define a new grading on $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell} = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} A(v)'_z$ by setting

$$A(v)'_{\phi(z)} := \pi_{z, \phi(z)}^u(A(v)_z)$$

for $z \preceq u$ and $A(v)'_z := A(v)_z$ if $z \not\preceq u$. We set $\pi^u := \bigoplus_{\substack{z \prec u \\ \ell(z)=i}} \pi_{z,\phi(z)}^u$ so that we get an endomorphism

$$\pi^u : A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} \longrightarrow A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell}.$$

Analogously we define an endomorphism $\pi^v := \bigoplus_{\substack{z \prec v \\ \ell(z)=i}} \pi_{z,\phi(z)}^v$ of $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ with $\pi_{z,\phi(z)}^v = \text{id}$. Here we have extended ϕ to a function on the set $\{z \preceq v, \ell(z) = i\}$ by $\phi(z) = z$ for $z \not\preceq u$. We get a commutative diagram

$$\begin{array}{ccc} A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell} & \xrightarrow{\pi^v} & A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell} \\ \uparrow f' & & \uparrow f' \\ A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} & \xrightarrow{\pi^u} & A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} \end{array}$$

Now the statement follows easily.

In general we divide the set $\{z \preceq u \mid \ell(z) = i\} = A \dot{\cup} B$ into two disjoint subsets where A consists of those $z \preceq u$ such that the map $A(u)_z \rightarrow A(v)_z$ is surjective. To define a new grading on $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ we proceed with the set A as above. In particular the set A covers the kernel of the map $f' : A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} \rightarrow A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$. Hence for $z \in B$ the map $A(u)_{\phi(z)} \rightarrow A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ is (strictly) injective. Then there is some induced representation $i_P^G \subset A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ which contains the image of the latter map ². We let i_P^G be a constituent of the disired grading. But the map $\bigoplus_{z \in B} A(u)_{\phi(z)} \rightarrow A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ is injective, as well. The claim of the lemma follows by applying the former procedure sucessively. \square

Proof. Part a) and b). The last statement of part b) is a consequence of Remark 2.2 ii). The remaining proof is by induction on $\ell(v)$. By Remark 3.19 and Künneth arguments we may suppose that v has full support. If v is a Coxeter element then the statement follows from Propositions 3.4, 3.13. So let $\text{ht}(v) \geq 1$.

1. Case. $v = sv's$ for some $s \in S$.

Hence $H^{2i}(\overline{X}(v)) = H^{2i}(\overline{X}(v's)) \oplus H^{2i-2}(\overline{X}(v's))(-1)$. As for part a) we may write by induction

$$H^*(\overline{X}(v's)) = \bigoplus_{z \preceq v's} i_{P_z^{v's}}^G(-\ell(z))[-2\ell(z)].$$

Then

$$H^*(\overline{X}(v)) = \bigoplus_{z \preceq v} i_{P_z^v}^G(-\ell(z))[-2\ell(z)]$$

where $P_z^v = P_z^{v's}$ if $z \preceq v's$ and $P_z^v = P_{z'}^{sv'}$ if $z \in Q(sv', v)$. Here $z = sz'$. Concerning part b) we distinguish the following cases:

²Indeed let $i_P^G \rightarrow i_Q^G \oplus i_R^G$ be an injective map. Then we may suppose w.l.o.g. that $i_P^G \rightarrow i_Q^G$ is injective, as well. We may extend the first map to an injection $i_Q^G \rightarrow i_Q^G \oplus i_R^G$ and the graph contains i_P^G .

Subcase a). $u = v's$ (The case $u = sv'$ is symmetric to this one).

Here the proof is trivial since the map $H^{2i}(\overline{X}(v)) \rightarrow H^{2i}(\overline{X}(u))$ identifies with the projection map.

Subcase b). $u = su's$.

Then $H^{2i}(\overline{X}(u)) = H^{2i}(\overline{X}(u's)) \oplus H^{2i-2}(\overline{X}(u's))(-1)$. The statements follow now by induction with respect to the homomorphism $H^j(\overline{X}(v's)) \rightarrow H^j(\overline{X}(u's))$ with $j \in \{2i - 2, 2i\}$.

2. Case. v is arbitrary.

Then we apply the operations (I) - (III) (introduced before Proposition 7.9) to arrange v in the shape as in the first case. Here we use inner induction on the necessary operations. So let $w \in F^+$ and suppose that the statements are true for w and for all $x \prec w$ with $\ell(x) = \ell(w) - 1$.

The operations (I) and (II) are easy to handle by Propositions 7.9 and 7.12:

(I) Let $w = sw'$, with $s \in S$ and $w' \in F^+$ and $w's = C(w)$. Let $\tilde{C} : H^{2i}(\overline{X}(w)) \rightarrow H^{2i}(\overline{X}(C(w)))$ be the cyclic shift isomorphism. Define for $z \preceq w$,

$$C(z) = \begin{cases} z's & \text{if } z = sz' \\ z & \text{if } z \preceq w'. \end{cases}$$

Then the assignment $H(C(w))_{C(z)} := \tilde{C}(H(w)_z)$ for $\ell(z) = i$, defines the desired grading on $H^{2i}(\overline{X}(C(w)))$. In the same way we get a grading on $H^{2i}(\overline{X}(C(x)))$. Moreover, the homomorphism $H^{2i}(\overline{X}(C(w))) \rightarrow H^{2i}(\overline{X}(C(x)))$ is graded.

(II) Let $w = w_1stw_2$ and $K(w) = w_1tsw_2$. Let $\tilde{K} : H^{2i}(\overline{X}(w)) \rightarrow H^{2i}(\overline{X}(K(w)))$ be the induced isomorphism. Define for $z \preceq w$,

$$K(z) = \begin{cases} v_1tstv_2 & \text{if } z = v_1stv_2 \\ z & \text{if } st \not\prec z. \end{cases}$$

Then the assignment $H(K(w))_{K(z)} := \tilde{K}(H(w)_z)$ for $\ell(z) = i$, defines the desired grading on $H^{2i}(\overline{X}(K(w)))$. Part b) is proved in the same way as above.

(III) So let $w = w_1stsw_2$ and $v = R(w) = w_1tstw_2$. Set $\hat{w} = \hat{v} = w_1\widehat{st}sw_2$.

We start with the observation that we have a geometrical grading on $H^{2i}(\overline{X}(\hat{w}))$. Indeed by induction hypothesis applied to w and $x = w_1s^2w_2$, we have gradings on $H^{2i}(\overline{X}(w))$ and $H^{2i}(\overline{X}(w_1s^2w_2))$ such that the natural map $H^{2i}(\overline{X}(w)) \rightarrow H^{2i}(\overline{X}(w_1s^2w_2))$ is in diagonal form. In particular it follows that $H(w)_z = H(w_1s^2w_2)_z$ for all $z \in Q(s, w_1s^2w_2)$ (since $\overline{X}(\hat{w}) \setminus \overline{X}(w_1sw_2) = \overline{X}(w) \setminus \overline{X}(w_1s^2w_2)$) and henceforth that we have a geometrical grading on $H^{2i}(\overline{X}(\hat{w}))$ by induced representations. Hence we see that we have such a grading

on $H^{2i}(\overline{X}(v)) = H^{2i}(\overline{X}(\hat{v})) \oplus H^{2i-2}(\overline{X}(w_1 t w_2))$. More precisely, we set for $z_1 \preceq w_1$ and $z_2 \preceq w_2$,

$$R(z) = \begin{cases} z_1 t s t z_2 & \text{if } z = z_1 s t s z_2 \\ z_1 t_l s z_2 & \text{if } z = z_1 t s_r z_2 \\ z_1 s t_r z_2 & \text{if } z = z_1 s_l t z_2 \\ z_1 z_2 & \text{if } z = z_1 z_2. \end{cases}$$

Then the assignment $H(R(w))_{R(z)} := H(w)_z$ for $z \notin Q(s_l, z_1 s^2 z_2)$ and $H(v)_z = H(v/t_r)_{z/t_r}$ for $z \in Q(t_r, w_1 t^2 w_2)$ with $\ell(z) = i$ defines the desired grading on $H^{2i}(\overline{X}(v))$.

Concerning part b) we distinguish the following situations.

Subcase a). $u = R(x) = w_1 t s w_2$. (The case $u = R(x) = w_1 s t w_2$ behaves symmetrically)

We consider the viewpoint of Chow groups. By induction hypothesis there are geometric gradings on $A_i(\overline{X}(w_1 w_2))_{\overline{\mathbb{Q}}_\ell}$ and $A_i(\overline{X}(w_1 t w_2))_{\overline{\mathbb{Q}}_\ell}$ such that the homomorphisms $A_i(\overline{X}(w_1 w_2))_{\overline{\mathbb{Q}}_\ell} \rightarrow A_i(\overline{X}(w_1 t w_2))_{\overline{\mathbb{Q}}_\ell}$ induced by the inclusion is in diagonal form. Further we may suppose that we have a geometric grading on $A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell}$ such that the map $A_i(\overline{X}(w_1 t w_2))_{\overline{\mathbb{Q}}_\ell} \rightarrow A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell}$ is in diagonal form, as well. Let $f_w : A_i(\overline{X}(u)) \rightarrow A_i(\overline{X}(w))$ resp. $f_v : A_i(\overline{X}(u)) \rightarrow A_i(\overline{X}(v))$ be the homomorphisms induced by the inclusion. Again by assumption there is a grading $A_i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} = \bigoplus_z A(w)_z$ such the map f_w is graded. We have a natural commutative diagram

$$\begin{array}{ccc} & \overline{X}(w) & \\ & \nearrow \downarrow \pi_w & \\ \overline{X}(w_1 s t w_2) & \longrightarrow & \overline{X}(\hat{w}). \end{array}$$

where $\pi_w : \overline{X}(w) \rightarrow \overline{X}(\hat{w})$ is the map of Lemma 7.13. It follows that the map $A_i(\overline{X}(u)) \rightarrow A_i(\overline{X}(\hat{w}))$ is graded, as well, if we consider on $A_i(\overline{X}(\hat{w}))_{\overline{\mathbb{Q}}_\ell}$ the induced grading, i.e., $A(\hat{w})_z = \max\{\pi_w(A(w)_{z_1 s_l z_2}), \pi_w(A(w)_{z_1 s_r z_2})\}$ for $z = z_1 s z_2 \preceq \hat{w}$ and $A(\hat{w})_z = \pi_w(A(w)_z)$ for the remaining $z \preceq \hat{w}$, cf. Remark 2.2 and [Fu, Prop. 6.7]. In order to define the grading on $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ we consider again the splitting

$$(7.5) \quad A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell} = A_i(\overline{X}(\hat{w}))_{\overline{\mathbb{Q}}_\ell} \oplus A_{i-1}(\overline{X}(w_1 t_r w_2))_{\overline{\mathbb{Q}}_\ell}.$$

Now we apply the induction hypothesis to deduce the existence of a grading on the vector space $A_{i-1}(\overline{X}(w_1 t_r w_2))_{\overline{\mathbb{Q}}_\ell}$ such that under the map

$$A_i(\overline{X}(w_1 t_l w_2))_{\overline{\mathbb{Q}}_\ell} \rightarrow A_{i-1}(\overline{X}(w_1 t_r w_2))_{\overline{\mathbb{Q}}_\ell}$$

we have

$$A(w_1 t_l w_2)_z = A(w_1 t_r w_2)_{z/t_l}$$

for all z with $t_l \mid z$. But the latter map factorizes over the map

$$r : A_i(\overline{X}(w_1 t s w_2))_{\overline{\mathbb{Q}}_\ell} \longrightarrow A_{i-1}(\overline{X}(w_1 t_r w_2))_{\overline{\mathbb{Q}}_\ell}.$$

Moreover, the contributions $A(w_1 t_l w_2)_{z/t_r}$ with $t_l \mid z$ do not lie in the image of the map r since the trivial subrepresentations $i_G^G \subset A(w_1 t_r w_2)_{z/t_l}$ are linearly independent from those of $A_i(\overline{X}(w_1 t s w_2))_{\overline{\mathbb{Q}}_\ell}$, as one deduces from [L4, Prop. 2.7]. The claim follows.

Subcase b). $u = R(x) = v_1 t s t w_2$ with $x = v_1 s t s w_2$ (or $u = R(x) = w_1 t s t v_2$).

Here the result follows by writing f_v as the sum of the homomorphisms

$$H^{2i}(\overline{X}(\hat{v})) \longrightarrow H^{2i}(\overline{X}(\hat{u}))$$

and

$$H^{2i-2}(\overline{X}(w_1 t w_2))(-1) \longrightarrow H^{2i-2}(\overline{X}(v_1 t w_2))(-1)$$

where $\hat{u} = v_1 \widehat{t s t} w_2$.

Subcase c). $u = w_1 t^2 w_2$.

Here the homomorphism $H^{2i}(\overline{X}(v)) \longrightarrow H^{2i}(\overline{X}(w_1 t^2 w_2))$ contracts by deleting the summand $H^{2i-2}(\overline{X}(w_1 t w_2))(-1)$ on both sides to the map $H^{2i}(\overline{X}(\hat{v})) \longrightarrow H^{2i}(\overline{X}(w_1 t w_2))$. The latter one factorizes over $H^{2i}(\overline{X}(w_1 t s w_2))$. By induction the maps $H^{2i}(\overline{X}(\hat{w})) \longrightarrow H^{2i}(\overline{X}(w_1 t s w_2))$ and $H^{2i}(\overline{X}(w_1 t s w_2)) \longrightarrow H^{2i}(\overline{X}(w_1 t w_2))$ are graded. By Lemma 7.19 the bases can be chosen in a compatible way. Hence the composite map is graded.

Alternatively one can avoid this case by the proceeding Lemma 7.20.

Part c). The remaining part of the theorem can be proved by induction. Here Lemma 6.7 serves as the start of the induction process. However, part c) is already a consequence of the diagram (7.4). To explain this we consider the viewpoint of Chow groups.

As in part b) we choose geometric gradings on $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ and $A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell}$ such that the natural homomorphism $A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell} \longrightarrow A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ is graded. We need to show that the intersection of $A_i(\overline{X}(v))_{\overline{\mathbb{Q}}_\ell}$ and $A_i(\overline{X}(\bar{v}))_{\overline{\mathbb{Q}}_\ell}$ within $A_i(\overline{X}(\check{v}))_{\overline{\mathbb{Q}}_\ell}$ is as small as possible, i.e., induced by the image of $A_i(\overline{X}(u))_{\overline{\mathbb{Q}}_\ell}$. For this, suppose that there is a constituent $A(v)_z = i_P^G$ with $r \mid z$ such that an irreducible subrepresentation j_λ of i_P^G lies in $A_i(\overline{X}(\bar{v}))_{\overline{\mathbb{Q}}_\ell}$. Let $x \in j_\lambda$ and write $x = y_1 + y_2$ with uniquely determined elements $y_1 \in A_i(\overline{X}(\bar{v}))_{\overline{\mathbb{Q}}_\ell}$ and $y_2 \in A_i(\overline{X}(r, \check{v}))_{\overline{\mathbb{Q}}_\ell}$. Here we have fixed the natural splittings. By what we saw in Section 4, it follows that $y_1 = \sigma_*(x)$ for the cyclic shift map $\sigma : \overline{X}(v) \longrightarrow \overline{X}(\bar{v})$. But the map σ is an isomorphism, more precisely we have by Proposition 4.3 that $\sigma \circ \tau$ is a scalar multiple of the identity. It follows that σ is up to a scalar a multiple of the morphism induced by the identity element $e \in W$. The upshot is that if j_λ lies in $A_i(\overline{X}(\bar{v}))_{\overline{\mathbb{Q}}_\ell}$ the same is true for the induced representation $i_{P_\lambda}^G \subset i_P^G$. But all the trivial subrepresentations

$i_G^G = [\overline{X}(y)]_{\overline{\mathbb{Q}}_\ell} \subset A(\check{v})_y$ induced by the cycle classes $\overline{X}(y)$ with $y \preceq \check{v}$ and $\ell(y) = i$ are linear independent in $A_i(\overline{X}(\check{v}))_{\overline{\mathbb{Q}}_\ell}$ by the result of Lusztig [L4, Prop. 2.7]. Hence we get a contradiction. \square

Let $u, v \in F^+$ with $u \prec v$ with $\ell(u) = \ell(v) - 1$. There is the obvious notion of simultaneous transformation applied to the pair (v, u) with respect to the operations (I) - (III), as long as the corresponding subword s, st, sts is part of u , as well. Apart from the critical situation where $v = v_1stsv_2$ and $u = v_1ssv_2$, we extend the simultaneous transformation to the tuple (v, u) by letting act C, K, R trivially on u .

Lemma 7.20. *Let $u, v \in F^+$ with $u \prec v$, $\ell(u) = \ell(v) - 1$ and $\text{ht}(v) \geq 1$. Then there exists a sequence $(v_i, u_i), i = 0, \dots, n$, of such tuples such that $(v, u) = (v_0, u_0)$, v_n is of the shape $sv's$ and (v_{i+1}, u_{i+1}) is induced simultaneously from (v_i, u_i) via one of the operations (I) - (III). Here the situation that $(v_i, u_i) = (v_1stsv_2, v_1s^2v_2)$ and $(v_{i+1}, u_{i+1}) = (v_1tstv_2, v_1t^2v_2)$ does not occur.*

Proof. By Theorem 4.2 resp. Lemma 5.2 we may transform v into the desired shape. We only have to analyse the critical case which we want to avoid. After a series of cyclic shifts we may then suppose that $v = tsv's$ and $u = sv's$. If $\gamma(u) \notin W$, then we may transform by Lemma 5.2 u without cyclic shifts into the shape $u_1r^2u_2$ for some $r \in S$. But then v can be transformed into the word $tu_1r^2u_2$ and the claim follows using cyclic shifts again. If $\gamma(u) \in W$ but $\gamma(v) \notin W$, then we may write $\gamma(u) = tw'$ for some $w' \in W$ and we are done again.

So it remains to consider the case where $\gamma(v) \in W$. We denote for any $z \in F^+$ and any $t = s_k \in S$ by $m_t(z) = m_k(z) \in \mathbb{Z}_{\geq 0}$ its multiplicity in z .

Let $s = s_{m+1}$ and $t = s_m$. Choose under all successive transformations (without using the forbidden replacement $sts \rightsquigarrow tst$) and all cyclic shifts (if we get an element which is not reduced, we are done by the first case) starting with v an element z such that $m(z) := (m_{n-1}(z), m_{n-2}(z), \dots, m_2(z), m_1(z))$ is maximal for the lexicographical order. Then let $1 \leq i \leq n - 1$ be the unique index with $m_i(z) > m_k(z)$ for all $k > i$ and $m_i(z) \geq m_k(z)$ for all $k \leq i$. Moreover, let $j \leq i$ such that $m_i(z) = m_{i-1}(z) = \dots = m_j(z)$. We claim that z has up to cyclic shift the desired shape $s_jz's_j$. Suppose that the claim is wrong, i.e. $z = z_1s_jz_2s_jz_3$ with $z_1 \neq e$ or $z_3 \neq e$ and $s_j \notin \text{supp}(z_1) \cup \text{supp}(z_3)$.

1st Case: $j > 1$. Consider two consecutive simple reflections s_j together with the corresponding subword $s_ju's_j$ of z . If there is no simple reflection s_{j-1} appearing in u' there

must be (as $\gamma(z) \in W$) some simple reflection s_{j+1} in between. Hence we can finally replace an expression of the shape $s_k s_{k+1} s_k$ in $s_j u s_j$ with $k \geq j$ by $s_{k+1} s_k s_{k+1}$. This gives a contradiction to the maximality of $m(z)$. On the other hand since $m_{j-1} < m_j$ there cannot be modulo cyclic shift such a reflection between any such consecutive pair.

2nd Case: $j = 1$. In this case we easily see that it is possible to increase the lexicographical order by a replacing an expression $s_1 s_2 s_1$ by $s_2 s_1 s_2$. Hence we get a contradiction, too.

The case $s = s_m$ and $t = s_{m+1}$ behaves symmetrically using the lexicographical order on $(m_1(z), m_2(z), \dots, m_{n-2}(z), m_{n-1}(z))$.

Example 7.21. a) Let $(1, 5) = (1, 2)(2, 3)(3, 4)(4, 5)(3, 4)(2, 3)(1, 2) = sw's$. Then $sw' = (1, 2)(2, 3)(3, 4)(4, 5)(3, 4)(2, 3)$. Shifting the simple reflection $(2, 3)$ from the RHS to the LHS we get from $(v, u) = (sw', w')$ the tuple

$$((2, 3)(1, 2)(2, 3)(3, 4)(4, 5)(3, 4), (2, 3)(2, 3)(3, 4)(4, 5)(3, 4))$$

which we want to avoid since the first entry has not desired shape. Instead we consider the sequence

$$\begin{aligned} (v_1, u_1) &= ((1, 2)(2, 3)(4, 5)(3, 4)(4, 5)(2, 3), (2, 3)(4, 5)(3, 4)(4, 5)(2, 3)) \\ &\quad \vdots \\ (v_4, u_4) &= ((4, 5)(1, 2)(2, 3)(3, 4)(2, 3)(4, 5), (4, 5)(2, 3)(3, 4)(2, 3)(4, 5)). \end{aligned}$$

b) Let $v = sw' = (3, 4)(2, 3)(1, 2)(3, 4)(4, 5)(2, 3)$ and $u = (2, 3)(1, 2)(3, 4)(4, 5)(2, 3)$. Again shifting the reflection $(2, 3)$ from the RHS to the LHS, would yield the (non-desirable) tuple. Instead we consider the sequence

$$\begin{aligned} (v_1, u_1) &= ((3, 4)(2, 3)(3, 4)(1, 2)(4, 5)(2, 3), (2, 3)(3, 4)(1, 2)(4, 5)(2, 3)), \\ (v_2, u_2) &= ((2, 3)(3, 4)(2, 3)(1, 2)(4, 5)(2, 3), (2, 3)(3, 4)(1, 2)(4, 5)(2, 3)). \end{aligned}$$

□

We believe that the above theorem can be stated with respect to words in \hat{F}^+ . In particular the following conjecture should hold true.

Conjecture 7.22. *Let $w \in \hat{F}^+$. The cohomology of $\overline{X}(w)$ in degree $2i$ can be written as*

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \leq w \\ \ell(v)=i}} H(w)_z$$

with $H(w)_z = i_{P_z^w}^G(-i)$ for certain standard parabolic subgroups $P_z^w \subset G$.

□

Remarks 7.23. i) The grading

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(v)=i}} H(w)_z$$

produced above is licentious since it depends among other things on the chosen gradings with respect to the (relative) Coxeter elements in Levi subgroups.

ii) We can use Theorem 7.18 in order to reprove the statement in Remark 5.10 concerning the appearance of the Steinberg representation in the cohomology of a DL-variety $X(w)$. Indeed, it is easily seen that the induced representation i_B^G occurs in the spectral sequence only in the contribution $H^0(\overline{X}(e))$. Hence the G -representation v_B^G occurs by Prop. 2.11 exactly in degree $\ell(w)$.

iii) In [DMR, Cor. 3.3.8] the authors determine the character of $H^{2i}(\overline{X}(w))$ as H -representation. But for the author it is not clear that their result leads to part a) of Theorem 7.18.

8. THE SPECTRAL SEQUENCE REVISITED

In this section we reconsider the spectral sequence

$$E_1^{p,q} = \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-p}} H^q(\overline{X}(v)) \implies H_c^{p+q}(X(w))$$

of the previous paragraph and treat the final aspect of the introduction.

Conjecture 8.1. *Let $w \in F^+$ and fix an integer $i \geq 0$. For $v \preceq w$, there are geometric gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} i_{P_z^G}(-\ell(z))$ such that the complex*

$$E_1^{\bullet, 2i} : H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-2}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(e))$$

is quasi-isomorphic to a direct sum $\bigoplus_{\substack{z \preceq w \\ \ell(z)=i}} H(\cdot)_z$ of complexes of the shape

$$H(\cdot)_z : i_{P_z^G} \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i_{P_v^G} \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i_{P_v^G} \rightarrow \cdots \rightarrow i_{P_e^G}$$

as in section 1, cf. (2.9). (Here the maps $i_{P_z^G} \rightarrow i_{P_{z'}^G}$ in the complex are induced - up to sign - by the double cosets of 1 in $W_{P_z^v} \setminus W/W_{P_{z'}^v}$ via Frobenius reciprocity. Further $i_{P_z^G} = (0)$ if $z \not\preceq v$.)

By Proposition 3.11 the conjecture is true for Coxeter elements. Thus we deal in what follows with elements of positive height.

Proposition 8.2. *Let $w \in F^+$ with $\text{ht}(w) \geq 1$. Then for proving the conjecture we may assume that w is of the form $w = sw's$ with $\ell(w) = \ell(w') + 2$.*

Proof. We apply again the operations (I) - (III) to the complex. Suppose that the statement is true for w . We need to show that the assertion is true for the transformed element in F^+ .

(I) Let $w = sw'$. Then we set $H(C(v))_{C(z)} = H(v)_z$ for $v \preceq w$ and $z \preceq v$. The corresponding complexes are clearly quasi-isomorphic.

(II) Let $w = w_1stw_2$ with $st = ts$. We set $H(K(v))_{K(z)} = H(v)_z$ for $v \preceq w$ and $z \preceq v$. The corresponding resulting complexes are clearly quasi-isomorphic.

(III) Let $w = w_1stsw_2$ and $R(w) = w_1tstw_2$. By Corollary 7.14 we know that for $z_i \preceq w_i$, $i = 1, 2$, the cohomology $H_c^{2i}(X(s_l, z_1stsz_2))$ is a direct factor of $H^{2i}(\overline{X}(z_1stsz_2))$ and that

$$\begin{aligned} & H^{2i}(\overline{X}(z_1stsz_2)) - H_c^{2i}(X(s_l, z_1s^2z_2)) = H^{2i}(\overline{X}(z_1\widehat{stsz_2})) \\ & = H^{2i}(\overline{X}(z_1tstz_2)) - H_c^{2i}(X(t_r, z_1t^2z_2)). \end{aligned}$$

Further $H^{2i}(\overline{X}(z_1s^2z_2)) = H^{2i}(\overline{X}(z_1sz_2)) \oplus H^{2i-2}(\overline{X}(z_1sz_2))(-1)$. Hence the complex $E_1^{\bullet, 2i}$ for w is quasi-isomorphic to

$$\begin{array}{ccc} & \bigoplus_{\substack{v_1 \prec w_1 \\ \ell(v_1) = \ell(w_1) - 1}} H^{2i}(\overline{X}(v_1\widehat{stsw_2})) & \longrightarrow \cdots \\ \nearrow & & \\ H^{2i}(\overline{X}(\hat{w})) & \longrightarrow H^{2i}(\overline{X}(w_1stw_2)) \oplus H^{2i}(\overline{X}(w_1tsw_2)) & \longrightarrow \cdots \\ \searrow & & \\ & \bigoplus_{\substack{v_2 \prec w_2 \\ \ell(v_2) = \ell(w_2) - 1}} H^{2i}(\overline{X}(w_1\widehat{stsv_2})) & \longrightarrow \cdots \end{array}$$

which coincides with

$$\begin{array}{ccc}
& \bigoplus_{\substack{v_1 \prec w_1 \\ \ell(v_1) = \ell(w_1) - 1}} H^{2i}(\overline{X}(v_1 \widehat{tst} w_2)) \longrightarrow \cdots \\
\nearrow & & \\
H^{2i}(\overline{X}(R(\widehat{w}))) & \longrightarrow & H^{2i}(\overline{X}(w_1 st w_2)) \oplus H^{2i}(\overline{X}(w_1 tsw_2)) \longrightarrow \cdots \\
& & \searrow \\
& \bigoplus_{\substack{v_2 \prec w_2 \\ \ell(v_2) = \ell(w_2) - 1}} H^{2i}(\overline{X}(w_1 \widehat{tst} v_2)) \longrightarrow \cdots
\end{array}$$

By reversing the above argument, i.e. by adding the contributions $H_c^{2i}(X(t_r, z_1 t^2 z_2))$ and $H^{2i-2}(X(z_1 t z_2))(-1)$ to the complex (with gradings chosen by induction) we see that the E_1 -term of the transformed element $R(w)$ in degree $2i$ is quasi-isomorphic to the complex $H(R(\cdot))$. \square

Remark 8.3. Let $w = sw's$ with $\ell(w) = \ell(w') + 2$. For every $v' \preceq w'$, we have the identity $H^{2i}(\overline{X}(sw's)) = H^{2i}(\overline{X}(v's)) \oplus H^{2i-2}(\overline{X}(v's))(-1)$. Hence the complex in Conjecture 8.1 is quasi-isomorphic to its subcomplex

$$\begin{array}{ccc}
(8.1) & & \\
& & H^{2i}(\overline{X}(sw's)) \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H^{2i}(\overline{X}(v's)) \oplus H^{2i}(X(w')) \\
& \nearrow & \nearrow \qquad \qquad \qquad \longrightarrow \cdots \\
H^{2i-2}(\overline{X}(w's))(-1) & \longrightarrow & \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H^{2i-2}(\overline{X}(v's))(-1).
\end{array}$$

Moreover, by considering the middle column in the diagram (7.3), we see that the lower line which we may identify with the complex

$$H_c^{2i}(X(s, sw's)) \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H_c^{2i}(X(s, v's)) \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s, ss))$$

is the direct sum of the complexes

$$H_c^{2i}(X(s^2, sw's))\langle -i \rangle \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H_c^{2i}(X(s^2, v's))\langle -i \rangle \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s^2))\langle -i \rangle$$

and

$$H_c^{2i}(X(s, sw's)) \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H_c^{2i}(X(s, v's)) \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s)).$$

Theorem 8.4. *Let $w \in F^+$. Then Conjecture 8.1 is true for $i = 0, 1, \ell(w) - 1, \ell(w)$.*

Proof. We may suppose that w has full support. If $\text{ht}(w) = 0$, then the claim follows from Proposition 3.11. So we assume in the sequel that $\text{ht}(w) \geq 1$.

If $i = 0$ then the complex coincides with the complex (2.7) of section 1 which yields the Steinberg representation v_B^G .

If $i = \ell(w)$, the claim is trivial.

If $i = \ell(w) - 1$ the assertion follows from Corollary 7.7.

So let $i = 1$. The proof is by induction on the length. By Proposition 8.2 we may assume that $w = sw's$. For the start of induction, let $w = sw's$ be as in Corollary 5.9 with $s = s_i$ and

$$w' = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{n-1}.$$

Here the Tate twist -1 contribution of the cohomology of $X(w)$ is given by

$$H_c^*(X(w))\langle -1 \rangle = H_c^n(X(w))\langle -1 \rangle = v_{P(s)}^G(-1) - v_{P(2,1,1,\dots,1)}^G(-1).$$

The lower line in the complex (8.1) is nothing else but the complex (2.8) which is a resolution of $v_{P(s)}^G$. In view of Theorem 7.18 we define a grading on the upper line as follows. Set for all $j \neq i$,

$$H(ss_j)_s = H(s_j s)_e(-1) = i_{P(s_j s)}^G(-1)$$

and

$$H(t)_t = H(t)_e(-1) = i_{P(t)}^G(-1) \quad \forall t \in S.$$

In particular, we have fixed $H(ss_j)_{s_j}$ in this way for all $j \neq i$. Further we set for $j < i - 1$,

$$H(s_j s_{j+1})_{s_j} = i_{P(s_j)}^G(-1)$$

and

$$H(s_j s_{j+1})_{s_{j+1}} = i_{P(s_{j+1})}^G(-1)$$

for $j > i$. If $s_j s_k = s_k s_j$, then there is a canonical grading on $H^2(\overline{X}(s_j s_k))$. Thus we have defined gradings for all subwords of sw' of length ≤ 2 . Now we extend the above gradings to the complex

$$0 \longrightarrow H^2(\overline{X}(w')) \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = \ell(w') - 1}} H^2(\overline{X}(v')) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v') = 1}} H^2(\overline{X}(v'))$$

which is induced by the Künneth formula and (compatible) gradings with respect to the the relative Coxeter elements $s_1 \cdots s_{i-1}$ and $s_{i+1} \cdots s_{n-1}$. More precisely, for $s_{i+1} \cdots s_{n-1}$ we consider the grading described by Proposition 3.4 whereas for $s_1 \cdots s_{i-1}$ we consider the dual

grading, i.e., induced by blowing up hyperplanes (For $v' = w'$ the resulting grading coincides with the one in Proposition 3.13). Finally we apply Theorem 7.18 b) once again in order to get the remaining gradings on the upper line in the complex (8.1). Here we have to make a choice for the grading on $H^2(\overline{X}(ss_{i-1}s_{i+1}))$, say $H(ss_{i-1}s_{i+1})_{s_{i-1}} \subset H(ss_{i-1}s_{i+1})_{s_{i+1}}$. The resulting graded complex satisfies the claim. Indeed, for any $t \mid s_{i+1} \cdots s_{n-1}$ the complex $H(\cdot)_t$ is acyclic as $H(sv')_t = H(sv'/s_{i+1})_t$ for all $v' \preceq w'$ with $s_{i+1} \mid v'$. Similarly, for $t \mid s_1 \cdots s_{i-1}$ the complex $H(\cdot)_t$ is acyclic as $H(sv')_t = H(sv'/s_{i-1})_t$ for all $v' \preceq w'$ with $s_{i-1} \mid v'$. Moreover $H(sv')_{s_{i+1}} = H(v')_{s_{i+1}}$ for all $v' \preceq w'$ which shows that the complex $H(\cdot)_{s_{i+1}}$ is acyclic. Finally, one checks that the complex $H(\cdot)_{s_{i-1}}$ is a resolution of the representation $v_{P(s)}^G(-1) - v_{P(2,1,1,\dots,1)}^G(-1)$. Moreover, we see that the differential

$$(8.2) \quad \bigoplus_{\substack{v \preceq w \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec w \\ \ell(t)=1}} H^2(\overline{X}(t))$$

is surjective.

Let's proceed with the induction step. So let $w = sw's \in F^+$ with $\text{ht}(sw') \geq 1$.

Claim: The map (8.2) is surjective, as well.

Here we may consider the complex (8.1) again. By induction hypothesis we deduce that the map $\bigoplus_{\substack{v \preceq sw' \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec sw' \\ \ell(t)=1}} H^2(\overline{X}(t))$ is surjective. On the other hand, we have a surjection $H^2(\overline{X}(s^2)) \longrightarrow H^2(\overline{X}(s_r))$. The claim follows.

We distinguish finally the following cases.

Case a). $s \in \text{supp}(w')$. Then the lower line in the complex (8.1) coincides with the complex (2.8). It is contractible by Proposition 2.13. By induction hypothesis the statement is true for the upper line. We extend the grading to the complex with respect to w in the obvious way.

Case b). $s \notin \text{supp}(w')$. Then the lower line in the complex (8.1) coincides with the complex (2.8) and gives a resolution of the generalized Steinberg representation $v_{P(s)}^G$. By induction hypothesis the statement is true for the upper line. As the map $\bigoplus_{\substack{v \preceq sw' \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec sw' \\ \ell(t)=1}} H^2(\overline{X}(t))$ is surjective the representation $v_{P(s)}^G(-1)$ occurs in the cohomology of $X(w)$. Here, we extend the grading to the complex with respect to w in the obvious way, as well. \square

By the proof of the preceding theorem we get an inductive formula for the Tate twist -1 contribution of the cohomology of DL-varieties.

Corollary 8.5. *Let $w = sw's \in F^+$ with $\text{ht}(sw') \geq 1$. Then*

$$H_c^*(X(w))\langle -1 \rangle = \begin{cases} H_c^*(X(sw'))\langle -1 \rangle[-1] & \text{if } s \in \text{supp}(w') \\ H_c^*(X(sw'))\langle -1 \rangle[-1] \oplus v_{P(s)}^G(-1)[-l(w)] & \text{if } s \notin \text{supp}(w') \end{cases}.$$

In view of the lower line in (8.1) we generalize Conjecture 3.12.

Conjecture 8.6. *Let $w \in F^+$ and fix $u \prec w$. For $u \preceq v \preceq w$, there are geometric gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} H(v)_z$ such that the complex*

$$(8.3) \quad 0 \longrightarrow H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{u \preceq v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(u))$$

is quasi-isomorphic to the graded direct sum of complexes of the shape

$$0 \rightarrow i_{P_z}^G \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i_{P_z}^G \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i_{P_z}^G \rightarrow \cdots \rightarrow i_{P_e}^G \rightarrow 0$$

cf. (2.9).

Remark 8.7. As in Proposition 8.2 one can reduce the conjecture to the case where w is of the shape $w = sw's$. Again we only have seriously to consider the operation (III) in this process of transformations. So let $w = w_1stsw_2$ and $R(w) = \overline{w} = w_1tstw_2$. If

$$u = v_1stsv_2, v_1s^2v_2, v_1stsv_2, v_1tstv_2, v_1v_2,$$

respectively, we set

$$\overline{u} = v_1tstv_2, v_1t^2v_2, v_1tstv_2, v_1stsv_2, v_1v_2.$$

Then the complex (8.3) is quasi-isomorphic to

$$(8.4) \quad 0 \longrightarrow H^{2i}(\overline{X}(\overline{w})) \longrightarrow \bigoplus_{\substack{u \preceq v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(\overline{v})) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(\overline{u})).$$

Similarly, we transform the lower complex in the conjecture above.

Remark 8.8. If Conjecture 8.1 is true for $w = sw's$ then the induced graded complex restricted to

$$0 \longrightarrow H^{2i}(\overline{X}(sw')) \longrightarrow \bigoplus_{\substack{v \prec sw' \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(e)).$$

is not necessarily quasi-isomorphic to the latter complex.

Remark 8.9. Suppose that Conjecture 8.1 is true. Then we can reprove the statement in Remark 5.10 concerning the appearing of the trivial representation in the cohomology of a DL-variety $X(w)$. Indeed, the multiplicity of the trivial representation i_G^G in an induced representation i_P^G is always 1. Hence for any $z \preceq w$, the v -contribution

$$0 \rightarrow i_{P_z}^G \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v) = \ell(w) - 1}} i_{P_v}^G \rightarrow \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v) = \ell(w) - 2}} i_{P_v}^G \rightarrow \cdots \rightarrow i_{P_e}^G \rightarrow 0$$

restricted to i_G^G is acyclic as the resulting index set is contractible (a lattice). It follows that the i_G^G occurs only in the top cohomology group of $X(w)$.

9. EXAMPLES

Here we present some examples concerning Theorem 8.4. In the following we omit the Tate twists for reasons of clarity. For $w \in F^+$ and $z \preceq w$, we write $i_P^G(z)$ instead of $H(w)_z = i_P^G$.

a) Let $G = \mathrm{GL}_3$ and let $w = (1, 2)(2, 3)(1, 2) = sw's \in F^+$ with $s = (1, 2)$ and $w' = (2, 3)$.

α) Let $i = 2$. Here the complex (8.1) is

$$\begin{array}{c} H^4(\overline{X}((1, 2)(2, 3))) = i_G^G \\ \nearrow \end{array}$$

$$H^2(\overline{X}((2, 3)(1, 2))) \longrightarrow H^2(\overline{X}((1, 2)_r)) = i_{P_{(2,1)}}^G.$$

We consider the grading $H^2(\overline{X}(2, 3)(1, 2)) = i_G^G(2, 3) \oplus i_{P_{(2,1)}}^G(1, 2)$. Thus we see that the above complex is contractible, as it should be by Example 4.16.

β) Let $i = 1$. Here the complex (8.1) is

$$\begin{array}{ccc} H^2(\overline{X}((1, 2)(2, 3))) & \longrightarrow & H^2(\overline{X}((1, 2)_l)) \oplus H^2(X((2, 3))) = i_{P_{(2,1)}}^G \oplus i_{P_{(2,1)}}^G \\ \nearrow & & \nearrow \end{array}$$

$$H^0(\overline{X}((2, 3)(1, 2))) = i_G^G \longrightarrow H^0(\overline{X}((1, 2)_r)) = i_{P_{(2,1)}}^G.$$

We consider the grading $H^2(\overline{X}((2, 3)(1, 2))) = i_G^G(1, 2) \oplus i_{P_{(2,1)}}^G(2, 3)$. Thus we see that the above complex is contractible, as it should be by Example 4.16.

b) Let $G = \mathrm{GL}_4$ and let $w = (3, 4)(1, 2)(2, 3)(3, 4) = sw's \in F^+$ with $s = (3, 4)$ and $w' = (1, 2)(2, 3)$.

α) Let $i = 2$. We have $H^4(\overline{X}((1, 2)(2, 3))) = i_{P_{(3,1)}}^G$. We consider the grading

$$H^4(\overline{X}((3, 4)(1, 2)(2, 3))) = i_G^G((3, 4)(2, 3)) \bigoplus i_{P_{(3,1)}}^G((1, 2)(2, 3)) \bigoplus i_{P_{(2,2)}}^G((1, 2)(3, 4)).$$

The reduced complex (8.1) is given by

$$\begin{array}{ccccc} & & & & H^4(\overline{X}((1, 2)(2, 3))) = i_{P_{(3,1)}}^G \\ & & & \nearrow & \\ H^4(\overline{X}((3, 4)(1, 2)(2, 3))) & \longrightarrow & H^4(\overline{X}((3, 4)(1, 2))) = i_{P_{(2,2)}}^G & & \\ & & \searrow & & \\ & & & & H^4(\overline{X}((3, 4)(2, 3))) = i_{P_{(1,3)}}^G \\ & & & & \\ & & & \uparrow & \\ & & & H^2(\overline{X}((1, 2)(3, 4))) & \\ \uparrow & \nearrow & & \searrow & \\ H^2(\overline{X}(\mathrm{Cox})) & & \uparrow & & H^2(\overline{X}((3, 4))) = i_{P_{(1,1,2)}}^G \\ & \searrow & & \nearrow & \\ & & H^2(\overline{X}((2, 3)(3, 4))) & & \end{array}$$

Here the upper arrows \uparrow go as in the complex (8.1) from $H^2(\overline{X}(v's))$ to $H^4(\overline{X}(sv'))$ for $v' \preceq w'$. We consider the gradings

$$\begin{aligned} H^2(\overline{X}((2, 3)(3, 4))) &= i_{P_{(1,3)}}^G(2, 3) \bigoplus i_{P_{(1,1,2)}}^G(3, 4), \\ H^2(\overline{X}((1, 2)(3, 4))) &= i_{P_{(2,2)}}^G(1, 2) \bigoplus i_{P_{(2,2)}}^G(3, 4) \end{aligned}$$

and

$$H^2(\overline{X}(\mathrm{Cox})) = i_G^G(2, 3) \bigoplus i_{P_{(3,1)}}^G(3, 4) \bigoplus i_{P_{(2,2)}}^G(1, 2).$$

We get $H_c^*(X(w))\langle -2 \rangle = j_{(2,2)}[-5]$, as it should be by Example 5.11.

β) Let $i = 1$. We consider for $w' = (1, 2)(2, 3)$ the graded complex

$$\begin{array}{ccc} & & H^2(\overline{X}((1, 2))) = i_{P_{(2,1,1)}}^G \\ & & \nearrow \\ H^2(\overline{X}((1, 2)(2, 3))) & & \\ & & \searrow \\ & & H^2(\overline{X}((2, 3))) = i_{P_{(1,2,1)}}^G \end{array}$$

with $H^2(\overline{X}((1, 2)(2, 3))) = i_{P_{(3,1)}}^G(2, 3) \oplus i_{P_{(2,1,1)}}^G(1, 2)$.

We consider further the gradings

$$\begin{aligned} H^2(\overline{X}((3, 4)(2, 3))) &= i_{P_{(3,1)}}^G(3, 4) \oplus i_{P_{(1,2,1)}}^G(2, 3), \\ H^2(\overline{X}((3, 4)(1, 2))) &= i_{P_{(2,2)}}^G(1, 2) \oplus i_{P_{(2,2)}}^G(3, 4) \end{aligned}$$

and

$$H^2(\overline{X}((3, 4)(1, 2)(2, 3))) = i_G^G(3, 4) \oplus i_{P_{(3,1)}}^G(2, 3) \oplus i_{P_{(2,2)}}^G(1, 2).$$

The reduced complex (8.1) is given by

$$\begin{array}{ccccc} & & H^2(\overline{X}((1, 2)(2, 3))) & \longrightarrow & H^2(X((1, 2))) = i_{P_{(2,1,1)}}^G \\ & \nearrow & & & \times \\ H^2(\overline{X}((3, 4)(1, 2)(2, 3))) & \longrightarrow & H^2(\overline{X}((3, 4)(1, 2))) & \longrightarrow & H^2(\overline{X}((2, 3))) = i_{P_{(1,2,1)}}^G \\ & \searrow & & & \times \\ & & H^2(\overline{X}((3, 4)(2, 3))) & \longrightarrow & H^2(\overline{X}((3, 4))) = i_{P_{(1,1,2)}}^G \end{array}$$

$$\begin{array}{ccccc} & & \uparrow & & \\ & & H^0(\overline{X}((1, 2)(3, 4))) = i_{P_{(2,2)}}^G & & \\ \uparrow & \nearrow & & \searrow & \uparrow \\ H^0(\overline{X}(\text{Cox})) = i_G^G & & \uparrow & & H^0(\overline{X}((3, 4))) = i_{P_{(1,1,2)}}^G \\ & \searrow & & \nearrow & \\ & & H^0(\overline{X}((2, 3)(3, 4))) = i_{P_{(1,3)}}^G & & \end{array}$$

Here the upper arrows \uparrow go as in the complex (8.1) from $H^0(\overline{X}(v's))$ to $H^2(\overline{X}(sv'))$ for $v' \preceq w'$. It follows that the complex is contractible, as it should be by Example 5.11.

c) Let $G = \text{GL}_4$ and let $w = (2, 3)(1, 2)(3, 4)(2, 3) \in sw'sF^+$ with $s = (2, 3)$ and $w' = (1, 2)(3, 4)$.

α) Let $i = 2$. We have $H^4(\overline{X}((1, 2)(3, 4))) = i_{P_{(2,2)}}^G$. We consider the grading

$$H^4(\overline{X}((2, 3)(1, 2)(3, 4))) = i_G^G((2, 3)(1, 2)) \oplus i_{P_{(3,1)}}^G((2, 3)(3, 4)) \oplus i_{P_{(2,2)}}^G((1, 2)(3, 4)).$$

The reduced complex (8.1) is given by

$$\begin{array}{ccccc}
 & & H^4(\overline{X}((1, 2)(3, 4))) = i_{P_{(2,2)}}^G & & \\
 & \nearrow & & & \\
 H^4(\overline{X}((2, 3)(1, 2)(3, 4))) & \longrightarrow & H^4(\overline{X}((2, 3)(1, 2))) = i_{P_{(3,1)}}^G & & \\
 & \searrow & & & \\
 & & H^4(\overline{X}((2, 3)(3, 4))) = i_{P_{(1,3)}}^G & & \\
 & & \uparrow & & \\
 & & H^2(\overline{X}((1, 2)(2, 3))) & & \\
 \uparrow & \nearrow & & \searrow & \\
 H^2(\overline{X}((1, 2)(3, 4)(2, 3))) & & \uparrow & & H^2(\overline{X}((2, 3))) = i_{P_{(1,2,1)}}^G \\
 & \searrow & & \nearrow & \\
 & & H^2(\overline{X}((3, 4)(2, 3))) & &
 \end{array}$$

Here the upper arrows \uparrow go as in the complex (8.1) from $H^2(\overline{X}(v's))$ to $H^4(\overline{X}(sv'))$ for $v' \preceq w'$. We consider the gradings

$$\begin{aligned}
 H^2(\overline{X}((3, 4)(2, 3))) &= i_{P_{(1,3)}}^G(3, 4) \oplus i_{P_{(1,2,1)}}^G(2, 3), \\
 H^2(\overline{X}((1, 2)(2, 3))) &= i_{P_{(3,1)}}^G(1, 2) \oplus i_{P_{(1,2,1)}}^G(2, 3)
 \end{aligned}$$

and

$$H^2(\overline{X}((1, 2)(3, 4)(2, 3))) = i_G^G(1, 2) \oplus i_{P_{(3,1)}}^G(3, 4) \oplus i_{P_{(2,2)}}^G(2, 3).$$

We get $H_c^*(X(w))\langle -2 \rangle = i_{P_{(2,1,1)}}^G / i_{P_{(2,2)}}^G[-5]$, as it should be by Example 5.12.

β) Let $i = 1$. We consider for $w' = (1, 2)(3, 4)$ the graded complex

$$\begin{array}{ccc}
 & & H^2(\overline{X}((1, 2))) = i_{P_{(2,1,1)}}^G \\
 & \nearrow & \\
 H^2(\overline{X}((1, 2)(3, 4))) & & \\
 & \searrow & \\
 & & H^2(\overline{X}((3, 4))) = i_{P_{(1,1,2)}}^G
 \end{array}$$

with $H^2(\overline{X}((1, 2)(3, 4))) = i_{P_{(2,2)}}^G(1, 2) \oplus i_{P_{(2,2)}}^G(3, 4)$. We consider further the gradings

$$\begin{aligned}
 H^2(\overline{X}((2, 3)(3, 4))) &= i_{P_{(1,3)}}^G(2, 3) \oplus i_{P_{(1,1,2)}}^G(3, 4), \\
 H^2(\overline{X}((2, 3)(1, 2))) &= i_{P_{(3,1)}}^G(2, 3) \oplus i_{P_{(2,1,1)}}^G(1, 2)
 \end{aligned}$$

and

$$H^2(\overline{X}((2, 3)(1, 2)(3, 4))) = i_G^G(2, 3) \bigoplus i_{P_{(3,1)}}^G(3, 4) \bigoplus i_{P_{(2,2)}}^G(1, 2).$$

The reduced complex (8.1) is given by

$$\begin{array}{ccccc}
& & H^2(\overline{X}((2, 3)(1, 2))) & \longrightarrow & H^2(X((1, 2))) = i_{P_{(2,1,1)}}^G \\
& \nearrow & & \searrow & \\
H^2(\overline{X}((2, 3)(1, 2)(3, 4))) & \longrightarrow & H^2(\overline{X}(((1, 2)(3, 4)))) & \longrightarrow & H^2(\overline{X}((2, 3))) = i_{P_{(1,2,1)}}^G \\
& \searrow & & \searrow & \\
& & H^2(\overline{X}((2, 3)(3, 4))) & \longrightarrow & H^2(\overline{X}((3, 4))) = i_{P_{(1,1,2)}}^G \\
& & \uparrow & & \\
& & H^0(\overline{X}((1, 2))(2, 3)) = i_{P_{(3,1)}}^G & & \\
\uparrow & \nearrow & & \searrow & \uparrow \\
H^0(\overline{X}((1, 2)(3, 4)(2, 3))) = i_G^G & & \uparrow & & H^0(\overline{X}((2, 3))) = i_{P_{(1,2,1)}}^G \\
& \searrow & & \nearrow & \\
& & H^0(\overline{X}((3, 4)(2, 3))) = i_{P_{(1,3)}}^G & &
\end{array}$$

Here the upper arrows \uparrow go as in the complex (8.1) from $H^0(\overline{X}(v's))$ to $H^2(\overline{X}(sw'))$ for $v' \preceq w'$. We get $H_c^*(X(w))\langle -1 \rangle = j_{(2,2)}[-4]$, as it should be by Example 5.12.

10. THE SECOND APPROACH

In this section we present a different but more vague method for determining the cohomology of DL-varieties attached to elements in the Weyl group. The analogous situation for elements of the monoid F^+ can be treated in the same way. Here the approach is the other way round compared to the foregoing version. Some results presented here might be also treated by using Demazure resolutions and the results of the previous sections.

Let $w = sw's \in W$ with $\ell(w) = \ell(w') + 2$ and $X(w, sw') = X(w) \cup X(sw') \subset X$ as before. Reconsider for $i \geq 0$, the natural map $r^i = r_{w,sw'}^i : H_c^i(X(w, sw')) \longrightarrow H_c^i(X(sw'))$. Now we write $H_c^i(X(w, sw')) = H_c^{i-2}(X(w's, w'))(-1) = A \bigoplus B$ where

$$A \cong \operatorname{coker}(H_c^{i-3}(X(w')) \longrightarrow H_c^{i-2}(X(w's)))(-1)$$

and

$$B = \ker(H_c^{i-2}(X(w')) \longrightarrow H_c^{i-1}(X(w's)))(-1).$$

By Remark 4.14 we know that $r_{|A}^i = 0$.

Motivated by the Examples in the GL_4 -case we pose the following conjecture.

Conjecture 10.1. *For $i \geq 0$, the map $r_{|B}^i : B \rightarrow H_c^i(X(sw'))$ has si-full rang.*

Remark 10.2. For our purpose it is even enough to have the validity of a weaker form of the above conjecture. More precisely, it suffices to know that for all irreducible H -representations V with $i = -2t(V)$ the map $r_{|B^V}^i : B^V \rightarrow H_c^{2i}(X(sw'))^V$ has full rang.

Indeed if the assumption $i = -2t(V)$ is not satisfied, then we proceed as follows to determine the V -isotypic part of the map $r_{w,sw'}^i$. We fix a reduced decomposition of w' . Since $i > -2t(V)$ there is by purity of $\overline{X}(w)$ some hypersquare $Q \subset F^+$ of dimension d (which is assumed to be minimal) with head w and with $\{w, sw'\} \subset Q$ and such that a given irreducible subrepresentation $Z \subset B^V \subset H_c^i(X(w) \cup X(sw'))$ is induced by an isomorphic subrepresentation $Z' \subset H_c^{i-1}(X(Q) \setminus X(w) \cup X(sw'))$ via the corresponding boundary map δ^{i-1} . A minimal hypersquare exists since the extreme case where $\text{tail}(Q) = e$ yields one. Then $\text{tail}(Q) = sv'$ or $\text{tail}(Q) = v'$ for some $v' \in F^+$ with $v' \preceq w'$.

Suppose that $d = 2$.

1. Case. $\text{tail}(Q) = sv'$. Thus $Z' \subset H_c^{i-1}(X(sv's) \cup X(sv'))$ maps onto $Z \subset H_c^i(X(w) \cup X(sw'))$ via the boundary map δ^{i-1} .

Subcase a) Z' is induced by $H_c^{i-1}(X(sv'))$.

Subsubcase i) $\delta_{sw'sv'}^{i-1}(Z') \neq 0$ where $\delta_{sw'sv'}^{i-1}$ is the boundary map $H_c^{i-1}(X(sv')) \rightarrow H_c^i(X(sw'))$ (which is known by induction, cf. the following pages). In this case $r_{w,sw'}^i$ maps $Z \subset H_c^i(X(w) \cup X(sw'))$ onto $\delta_{sw'sv'}^{i-1}(Z') \subset H_c^i(X(sw'))$ by considering the commutative diagram:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \cdots \longrightarrow & H_c^i(X(w)) & \longrightarrow & H_c^i(X(w) \cup X(sw')) & \longrightarrow & H_c^i(X(sw')) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow \delta^{i-1} & & \uparrow \delta_{sw'sv'}^{i-1} & \\
 \cdots \longrightarrow & H_c^{i-1}(X(v)) & \longrightarrow & H_c^{i-1}(X(v) \cup X(sv')) & \longrightarrow & H_c^{i-1}(X(sv')) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \vdots & & \vdots & & \vdots &
 \end{array}$$

Subsubcase ii) $\delta_{sw'sv'}^{i-1}(Z') = 0$. Then $r^i(Z) = 0$.

Subcase b) Z' is induced by $H_c^{i-1}(X(v))$. In this case $r^i(Z) = 0$.

2. Case. $\text{tail}(Q) = w'$. This case yields a contradiction since the boundary map $H_c^{i-1}(X(w's) \cup X(w')) \rightarrow H_c^i(X(w) \cup X(sw'))$ vanishes by Corollary 4.12.

Suppose that $d = 3$.

1. Case $\text{tail}(Q) = sv'$ for some $v' \prec w'$ with $\ell(w') = \ell(v') + 1$, i.e. Q has the shape

$$Q: \begin{array}{ccccc} & & w & & \\ & \nearrow & \uparrow & \nwarrow & \\ sw' & & sv'_1s & & sv'_2s \\ \uparrow & \nearrow \nwarrow & & \nearrow \nwarrow & \uparrow \\ sv'_1 & & sv'_2 & & sv's \\ & \nwarrow & \uparrow & \nearrow & \\ & & sv' & & \end{array}$$

for some $v'_1, v'_2 \in F^+$. We set $A = \{sv'_1, sv'_2, sv'\}$ and $B = \{sv'_1s, sv'_2s, sv's\}$. Then $X(A)$ is closed in $X(Q) \setminus (X(w) \cup X(sw'))$ whereas $X(B)$ is open in the latter space. Now we may imitate the procedure of the case $d = 2$. The variety $X(A)$ corresponds to $X(sv')$ whereas $X(B)$ plays the role of $X(v)$.

2. Case $\text{tail}(Q) = v'$ for some $v' \prec w'$ with $\ell(w') = \ell(v') + 1$, i.e., Q has the shape

$$Q: \begin{array}{ccccc} & & w & & \\ & \nearrow & \uparrow & \nwarrow & \\ sw' & & sv's & & w's \\ \uparrow & \nearrow \nwarrow & & \nearrow \nwarrow & \uparrow \\ sv' & & w' & & v's \\ & \nwarrow & \uparrow & \nearrow & \\ & & v' & & \end{array}$$

We claim that $r_{w,sw'}^i$ is trivial. Indeed, as $X(sv's) \cup X(sv')$ and $X(w's) \cup X(w')$ are both open in $X(Q) \setminus (X(w) \cup X(sw'))$ whereas $X(v's) \cup X(v')$ is closed we see that

- $Z' \subset H_c^{i-1}(X(sv's) \cup X(sv'))$ gives a contradiction to the minimality with respect to d .
- $Z' \subset H_c^{i-1}(X(w's) \cup X(w'))$ gives a contradiction as the boundary map is trivial.

So Z' is induced by $H_c^{i-1}(X(v's) \cup X(v'))$ and it is mapped to $Z \subset H_c^i(X(Q(w', w)))$ via the boundary map. But the latter one is given by the diagram

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \uparrow & & \uparrow & & \uparrow \\
 H_c^i(X(Q(w', w))) & = & H_c^i(X(w) \cup X(sw')) & \oplus & H_c^i(X(w's) \cup X(w')) \\
 \uparrow & & \uparrow & & \uparrow \\
 H_c^{i-1}(X(Q(v', v))) & = & H_c^{i-1}(X(v) \cup X(sv')) & \oplus & H_c^{i-1}(X(v's) \cup X(v')) \\
 \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

i.e. it is for trivial reasons the direct sum of the summands. Hence we get a contradiction.

The higher dimensional cases $d \geq 4$ behave as above whether $\text{tail}(Q) = sv'$ or $\text{tail}(Q) = v'$. The latter case gives a contradiction. \square

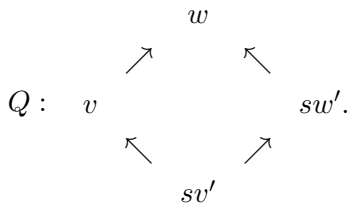
Thus under the validity of the above conjecture for determining the cohomology of $X(w)$, it remains to compute the cohomology of the edge $X(w') \cup X(w's)$ which we explain now.

Let $w, v \in W$ with $\ell(w) = \ell(v) + 1$. We want to determine the cohomology of the locally closed subvariety $X(w) \cup X(v) \subset X$. Suppose that we may write $w = sw's$ as in the previous sections. If $v \in \{sw', w's\}$ then $H_c^i(X(w) \cup X(v)) = H_c^{i-2}(X(w') \cup X(w's))(-1)$ and we may suppose by induction on the length of w that these groups are known.

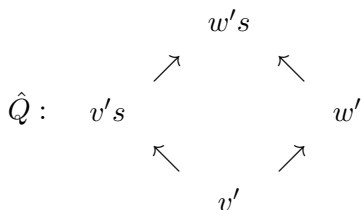
So let $v = sv's$ with $v' < w'$. Then the cohomology of $X(w) \cup X(v)$ sits in a long exact cohomology sequence

$$\dots \longrightarrow H_c^i(X(w) \cup X(v)) \longrightarrow H_c^i(X(Q)) \longrightarrow H_c^i(X(sw') \cup X(sv')) \longrightarrow \dots$$

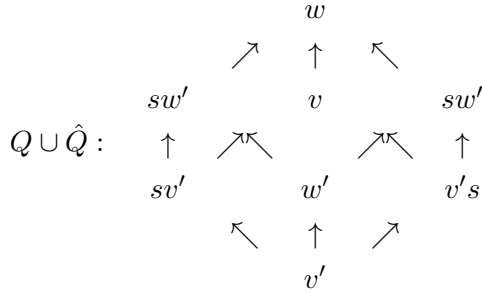
where Q is the square $Q = \{w, v, sw', sv'\} \subset W$. The cohomology of $X(sw') \cup X(sv')$ is known by induction on the length of $w's$. On the other hand, the square



is induced by the square



via multiplication with $s \in S$ from the left. The union $Q \cup \hat{Q}$ gives rise to a cube or a 3-dimensional hypersquare

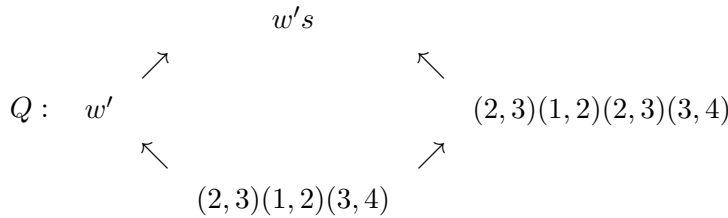


Then by Proposition 6.5 the variety $X(Q)$ is an \mathbb{A}^1 -bundle over $X(\hat{Q})$ and $X(Q \cup \hat{Q})$ is a \mathbb{P}^1 -bundle over $X(\hat{Q})$. Further the restriction map in cohomology

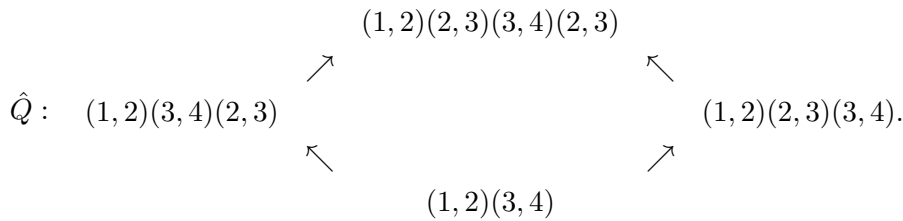
$$r_{Q, \{sw', sv'\}}^i : H_c^i(X(Q)) \longrightarrow H_c^i(X(sw') \cup X(sv'))$$

can be computed in the same way as in Conjecture 10.1 resp. Remark 10.2. Thus if we are able to determine the cohomology group $H_c^*(X(\hat{Q}))$ we have knowledge of the cohomology of $X(w) \cup X(v)$. Hence we have reduced the question of determining the cohomology of the edge $X(w) \cup X(v)$ by the prize of enlarging the square but where the head has smaller length.

Example 10.3. Let $G = GL_4$. Let $w = (1, 4)(2, 3) \in W$. Here we write $w = sw's = (3, 4)(1, 3)(2, 4)(3, 4)$ so that $sw' = (2, 3)(3, 4)(2, 3)(1, 2)(2, 3)$, $w' = (2, 3)(1, 2)(3, 4)(2, 3)$ and $w's = (2, 3)(1, 2)(2, 3)(3, 4)(2, 3)$. Hence for computing the cohomology of the edge $X(w's) \cup X(w')$ we consider the square



and henceforth the square



The cohomology of $X(\hat{Q})$ is by using the results in section 5 easily computed as

$$\begin{aligned} H_c^*(X(\hat{Q})) &= v_{P_{(1,2,1)}}^G[-2] \oplus j_{(3,1)}(-1)[-3]^2 \oplus i_G^G(-2)[-4] \\ &\quad \oplus j_{(3,1)}(-2)[-5] \oplus i_G^G(-3)[-6]^2 \oplus i_G^G(-4)[-8]. \end{aligned}$$

Hence

$$\begin{aligned} H_c^*(X(Q)) &= v_{P_{(1,2,1)}}^G(-1)[-4] \oplus j_{(3,1)}(-2)[-5]^2 \oplus i_G^G(-3)[-6] \\ &\quad \oplus j_{(3,1)}(-3)[-7] \oplus i_G^G(-4)[-8]^2 \oplus i_G^G(-5)[-10]. \end{aligned}$$

On the other hand the cohomology of $Y = X((2, 3)(1, 2)(2, 3)(3, 4)) \cup X((2, 3)(1, 2)(3, 4))$ is given (using loc.cit.) by

$$H_c^*(Y) = v_{P_{(2,1,1)}}^G(-1)[-4] \oplus j_{(2,2)}(-2)[-5] \oplus j_{(3,1)}(-2)[-5] \oplus i_G^G(-3)[-6] \oplus i_G^G(-4)[-8].$$

By a careful study of the restriction map we compute that the cohomology of $X(w's) \cup X(w')$ is hence given by

$$\begin{aligned} H_c^*(X(w's) \cup X(w')) &= j_{(2,2)}(-1)[-4] \oplus j_{(3,1)}(-2)[-5] \oplus j_{(2,2)}(-2)[-6] \\ &\quad \oplus j_{(3,1)}(-3)[-7] \oplus i_G^G(-4)[-8] \oplus i_G^G(-5)[-10]. \end{aligned}$$

Exemplarily, the contribution $j_{(3,1)}(-2)[-5]$ in $H_c^*(Y)$ is induced by the Coxeter element $(2, 3)(1, 2)(3, 4)$. Further one checks that this representation lies in the image of the restriction map $H_c^*(X((1, 3, 4)) \cup X((2, 3)(1, 2)(3, 4))) \rightarrow H_c^*(X((2, 3)(1, 2)(3, 4)))$. Finally one verifies that the representation on the LHS is induced by $H^5(X(Q))$ which implies that $j_{(3,1)}(-2) \subset H_c^5(Y)$ is killed by $H_c^5(X(Q))$. Alternatively, one can apply the method presented in Remark 10.2. With the same methods one deduces that

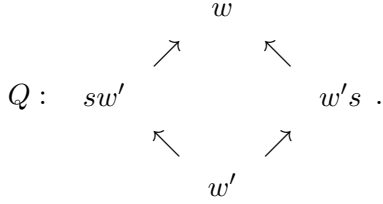
$$\begin{aligned} H_c^*(X((1, 4)(2, 3))) &= v_B^G[-6] \oplus j_{(2,1,1)}(-2)[-7] \oplus j_{(2,2)}(-3)[-8]^2 \\ &\quad \oplus j_{(3,1)}(-4)[-9] \oplus i_G^G(-6)[-12]. \end{aligned}$$

Again we consider exemplarily the contribution $j_{(2,2)}(-2) \subset H_c^6(X(sw'))$. It is induced by $H_c^5(X((1, 4, 3)))$. The set $Q' = \{w, sw', (1, 4), (1, 4, 3)\}$ is a square and the restriction map $H_c^5(X((1, 4)) \cup X((1, 4, 3))) \rightarrow H_c^5(X((1, 4, 3)))$ is surjective. By computing $H_c^4(X(Q')) = H_c^2(X(\hat{Q}'))(-1)$ with $Q' = \{w's, w', (1, 3, 4), (1, 3)\}$ one verifies that $j_{(2,2)}(-2) \subset H_c^6(X(sw'))$ is killed by the restriction map.

Remark 10.4. In general we have to apply to $w \in W$ the operations (I) - (III) (in the sense of Weyl groups) of the previous section in order to write it in the shape $w = sw's$, cf. Lemma 7.20. In what follows, we hope that this lemma (or rather a variant) generalizes somehow to arbitrary hypersquares.

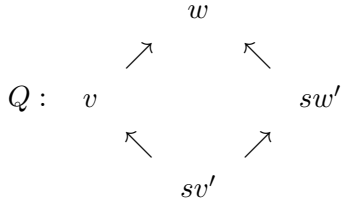
We now explain how the strategy works in higher dimensions. Here we suppose that a similar conjecture as above holds true. Hence we have to determine the cohomology of $X(Q)$ for squares $Q \subset W$. So let Q be such a square with head $w = sw's$.

Case 1: Q is of the shape



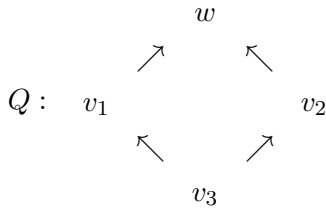
In this case we have by Proposition 4.12 a splitting $H_c^i(X(Q)) = H_c^i(X(w's) \cup X(w')) \oplus H_c^{i-2}(X(w's) \cup X(w'))(-1)$. By induction the cohomology of $X(w's) \cup X(w')$ and hence of $X(Q)$ is known.

Case 2: Q is of the shape



with $v = sv's$. In this case we have $H_c^i(X(Q)) = H_c^{i-2}(X(\hat{Q}))(-1)$ where $\hat{Q} = \{w's, v's, w', v'\}$. By induction the cohomology of $X(\hat{Q})$ and hence of $X(Q)$ is known.

Case 3: Q is of the shape



with $v_i = sv'_i s$ for $i = 1, 2, 3$ and where $Q' := \{w', v'_1, v'_2, v'_3\} \subset W$ is a square. In this case we consider as in the case of edges above the square $sQ' = \{sw', sv'_1, sv'_2, sv'_3\}$. Then the cohomology of $X(Q)$ sits in a long exact cohomology sequence

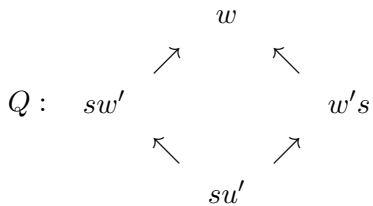
$$\cdots \longrightarrow H_c^{i-1}(X(sQ')) \longrightarrow H_c^i(X(Q)) \longrightarrow H_c^i(X(Q) \cup X(sQ')) \longrightarrow H_c^i(X(sQ')) \longrightarrow \cdots$$

Again by induction on the length of the head of a cube the cohomology of $X(sQ')$, $X(Q's)$ and $X(Q')$ are known. Further we have $H_c^i(X(Q) \cup X(sQ')) = H_c^{i-2}(X(Q's) \cup X(Q'))(-1)$,

where $Q's := \{w's, v'_1s, v'_2s, v'_3s\}$. Thus we may compute $r_{Q \cup sQ', sQ'}^i$ as in the lower dimensional cases, i.e. as in Conjecture 10.1 resp. Remark 10.2.

But there are yet two other kind of squares.

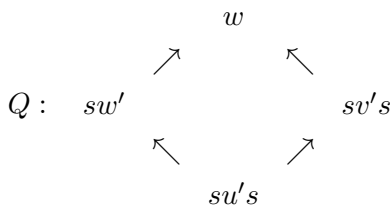
Case 4: Q is of the shape



with $su' = u's$.

and

Case 5: Q is of the shape



with $u' < v' < w'$ and $\ell(w') = \ell(u') + 2$.

Unfortunately, as we see, the higher the dimension of a square is the more complicated the situation behaves. This is of course due to the relations which exist in the Weyl group W . For this reason, we also consider the monoid F^+ in the sequel where this phenomenon does not appear.

Before we proceed we recall the following well-known fact.

Lemma 10.5. *Let $w \in W$ and $s, t \in S$. Suppose that $\ell(sw) = \ell(w) + 1$, $\ell(wt) = \ell(w) + 1$ and $\ell(swt) = \ell(w)$. Then $w = swt$.*

Proof. This is [DL, Lemma 1.6.4] □

Thus if $\ell(sw') = \ell(w's) = \ell(w') + 1$ and $\ell(sw's) = \ell(w')$ we have $sw' = w's$.

Let $Q \subset W$ be a hypersquare with head $w = sw's$. There are a priori for the tail of Q the following possibilities (mod symmetry) where $u' \leq w'$.

Case A: $\text{tail}(Q) = u'$ with $\ell(su's) = \ell(u') + 2$.

Case B: $\text{tail}(Q) = u'$ with $\ell(su') = \ell(u's) = \ell(u') + 1$ and $\ell(su's) = \ell(u')$.

Case C: $\text{tail}(Q) = su'$ with $\ell(su') = \ell(u') + 1$ and $\ell(su's) = \ell(u')$.

Case D: $\text{tail}(Q) = su'$ with $\ell(su') = \ell(u') + 1$ and $\ell(su's) = \ell(u') + 2$.

Case E: $\text{tail}(Q) = su's$ with $\ell(su's) = \ell(u') + 2$.

We shall examine in all cases the structure of Q and the cohomology of $X(Q)$.

Case A: $\text{tail}(Q) = u'$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

Let $\dim(Q) = d$, so that $\#Q = 2^d$. We consider the subintervals $I(u, w), I(su', sw'), I(u's, w's), I(u', w')$. Since Q is a square each of them is a square as well and has consequently 2^{d-2} elements. We shall see that the union of them is Q . Indeed, let $v \in Q$.

Case 1) If $v \leq w'$ then $v \in I(u', w')$.

Case 2) Let $v \leq sw'$ and $v \not\leq w'$. Thus we may write $v = sv'$ with $v' \leq w'$. As $\ell(su') = \ell(u') + 1$ we see that by considering reduced decompositions that we must have $v \geq su'$. Thus $w \in I(su', sw')$.

Case 3) Let $v \leq w's$ and $v \not\leq w'$. This case is symmetric to Case 2, hence $w \in I(u's, w's)$.

Case 4) Let $v \leq sw's$ and $v \not\leq sw'$ and $v \not\leq w's$. Then as in Case 2 we argue that $v = sv's$ with $v' \geq u'$. Hence $v \in I(su's, sw's)$.

As $4 \cdot 2^{d-2} = 2^d$, we see that the pairwise intersection of the above 4 subsquares is empty and that $I(su', sw')$ (resp. $I(u, w), I(u's, w's)$) is induced by $I(u', w')$ by multiplying with s from the left (resp. conjugating with s , multiplying with s from the right). Hence Q is a union of special squares (cf. Definition 6.4) and the cohomology is consequently given by

$$H_c^i(X(Q)) = H_c^i(X(Q(u', w's))) \oplus H_c^{i-2}(X(Q(u', w's)))(-1)$$

which is known by induction on the length.

Case B. $\text{tail}(Q) = u'$ with $u' < w'$ and $\ell(su's) = \ell(u')$. It follows that $su' = u's$ by Lemma 10.5. We claim that this case does not appear. The case where $\dim(Q) = 2$ does not occur. Let $\dim(Q) = 3$. Then Q must have the shape

$$I(w, u') : \begin{array}{ccccc} & & w & & \\ & & & & \\ & sw' & sv's & w's & \\ & sv' & w' & v's & \\ & & & & u' \end{array}$$

for some $v' \geq u'$ since $\ell(sv') = \ell(u') + 1$ (consider a reduced decomposition of v'). Hence $v' = u'$. But then $sv's = su's$, a contradiction as $\ell(sv's) = \ell(u')$.

If $\dim Q > 3$ then we argue by induction. Indeed in Q there must be a subsquare of dimension $\dim(Q) - 1$ with $\text{head}(Q) = sv's$ and $\text{tail}(Q) = u'$. By induction this is not possible.

Case C: $\text{tail}(Q) = su'$ with $\ell(us') > \ell(u') < \ell(sv')$.

We shall see that this case behaves very rigid. More precisely, we shall see that Q is paved by squares of type Case 4. Here we make usage of the following statement.

Lemma 10.6. *Let $w' \in W$, $s \in S$ with $\ell(sw's) = \ell(w') + 2$. Then there is no $v' \leq w'$ with $\ell(v') = \ell(w') - 1$ and such that $v' = su' = u's$ for some $u' \leq v'$.*

Proof. Let $w' = s_1 \cdots s_r$ be a reduced decomposition. Suppose that there exists such a $v' = su' = u's$ as above. Then there is some index $1 \leq i \leq r$ with $su' = s_1 \cdots \hat{s}_i \cdots s_r$. On the other hand, since $\ell(sv') < \ell(v')$ there exists by the *Exchange Lemma* some integer $1 \leq m \leq r$ with $ss_1 \cdots s_{m-1} = s_1 \cdots s_m$ (with s_i omitted depending on whether $i < m$ or $i > m$). If $m < i$, then $w' = s_1 \cdots s_r = ss_1 \cdots \hat{s}_m \cdots s_r$ a contradiction to the assumption that $\ell(sw') = \ell(w') + 1$. If $m > i$, then $ss_1 \cdots \hat{s}_i \cdots s_{m-1} = s_1 \cdots \hat{s}_i \cdots s_m$. But $su' = s \cdot s_1 \cdots \hat{s}_i \cdots \hat{s}_m \cdots s_r = s_1 \cdots \hat{s}_i \cdots \hat{s}_m \cdots s_r \cdot s$ as $su' = u's$. Hence we deduce that $s_{m+1} \cdots s_r \cdot s = s_m \cdot s_{m+1} \cdots s_r$. Again by plugging this expression into the reduced decomposition for w' , we obtain a contradiction to the assumption that $\ell(w's) = \ell(w') + 1$. \square

We start with the case of a square. Here it is as in Case 4 before. Consider now a hypersquare Q of dimension 3. Thus it must have the shape

$$Q: \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ ? & ? & ? \\ & su' & \end{array}$$

with $su' = u's$, $v' < w'$ and $u' < w'$ for certain elements $? \in W$. As $\ell(sv') = \ell(u') + 1$ we deduce that $v' \geq u'$. Hence we can make the structure of Q more precise, i.e.

$$Q: \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ sv' & ? & v's \\ & su' & \end{array}$$

As one verifies there are a priori for $? \in W$ only 2 possibilities : $? \in \{w', sz'\}$ with $sz' = z's$ and $z' \in I(u', w')$. But by Lemma 10.6 we must have $? = sz'$. Hence Q is the union of two squares of the shape as in Case 4.

Let $\dim(Q) = d > 3$. The square Q begins with the following elements

$$Q : \begin{array}{cccc} & & w & \\ & & \nearrow & \nwarrow \\ sw' & & sv'_1s, \dots, sv'_{d-2}s & & w's \\ \vdots & & \vdots & & \vdots \\ & & su' & & \end{array}$$

By induction on the size of Q we know that all subsquares of Q of the shape $Q(su', sv'_i s)$ are union of squares of the the desired shape. Now one verifies that square Q ends as follows

$$Q : \begin{array}{ccc} & w & \\ \vdots & \vdots & \vdots \\ sx' & sy'_1, \dots, sy'_{d-2} & x's \\ & su' & \end{array}$$

with $sy_i = y_i s$, $i = 1, \dots, d$. There must be some y_i with $sy'_i \leq w$ and again by induction the statement follows.

Next we turn to the cohomology with respect to these hypersquares. We start again with the 2-dimensional case. We fix reduced decompositions of w' and u' . We consider the hypersquare $Q^{F^+}(u', w) \subset F^+$

$$Q^{F^+}(u', w) : \begin{array}{ccccc} & & w & & \\ & & \nearrow & \uparrow & \nwarrow \\ sw' & & su's & & sw' \\ \uparrow & \nearrow \nwarrow & & \nearrow \nwarrow & \uparrow \\ su' & & w' & & u's \\ & \nwarrow & \uparrow & \nearrow & \\ & & u' & & \end{array}$$

and the interval $I(u', w)$ in W . A case by case study together with the fact (which follows by Lemma 10.5) that apart from u' there is no $z' \prec w'$ with $\ell(z') = \ell(w') - 1$ and $\gamma(z') = \gamma(u')$ it is seen that the preimage of $I(u', w)$ under the proper map $\pi : X^{\ell(w)+1} \rightarrow X$ is just $X(Q^{F^+}(u', w))$. Hence we get a proper surjective map

$$\pi : X(Q^{F^+}(u', w)) \rightarrow X(I(u', w)).$$

The hypersquare Q is an open subset of the interval $I(u', w)$ so that $U := X(Q)$ is an open subvariety of $X(I(u', w))$. The closed complement is given by $Y := X(I(u', w)) \setminus X(Q)$. We consider their preimages $U' := \pi^{-1}(X(Q))$ and $Y' := \pi^{-1}(Y)$ in $X^{\ell(w)+1}$. We get a

commutative diagram of long exact cohomology sequences

$$(10.1) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H_c^i(U') & \longrightarrow & H_c^i(X(Q^{F^+}(u', w))) & \longrightarrow & H_c^i(Y') \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & H_c^i(X(Q)) & \longrightarrow & H_c^i(X(I(u', w))) & \longrightarrow & H_c^i(Y) \longrightarrow \cdots \end{array}$$

We claim that we can recover the cohomology of $X(Q)$ by this diagram. In fact, the cohomology of $X(Q^{F^+}(u', w))$ is known by its particular structure since it is the union of special squares, i.e.

$$H_c^i(X(Q^{F^+}(u', w))) = H_c^i(X(Q^{F^+}(u', w's))) \oplus H_c^{i-2}(X(Q^{F^+}(u', w's)))(-1).$$

Further $Y = X(w') \cup X(u')$ whereas $Y' = X_1^{F^+}(su's) \cup Y^{F^+}$ with the obvious meaning for Y^{F^+} and $X_1^{F^+}(su's)$ is the closed subset of $X^{F^+}(su's)$ in Remark 4.15. Then

$$H_c^i(Y') = H_c^i(Y) \oplus H_c^{i-2}(X(u'))(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(u's))(-1).$$

Hence the restriction map $H_c^i(X(Q^{F^+}(u', w))) \longrightarrow H_c^i(Y')$ is induced by the sum of the maps

$$H_c^i(X(Q(u', w's))) \longrightarrow H_c^i(Y) \text{ and } H_c^{i-2}(X(Q(u', w's)))(-1) \longrightarrow H_c^{i-2}(X(u'))(-1).$$

The latter one factories over the representation $H_c^{i-2}(X(w') \cup X(v'))(-1)$. Hence both maps are known by induction. Thus we deduce the cohomology of U' . By factoring out the second summand in $H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(u's))(-1)$ we get the cohomology of $X(Q)$. Furthermore, the boundary map $H_c^{i-2}(X(u')) \longrightarrow H_c^{i-1}(X(u's))$ which appears in the boundary map $H_c^i(Y') \longrightarrow H_c^{i+1}(U')$ is known, as well.

If $\dim(Q) > 2$ then one verifies that the above description generalizes to the higher dimensional setting. In particular, we get

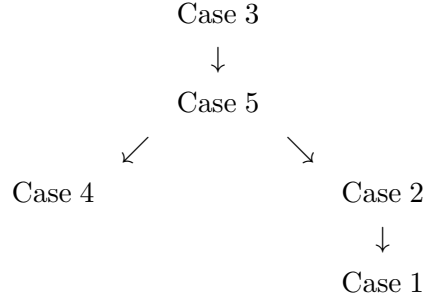
$$H_c^i(Y') = H_c^i(Y) \oplus H_c^{i-2}\left(\bigcup_{\substack{sv' \in Q \\ sv' = v's}} X(v')\right)(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}\left(\bigcup_{\substack{sv' \in Q \\ sv' = v's}} X(sv')\right)(-1).$$

The second summand is known by induction as the set $\{sv' \in Q \mid sv' = v's\}$ forms a subsquare in W .

For the remaining two cases (D and E), we introduce the following partial order on the sets of squares of type Case 1 - 5 via the following pre-order diagram.



Here the arrow Case $i \longrightarrow$ Case j means that Case $j <$ Case i .

Case D: $\text{tail}(Q) = su'$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

We start with the case of a square. Here it has the shape as in Case 2 before. Consider now a hypersquare Q of dimension 3. Thus it must have the shape

$$Q : \begin{array}{ccccc}
 & & w & & \\
 & sw' & sv's & ? & \\
 & sv' & ? & su's & \\
 & & su' & &
 \end{array}$$

for certain elements $? \in W$ where $v' \geq u'$ as $\ell(u's) = \ell(u') + 1$. There are two possibilities for completing Q . If $w' \geq su'$, then we get

$$Q : \begin{array}{ccccc}
 & & w & & \\
 & sw' & sv's & w's & \\
 & sv' & w' & su's & \\
 & & su' & &
 \end{array} .$$

On the other hand, if $w' \not\geq su'$, then we get

$$Q : \begin{array}{ccccc}
 & & w & & \\
 & sw' & sv's & sz's & \\
 & sv' & sz' & su's & \\
 & & su' & &
 \end{array} .$$

Hence if we write $Q = Q(su', sv's) \dot{\cup} Q'$ then Q' is a specialization of $Q(su', sv's)$, i.e. $Q' \leq Q(su', sv's)$. For $\dim(Q) = d > 3$, we claim that Q is paved by 3-dimensional hypersquares of this kind. More precisely, if $Q' \subset Q$ is a 3-dimensional subsquare which we write as $Q = Q_1 \dot{\cup} Q_2$ where $\text{head}(Q_i) = sv_i s$ with $v_2 \prec v_1$, then $Q_1 \leq Q_2$. Indeed, the square Q begins with the following elements

$$Q : \begin{array}{cccc} & & w & \\ & sw' & sv'_1s, \dots, sv'_{d-2}s & ? \\ & \vdots & \vdots & \vdots \\ & & su' & \end{array}$$

with $? \in \{w's, sv'_{d-1}s\}$. By induction on the size of Q we know that all subsquares of Q of the shape $Q(su', sv'_i s)$ are union of squares of the the desired shape. But the union over all these squares exhaust all apart from $\{w, sw', ?, ?\}$. Indeed the number of elements in this union is $2^{d-1} + 2^{d-2} + \dots + 2^2 = 2^d - 4$. Again if $w' \geq su'$, then $\{w, sw', ?, ?\} = Q_w$ and the claim follows. On the other hand if $w' \not\geq su'$, then we can even says that Q is paved by squares of type Case 2, since there cannot be an element $v' \leq w'$ with $v' \in Q$ and $v' \geq su'$.

Let's determine the cohomology of $X(Q)$. If $\dim(Q) = 2$, then we get

$$H_c^i(Q(su', w)) = H_c^{i-2}(Q(u', w's))(-1).$$

If $\dim(Q) = 3$ and $w' \not\geq su'$ then again - by the observation above - we get $H_c^i(Q(su', w)) = H_c^{i-2}(Q(u', w's))(-1)$. Consider the other possibility of a cube. Here we consider as in the previous case the hypersquare $\hat{Q} := Q^{F^+}(u', w)$ in F^+ and the interval $I(u', w)$ in W .

$$\hat{Q} : \begin{array}{cccccc} & & & w & & \\ & & sw' & sv's & w's & sz's \\ & sv' & w' & su's & v's & sz' & z's \\ & & su' & v' & u's & z' & \\ & & & u' & & & \end{array}$$

with $z' = su'$. The map $\pi : X^{\ell(w)+1} \longrightarrow X$ induces surjective map

$$\pi : X(Q^{F^+}(u', w)) \longrightarrow X(I(u', w))$$

which is even proper although $X(Q^{F^+}(u', w))$ might be strictly contained in the subset $\pi^{-1}(X(I(u', w)))$. (The reason is that for any closed subset $A \subset \pi^{-1}(X(I(u', w)))$ the identity $\pi(A) = \pi(A \cap X(Q^{F^+}(u', w)))$ holds). We set $U := X(Q) \subset X(I(u', w))$ and $Y := X(I(u', w)) \setminus X(Q)$. We consider their preimages $U' := \pi^{-1}(X(Q))$ and $Y' := \pi^{-1}(Y)$ in $X(Q^{F^+}(u', w))$. Again we claim that we can recover the cohomology of $X(Q)$ by the diagram 10.1. The reasoning is similar to Case C. In fact, we have

$$H_c^i(X(Q^{F^+}(u', w))) = H_c^i(X(Q^{F^+}(u', w's))) \oplus H_c^{i-2}(X(Q^{F^+}(u', w's)))(-1)$$

since $Q^{F^+}(u', w)$ is paved by special squares. Further

$$Y = X(u') \cup X(v') \cup X(u's) \cup X(v's)$$

and

$$Y' = Y^{F^+} \cup X_1^{F^+}(sz') \cup X^{F^+}(sz's)$$

whereas

$$U' = U^{F^+} \cup X^{F^+}(z') \cup X^{F^+}(z's) \cup X_2^{F^+}(sz') \cup X_2^{F^+}(sz's).$$

Here $X_2^{F^+}(su's)$ is the open subset of $X^{F^+}(su's)$ in Remark 4.15. Then

$$H_c^i(Y') = H^i(Y) \oplus H_c^{i-2}(X(u') \cup X(u's))(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(su's) \cup X(su'))(-1).$$

The restriction map $H_c^i(X(Q^{F^+}(u', w))) \rightarrow H_c^i(Y')$ is given by the sum of the maps

$$H_c^i(X(Q(u', w's))) \rightarrow H_c^i(Y) \text{ and } H_c^{i-2}(X(Q(u', w's)))(-1) \rightarrow H_c^{i-2}(X(u's) \cup X(u'))(-1).$$

Again both maps are known by induction. Thus we deduce the cohomology of U' . By factoring out the second summand in $H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(su's) \cup X(su'))(-1)$ we get the cohomology of $X(Q)$.

The higher dimensional case is treated similar as in Case C.

Case E: $\text{tail}(Q) = su's$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

We start with the case of a square. Here it has the shape as in Case 3 or Case 5 before. Consider now a hypersquare Q of dimension 3. Thus if its lower subsquare is as in Case 5 must have the shape

$$Q : \begin{array}{ccccc} & & w & & \\ & ? & sv's & ? & \\ & sv' & ? & sz's & \\ & & su's & & \end{array}$$

for certain elements $? \in W$. There are three possibilities for completing Q . If $w' \geq su'$, then we get

$$Q : \begin{array}{ccccc} & & w & & \\ & sw' & sv's & w's & \\ & sv' & sy' & sz's & \\ & & su's & & \end{array}$$

with $sy' = y's$. On the other hand, if $w' \not\geq su'$, then we get

$$Q : \begin{array}{ccccc} & & w & & \\ & sw' & sv's & sy's & \\ & sv' & ? & sz's & \\ & & su's & & \end{array}$$

with $? = sy'$ or $? = sx's$ for some $x' < v'$. Hence if we write $Q = Q(su's, sv's) \dot{\cup} Q'$ then Q' is a specialization of $Q(su's, sv's)$, i.e. $Q' \leq Q(su', sv's)$. For $\dim(Q) = d > 3$, one proves as in Case D that is paved by 3-dimensional hypersquares of this kind.

Consider now a hypersquare Q of dimension 3 such that its lower subsquare is as in Case 3,

$$Q : \begin{array}{ccccc} & & w & & \\ & ? & sv's & ? & \\ & sx's & ? & sy's & \\ & & su's & & \end{array}$$

for certain elements $? \in W$. This is the most generic case in the sense that the upper square in Q can be arbitrary, i.e. if we write $Q = Q(su's, sv's) \dot{\cup} Q'$ then Q' is a any specialization of $Q(su', sv's)$.

Now we consider the cohomology and discuss only the case of a square Q . The higher dimensional cases are treated as before, see also Remark 10.7 for a general approach. So let Q be a square as in Case 5 (Case 3 has been already explained)

$$Q : \begin{array}{ccccc} & & w & & \\ & \nearrow & & \nwarrow & \\ sw' & & & & sv's \\ & \nwarrow & & \nearrow & \\ & & su's & & \end{array}$$

where $\begin{array}{ccccc} & & w' & & \\ & \nearrow & & \nwarrow & \\ y' & & & & v' \\ & \nwarrow & & \nearrow & \\ & & u' & & \end{array}$

is a square with $y' = u's$. We consider the extended interval $I(su', w) \subset W$ resp. the cube $\hat{Q} := Q^{F^+}(su', w) \subset F^+$.

$$\hat{Q} : \begin{array}{ccccc} & & w & & \\ & \nearrow & \uparrow & \nwarrow & \\ sw' & & sv's & & sy's \\ \uparrow & \nearrow \nwarrow & & \nearrow \nwarrow & \uparrow \\ sv' & & su's & & sy' \\ & \nwarrow & \uparrow & \nearrow & \\ & & su' & & \end{array} .$$

Here

$$Y = X(su') \cup X(sv').$$

$$Y' = Y^{F^+} \cup X_1^{F^+}(sy's).$$

$$U = X(w) \cup X(sw') \cup X(sv's) \cup X(su's).$$

$$U' = U^{F^+} \cup X^{F^+}(sy') \cup X_2^{F^+}(sy's).$$

Hence we get

$$H_c^i(Y') = H_c^i(Y) \oplus H_c^{i-2}(X(u's))(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(su's))(-1).$$

The restriction map $H_c^i(X(Q^{F^+}(su', w))) \rightarrow H_c^i(Y')$ is given as follows. First note that $H_c^i(X(\hat{Q})) = H_c^{i-2}(X(s \setminus Q))(-1)$, where $s \setminus Q$ is the cube such that $s \cdot (s \setminus Q) = Q$. With respect to the summand $H_c^i(Y)$ we know the map factorizes over $H_c^i(X(Q^{F^+}(su', sw'))) \rightarrow H_c^i(X(sv') \cup X(su'))$. As for the summand $H_c^{i-2}(X(su'))(-1)$ the necessary information follows from the identity $H_c^i(X(\hat{Q})) = H_c^{i-2}(X(s \setminus Q))(-1)$. All maps are known. On the other hand we know by induction the boundary map $H_c^{i-2}(X(su')) \rightarrow H_c^{i-1}(X(su's))$. Again we deduce the cohomology of U' and by factoring out the summand $H_c^{i-2}(X(su's))(-1)$ we get the cohomology of U .

Thus we have examined all cases. It remains to say that the start of induction is the situation where $\text{head}(Q)$ is minimal in its conjugacy class. This case can be handled explicitly using successively Proposition 5.7.

Remark 10.7. Let $I = I(u, w) \subset W$ be any interval. The map $\pi : X^{\ell(w)+1} \rightarrow X$ induces a proper map

$$\pi : \pi^{-1}(X(I)) \rightarrow X(I).$$

The following lines give a description of the preimage $Z = \pi^{-1}(X(I)) \subset X^{\ell(w)+1}$. Let $v \in Q^{F^+}(1, w)$.

1. Case. $\ell(\gamma(v)) = \ell(v)$.

Subcase a) $\gamma(v) \not\geq u$. In this case $X^{F^+}(v) \cap Z = \emptyset$.

Subcase b) $\gamma(v) \geq u$. In this case $X^{F^+}(v) \subset Z$ and the restriction of π to $X^{F^+}(v)$ induces an isomorphism $X^{F^+}(v) \xrightarrow{\sim} X(\gamma(v))$.

2. Case. $\ell(\gamma(v)) < \ell(v)$. By Lemma 5.2 we may suppose that $v = v_1 \cdot t \cdot t \cdot v_2$. Thus we may write $X^{F^+}(v) = X_1^{F^+}(v) \cup X_2^{F^+}(v)$ where $X_1^{F^+}(v)$ is closed and $X_2^{F^+}(v)$ is open. We have \mathbb{A}^1 -bundles $X_1^{F^+}(v) \rightarrow X^{F^+}(v_1v_2)$ and $X_2^{F^+}(v) \cup X^{F^+}(v_1tv_2) \rightarrow X^{F^+}(v_1tv_2)$. The map $\pi|_{X^{F^+}} : X^{F^+}(v) \rightarrow X$ factorizes through $X^{F^+}(v_1tv_2) \cup X^{F^+}(v_1v_2)$. Hence we have reduced the question to elements of lower length.

Suppose additionally that $w = sw's$. Then $Q^{F^+}(1, w)$ is paved by special squares. Then it is possible to say what the image of such a special square $Q_v = \{v, sv', v's, v'\}$ under the map π is. But we do not carry out this since there are too many cases.

11. APPENDIX B

Here we give summarizing tables of the cohomology of DL-varieties with respect to Weyl group elements of full support in GL_3 and GL_4 . We list only representatives of cyclic shift classes.

GL_3	$H_c^*(X(w))$
$(1, 2, 3)$	$j_{(1,1,1)}[-2] \oplus j_{(2,1)}(-1)[-3] \oplus j_{(3)}(-2)[-4]$
$(1, 3)$	$j_{(1,1,1)}[-3] \oplus j_{(3)}(-3)[-6]$

GL_4	$H_c^*(X(w))$
$(1, 2, 3, 4)$	$j_{(1,1,1,1)}[-3] \oplus j_{(2,1,1)}(-1)[-4] \oplus j_{(3,1)}(-2)[-5] \oplus j_{(4)}(-3)[-6]$
$(1, 2, 4)$	$j_{(1,1,1,1)}[-4] \oplus j_{(2,2)}(-2)[-5] \oplus j_{(4)}(-4)[-8]$
$(1, 3)(2, 4)$	$j_{(1,1,1,1)}[-4] \oplus j_{(2,2)}(-1)[-4] \oplus j_{(2,1,1)}(-2)[-5] \oplus j_{(3,1)}(-2)[-5] \oplus$ $j_{(2,2)}(-3)[-6] \oplus j_{(4)}(-4)[-8]$
$(1, 3, 2, 4)$	$j_{(1,1,1,1)}[-5] \oplus j_{(2,2)}(-2)[-6] \oplus j_{(2,1,1)}(-2)[-6] \oplus j_{(2,2)}(-3)[-7] \oplus$ $j_{(3,1)}(-3)[-7] \oplus j_{(4)}(-5)[-10]$
$(1, 4)$	$j_{(1,1,1,1)}[-5] \oplus j_{(2,1,1)}(-1)[-5] \oplus j_{(2,2)}(-2)[-6] \oplus j_{(2,2)}(-3)[-7] \oplus$ $j_{(3,1)}(-4)[-8] \oplus j_{(4)}(-5)[-10]$
$(1, 4)(2, 3)$	$j_{(1,1,1,1)}[-6] \oplus j_{(2,1,1)}(-2)[-7] \oplus j_{(2,2)}(-3)[-8]^2 \oplus j_{(3,1)}(-4)[-9] \oplus j_{(4)}(-6)[-12].$

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