Norm-Based Approximation in Multicriteria Programming[1]

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Abstract

Based on new theoretical results on norms, heuristic algorithms to approximate the nondominated set of multicriteria programs are proposed. By automatically adapting to the problem's structure and scaling, the approximation is constructed objectively without interaction with the decision maker. As the algorithms extend the results obtained for bicriteria programs, difficulties encountered when dealing with more than two criteria are discussed.

Keywords: multicriteria optimization, nondominated set, approximation, norms

1 Introduction

Decision making with respect to several conflicting criteria and constraints has become a central problem in management and technology. Trade-off information plays a central role in decision making since it facilitates the comparison of different alternatives. Approximations of the nondominated set visualize the alternatives for the decision maker and provide this trade-off information in a simple and understandable way.

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In this paper we suggest to use cones and norms, two concepts well-known in convex analysis, to construct piecewise linear approximations of the nondominated set of general multicriteria programming problems. Both cones and norms have been used in multicriteria programming quite extensively but, to our knowledge, Kaliszewski (1994) is the only other source to simultaneously combine both concepts in order to describe and solve multicriteria programs.

Norms have frequently been used in multicriteria programming to measure the distance between the solutions and the utopia point. In particular, the l_∞ norm and the augmented l_∞ norm were used for generating nondominated solutions of general continuous or discrete multicriteria programs and led to the well known weighted (augmented) Tchebycheff scalarization and its variations, see Steuer and Choo (1983). Kaliszewski (1987) introduced a modified l_∞ norm and showed its applicability in generating nondominated solutions. Carrizosa *et al.* (1997) suggested to use a class of norms that contains the family of l_p norms to generate the set of points that have minimal distance to the utopia point with respect to at least one norm within this class of norms. Further applications of norms in the context of multicriteria decision making can be found, among others, in Yu (1973), Zeleny (1973), Wierzbicki (1980), Steuer and Choo (1983) and Steuer (1986).

The literature on approximation of the nondominated set of general multicriteria problems is not rich in comparison to the literature devoted to the bicriteria case. The former is reviewed below while for an overview of the latter we refer to Schandl *et al.* (1999). Properties that hold quite naturally in the bicriteria case, do not hold in general in higher dimensions. Therefore many complex issues concerning the approximation arise only in multicriteria problems. We discuss some of these problems in the following sections.

Polak (1976) proposes to approximate the nondominated set by generating nondominated points as minimizers of constrained single criterion problems and constructing a piecewise linear or spline approximation from the determined candidates. This algorithm is later modified in Jahn and Merkel (1992), where special attention is given to the bicriteria case. Helbig

(1991) uses a slight variation of the direction method proposed in Pascoletti and Serafini (1984) to calculate a discrete approximation of a nondominated set in $\mathbb{R}^n$.

An approximation method based on the Tchebycheff approach is proposed in Kaliszewski (1994). Using a modified weighted Tchebycheff norm, several nondominated points are generated and then used for the generation of an approximation of the nondominated set. In Kostreva *et al.* (1995), the weighted l_∞ distance to the utopia point is minimized for a set of weights. The calculated points are then used to construct a linear approximation of the nondominated set in $\mathbb{R}^n$. Special attention is given to noncontinuous objective functions.

Sobol' and Levitan (1997) develop an approximation method based on the parameter space investigation introduced in Statnikov (1978). Benson and Sayin (1997) propose a global shooting procedure to find a global representation of the nondominated set of a general multicriteria problem with a compact feasible set.

Das (1999) briefly discusses an approach based on the normal-boundary intersection technique, introduced in Das and Dennis (1998). Using the hyperplane defined by the individual minimizers of the criteria, the nondominated points with maximal distance from this hyperplane in some specified directions are determined.

Some authors developed probability-based approaches to the approximation. Among others, genetic algorithms were developed by Fonseca and Fleming (1995) while simulated annealing was studied by Czyzak and Jaszkiewicz (1998) and Ulungu *et al.* (1999).

We propose a methodological framework for approximating continuous (convex and nonconvex) and discrete problems. Our assumptions are mild since we only require that the set of all feasible criterion vectors be $\mathbb{R}^n_{\geqq}$-closed and that the set of nondominated solutions be $\mathbb{R}^n_{\geqq}$-bounded. We show that the combination of norms and cones is a very powerful tool, on one hand, to perform the approximation in a very objective, neutral and efficient way and, on the other hand, to gain important information concerning the structure of the nondominated set and the trade-offs between the criteria in different regions of the nondominated set.

In the next section we state the multicriteria programming problem and give some general definitions and notations. The theoretical basis for the approximation algorithms is discussed in Section 3. Approximation approaches for problems with $\mathbb{R}^n_{\geqq}$-convex, $\mathbb{R}^n_{\geqq}$-nonconvex and discrete sets of feasible criterion vectors are presented in Sections 4, 5, and 6, respectively. The last section includes a short summary and some concluding remarks.

2 Problem Formulation

To facilitate further discussions, the following notation is used throughout the paper.

Let $u, w \in \mathbb{R}^n$ be two vectors. We denote components of vectors by subscripts and enumerate vectors by superscripts. $u < w$ denotes $u_i < w_i$ for all $i = 1, \ldots, n$. $u \leq w$ denotes $u_i \leq w_i$ for all $i = 1, \ldots, n$, but $u \neq w$. $u \leqq w$ allows equality. The symbols $>, \geq, \geqq$ are used accordingly.

Let $\mathbb{R}^n_{\geqq} := \{x \in \mathbb{R}^n : x \geqq 0\}$. If $S \subseteq \mathbb{R}^n$, then $S_{\geqq} := S \cap \mathbb{R}^n_{\geqq}$. The sets $\mathbb{R}^n_{\geq}$, $\mathbb{R}^n_{>}$, $S_{\geq}$ and $S_{>}$ are defined accordingly.

A set $C \in \mathbb{R}^n$ is called a *cone* if for all $u \in C$ and $\alpha > 0$ we also have $\alpha u \in C$. The origin may or may not belong to C. If $U = \{u^1, \ldots, u^k\} \subseteq \mathbb{R}^n$ is a set of vectors, then

$$\mathrm{cone}(U) := \left\{ v \in \mathbb{R}^n : v = \sum_{i=1}^{k} \alpha_i u^i, \alpha_i \geq 0, u^i \in U \right\}$$

is the cone generated by U.

We consider the following general multicriteria program

$$\begin{array}{ll} \min & \{z_1 = f_1(x)\} \\ & \vdots \\ \min & \{z_n = f_n(x)\} \\ \text{s.t.} & x \in X, \end{array} \tag{1}$$

where $X \subseteq \mathbb{R}^m$ is the *feasible set* and $f_i(x), i = 1, \dots, n$, are real-valued functions. We define the *set of all feasible criterion vectors* Z, the *set of all (globally) nondominated criterion vectors* N and the *set of all efficient points* E of (1) as follows

$$\begin{aligned} Z &= \{z \in \mathbb{R}^n : z = f(x), x \in X\} = f(X) \\ N &= \{z \in Z : \nexists \tilde{z} \in Z \text{ s.t. } \tilde{z} \leq z\} \\ E &= \{x \in X : f(x) \in N\}, \end{aligned}$$

where $f(x) = \big(f_1(x), \dots, f_n(x)\big)^T$. We assume that the set Z is $\mathbb{R}^n_{\geqq}$-closed and that we can find $u \in \mathbb{R}^n$ so that $u + Z \subseteq \mathbb{R}^n_{\geqq}$.

The set of properly nondominated solutions is defined according to Geoffrion (1968): A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$ such that for each $i = 1, \dots, n$ and each $z \in Z$ satisfying $z_i < \bar{z}_i$ there exists a $j \neq i$ with $z_j > \bar{z}_j$ and

$$\frac{z_i - \bar{z}_i}{\bar{z}_j - z_j} \leq M.$$

Otherwise $\bar{z} \in N$ is called *improperly nondominated*. The set of all properly nondominated points is denoted by N_p.

The point $z^* \in \mathbb{R}^n$ with

$$z_i^* = \min\{f_i(x) : x \in X\} - \epsilon_i \qquad i = 1, \dots, n$$

is called the *ideal (utopia) criterion vector*, where the components of $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$ are small positive numbers.

For bicriteria problems, the point $z^\times \in \mathbb{R}^2$ with

$$z_i^\times = \min\Big\{ f_i(\bar{x}) : f_j(\bar{x}) = \min_{x \in X} f_j(x), j \neq i \Big\} \qquad i = 1, 2$$

is called the *nadir point*. Note that this definition cannot be directly generalized to multicriteria problems.

3 Oblique Norms

The concept of oblique norms was introduced in Schandl *et al.* (1998) and Schandl (1999). Since oblique norms can be viewed as a special class of block norms, we first review some basic definitions about block norms and, more general, polyhedral gauges. Then oblique norms are discussed in the context of multicriteria programming. For a detailed introduction to norms and their properties we refer to Rockafellar (1970), Hiriart-Urruty and Lemaréchal (1993a) and Hiriart-Urruty and Lemaréchal (1993b). An overview of basic properties of block norms is also given in Schandl (1998).

We define polyhedral gauges according to Minkowski (1911):

Definition 3.1 Let B be a polytope in $\mathbb{R}^n$ containing the origin in its interior and let $z \in \mathbb{R}^n$. The *polyhedral gauge* $\gamma : \mathbb{R}^n \to \mathbb{R}$ of z is defined as

$$\gamma(z) := \min\{\lambda \geq 0 \, : \, z \in \lambda B\}.$$

If B is symmetric with respect to the origin, then γ is called a *block norm.*

The vectors defined by the extreme points of the unit ball B of γ are called *fundamental vectors* and are denoted by v^i. The fundamental vectors defined by the extreme points of a facet of B span a *fundamental cone.*

If z is in a fundamental cone C of a polyhedral gauge γ then one needs to consider only the fundamental vectors generating this cone to calculate the gauge of z. This result was proven in Hamacher and Klamroth (2000) for the two-dimensional case. In Theorem 3.2, this result is generalized to the multicriteria case.

Theorem 3.2 Let γ be a polyhedral gauge with the unit ball $B \subseteq \mathbb{R}^n$. Let $\bar{z} \in C$ where C is the fundamental cone generated by the fundamental vectors $v^1, \ldots, v^k$, $k \geq n$. Let $\bar{z} = \sum_{i=1}^k \lambda_i v^i$ be a representation of $\bar{z}$ in terms of $v^1, \ldots, v^k$. Then $\gamma(\bar{z}) = \sum_{i=1}^k \lambda_i$.

Proof. By definition, all fundamental vectors generating a fundamental cone are extreme points of the same facet of the unit ball B. Thus $v^1, \ldots, v^k$ are all located on a common hyperplane defined by, say, $\langle \mathfrak{n}, z \rangle = d$ where $\mathfrak{n} \in \mathbb{R}^n$ is the normal of the hyperplane and $d \in \mathbb{R}$. Since γ is a gauge, the origin is in the interior of the unit ball B and therefore $d \neq 0$.

Since $\bar{z} \in C$, the point $\bar{z}$ can be written as $\bar{z} = \gamma(\bar{z})\tilde{z}$ where $\tilde{z}$ is located on the same facet as $v^1, \ldots, v^k$. It follows that $\langle \mathfrak{n}, \tilde{z} \rangle = d$ and therefore

$$\begin{aligned}\gamma(\bar{z}) &= \gamma(\bar{z})\frac{\langle \mathfrak{n}, \tilde{z} \rangle}{d} = \frac{1}{d}\langle \mathfrak{n}, \gamma(\bar{z})\tilde{z} \rangle = \frac{1}{d}\langle \mathfrak{n}, \bar{z} \rangle = \frac{1}{d}\left\langle \mathfrak{n}, \sum_{i=1}^{k} \lambda_i v^i \right\rangle \\ &= \frac{1}{d}\sum_{i=1}^{k} \lambda_i \langle \mathfrak{n}, v^i \rangle = \frac{1}{d}\sum_{i=1}^{k} \lambda_i d = \sum_{i=1}^{k} \lambda_i. \qquad \square\end{aligned}$$

Note that *all* representations $\bar{z} = \sum_{i=1}^{k} \lambda_i v^i$ can be used to calculate $\gamma(\bar{z})$, even combinations where one or more λ_i's are negative which is only possible if $k > n$. If C is generated by n fundamental vectors though, the representation of $\bar{z}$ in terms of $v^1, \ldots, v^n$ is unique and all corresponding λ_i's are nonnegative.

For the definition of oblique norms we additionally need the concepts of reflection sets and of absolute norms.

Let $u \in \mathbb{R}^n$. The *reflection set* of u is defined as

$$R(u) := \{w \in \mathbb{R}^n : |w_i| = |u_i| \quad \forall i = 1, \ldots, n\}. \tag{2}$$

Using (2) we define absolute norms analogously to Bauer *et al.* (1961).

Definition 3.3 A norm γ is said to be *absolute* if for any given $u \in \mathbb{R}^n$, all elements of the reflection set $R(u)$ of u have the same distance from the origin with respect to γ, i. e.

$$\gamma(w) = \gamma(u) \quad \forall w \in R(u).$$

Note that the unit ball of an absolute norm has the same structure in every orthant of the coordinate system.

Definition 3.4 A block norm γ with a unit ball B is called *oblique* if it has the following properties:

(i) γ is absolute,

(ii) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geqq}$.

Observe that Definition 3.4 implies that an oblique norm is a block norm where no facet of the unit ball is parallel to any coordinate axis. Moreover, since an oblique norm is also an absolute norm, the structure of the norm's unit ball is the same in every orthant of the coordinate system. This property is convenient for the generation of nondominated solutions of (1) since they may only occur in $z^* + \mathbb{R}^n_{\geqq}$. An example of an oblique norm in $\mathbb{R}^2$ is given in Figure 1.

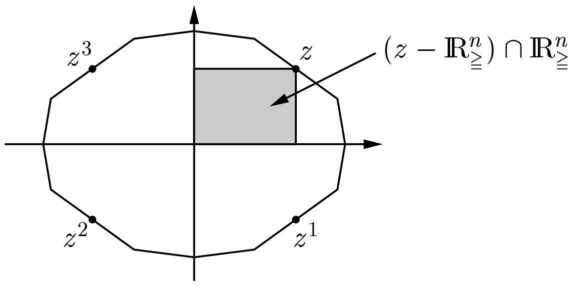


Figure 1: Example of the unit ball of an oblique norm with $R(z) = \{z, z^1, z^2, z^3\}$

In Schandl *et al.* (1998) it was shown that oblique norms centered at the utopia point z^* can be used to generate all properly nondominated solutions of (1). Moreover, every solution of such an oblique norm scalarization of (1) yields a nondominated solution. However, in the bicriteria case (see Schandl *et al.*, 1999) it turned out useful to use an oblique norm centered at a point in $Z + \mathbb{R}^n_{\geqq}$ for the approximation of the nondominated set. The theoretical foundation for this approach for higher dimensional problems is given by the following result showing that every oblique norm scalarization of (1) with an oblique norm centered at an arbitrary point in $Z + \mathbb{R}^n_{\geqq}$ yields a nondominated solution.

Theorem 3.5 Assume without loss of generality that $0 \in Z + \mathbb{R}^n_{\geqq}$. Let γ be an oblique norm with the unit ball B. If $\bar{z} \in \mathbb{R}^n$ is a solution of

$$\begin{array}{ll} \max & \gamma(z) \\ \text{s.t.} & z \in -\mathbb{R}^n_{\geqq} \cap Z \end{array} \tag{3}$$

then $\bar{z}$ is nondominated.

Proof. Assume $\bar{z} \notin N$, that is, there exists $\tilde{z} \in Z$ with $\tilde{z} \leq \bar{z}$. Since $\bar{z}$ is feasible for (3), we have $\tilde{z} \in -\mathbb{R}^n_{\geqq}$, and it follows that

$$\bar{z} \in -\mathbb{R}^n_{\geqq} \cap (\tilde{z} + \mathbb{R}^n_{\geqq}).$$

Since γ is oblique and therefore absolute, we can use the fact that an oblique norm γ with the unit ball B has the following properties:

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial(\gamma(z)B) = \{z\} \quad \forall z \in \mathbb{R}^n_{\geqq}\ , \tag{4}$$

and

$$(z - \mathbb{R}^n_{\geq}) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(z)B_{\geqq} \quad \forall z \in \mathbb{R}^n_{\geqq}\ , \tag{5}$$

see Schandl *et al.* (1998) or Schandl (1999). Using (4) and (5) in $-\mathbb{R}^n_{\geqq}$ instead of $\mathbb{R}^n_{\geqq}$ we can infer that

$$\bar{z} \in \operatorname{int}\big(\gamma(\tilde{z})B\big)$$

which implies $\gamma(\bar{z}) < \gamma(\tilde{z})$, a contradiction to the optimality of $\bar{z}$. $\square$

Unfortunately, we cannot guarantee to find all nondominated points using an oblique norm with its unit ball's center in $Z+\mathbb{R}^n_{\geqq}$ in the general setting of Theorem 3.5. Therefore the next theorem applies only to problems with an $\mathbb{R}^n_{\geqq}$-convex feasible set Z.

Theorem 3.6 Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}^n_{\geqq}$-convex and assume without loss of generality that $0 \in Z + \mathbb{R}^n_{\geqq}$. Let $\bar{z}$ be properly nondominated with $\bar{z} \in -\mathbb{R}^n_{\geqq} \cap N_p$.

Then there exists an oblique norm γ so that $\bar{z}$ solves the following problem:

$$\begin{array}{ll} \max & \gamma(z) \\ \text{s.t.} & z \in -\mathbb{R}^n_{\geqq} \cap Z. \end{array} \tag{6}$$

Proof. From Geoffrion (1968), we know that there exists a weight vector $w \in \mathbb{R}^n_{>}$ with $\sum_{i=1}^n w_i = 1$ so that $\bar{z}$ solves

$$\min_{z \in Z} \sum_{i=1}^n w_i z_i.$$

Let H be the hyperplane defined by the normal w and the point $\bar{z}$, see Figure 2, and H^+ be the halfspace defined as

$$H^+ := \{z \in \mathbb{R}^n : \langle w, z \rangle \geq \langle w, \bar{z} \rangle\}.$$

Then the set $R(-\mathbb{R}^n_{\geqq} \cap H^+)$ is the unit ball of an oblique norm γ.

Since $\bar{z} \in H$, it follows that $\bar{z}$ is located on a facet of the unit ball and thus $\gamma(\bar{z}) = 1$. So there cannot exist $z \in Z$ with $\gamma(z) > 1$, because H is a tangent plane of Z. Therefore $\bar{z}$ solves (6). $\square$

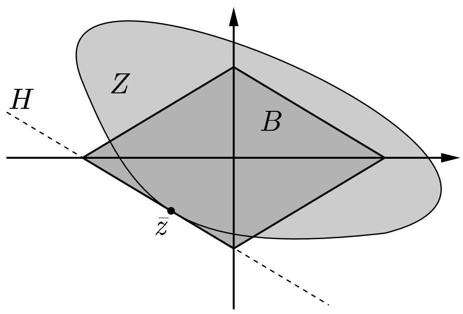


Figure 2: Illustration of the proof of Theorem 3.6

Note that $\bar{z}$ is in general not a *unique* solution of (6) with the constructed oblique norm. If $\bar{z}$ is, for example, in the interior of a facet of a polyhedral set Z, then all elements of that facet are solutions of (6).

4 Convex Multicriteria Problems

To keep explanations straightforward, the proposed approximation approach is illustrated using an $\mathbb{R}^2_{\geqq}$-convex example as given in Figure 3. We emphasize that higher dimensional problems cannot be illustrated and explained with bicriteria cases, see Schandl (1999), and require a more sophisticated treatment.

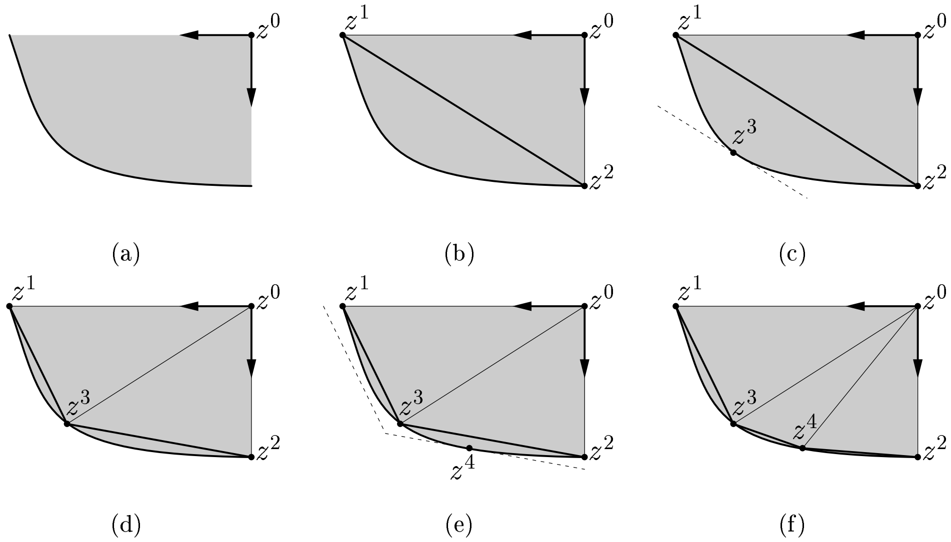


Figure 3: The steps of the approximation algorithm

The approximation process is started by choosing a reference point $z^0 \in Z + \mathbb{R}^n_{\geqq}$ and defining $z^0 - \mathbb{R}^n_{\geqq}$ as the region in which the nondominated set N is approximated. The reference point might be a currently implemented (not nondominated) solution or just a (not necessarily feasible) guess. Without loss of generality, we assume throughout this section that the reference point is located at the origin.

A first approximation is obtained by exploring the feasible set along $m \geq n$ search directions $d^1, \ldots, d^m \in -\mathbb{R}^n_{\geq}$. In the example given in Figure 3 the search directions are chosen as the negative unit vectors in $\mathbb{R}^2$, $d^1 = (-1, 0)$ and $d^2 = (0, -1)$, yielding the points z^1 and z^2. These two points together with the reference point z^0 are used to define a cone and a first approximation, see Figure 3(b). Interpreting this approximation as the lower left part of the unit ball of a polyhedral gauge γ (or an oblique norm) with z^0 as its center, this gauge is then maximized in $Z \cap (z^0 - \mathbb{R}^n_{\geqq})$. Consequently the next point (z^3 in the example problem) is found as a solution of problem (3), where γ is a polyhedral gauge. Observe that according to Theorem 3.6 the new point z^3 is always nondominated if γ is also an oblique norm.

The point z^3 is added to the approximation by building the convex hull of the candidate points generated so far and thus updating the approximation and the underlying gauge simultaneously, see Figure 3(d). Continuing this process, we get a finer approximation of the nondominated set while generating candidates for nondominated points and updating the unit ball of the polyhedral gauge. In each iteration, the candidate point of maximal gauge is added. Since this point is "farthest away" from the approximation with respect to the current gauge, we always add the point of worst approximation with respect to this gauge.

The following theorem shows that the quality of the approximation improves with each new point.

Theorem 4.1 Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}^n_{\geqq}$-convex and γ^k be an approximating gauge constructed from k nondominated points or points on the boundary of Z. Let $\bar{z}$ be a solution of

$$\begin{array}{ll} \max & \gamma^k(z) \\ \text{s.t.} & z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq}). \end{array} \tag{7}$$

Let γ^{k+1} be the updated gauge including the new point $\bar{z}$. Then

$$\gamma^{k+1}(z) \leq \gamma^k(z) \qquad \forall z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq}).$$

Proof. Let B^k and B^{k+1} be the unit balls of γ^k and γ^{k+1}, respectively. Since Z is $\mathbb{R}^n_{\geqq}$-convex, it follows that $\gamma^k(\bar{z}) \geq 1$ and therefore $B^k \subseteq B^{k+1}$. Thus for every $z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq})$ we have

$$\gamma^{k+1}(z) = \min\{\lambda \geq 0 : z \in \lambda B^{k+1}\} \leq \min\{\lambda \geq 0 : z \in \lambda B^k\} = \gamma^k(z). \qquad \square$$

In the following paragraphs the details of the resulting approximation algorithm are outlined.

One of the difficulties arising in higher dimensional problems is the fact that the nadir point concept does not directly generalize from the bicriteria case. Consequently, the nadir point is not available as a default reference point. If no starting solution is provided by the decision maker, a reference point z^0 can be constructed by the following method: The utopia point z^* is determined by calculating minimizers for each component, that is, points $\tilde{z}^i$ with $\tilde{z}^i_i = \min\{z_i : z \in Z\}$ are found for each $i = 1, \ldots, n$. Let $Z^* := \{\tilde{z}^i, i = 1, \ldots, n\}$. Then the i^{th} component, $i = 1, \ldots, n$, of the default reference point is defined as

$$z_i^0 = \max\{z_i : z \in Z^*\},$$

which can be viewed as a possible generalization of the nadir point for bicriteria problems.

The negative unit vectors can be used as default for the search directions $d^1, \ldots, d^m \in -\mathbb{R}^n_{\geqq}$ $(m \geq n)$ needed for the first approximation of the nondominated set. Then an adaptation of the *direction method* introduced in Pascoletti and Serafini (1984) is utilized to search for globally nondominated points in the entire set Z along the search directions. The following theorem adds a second step to their original problem formulation and guarantees to generate a nondominated point for general multicriteria problems.

Theorem 4.2 Let $z^0 \in Z + \mathbb{R}^n_{\geqq}$, $d \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geqq}$ and $1 \leq p < \infty$. Then the problem

$$\begin{array}{ll} \text{lex max} & (\alpha, \|q\|_p) \\ \text{s. t.} & z = z^0 + \alpha d + q \\ & q \in -\mathbb{R}^n_{\geqq} \\ & z \in Z, \end{array} \tag{8}$$

has a finite solution $(\bar{\alpha}, \bar{z}, \bar{q})$ where $\bar{z}$ is a globally nondominated point.

Proof. Since $z^0 \in Z + \mathbb{R}^n_{\geqq}$, there exists $\tilde{z} \in Z$ and $\tilde{q} \in \mathbb{R}^n_{\geqq}$ such that $z^0 = \tilde{z} + \tilde{q}$.

Thus $(\alpha, z, q) = (0, \tilde{z}, -\tilde{q})$ is a feasible solution of (8) where d is arbitrary. Therefore problem (8) is feasible.

To prove that (8) is not unbounded consider the utopia point z^*. We have $z^* - z^0 < 0$. Since $d \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geqq}$, there exists $i \in \{1, \ldots, n\}$ so that $d_i < 0$. Choosing $\alpha > \frac{z_i^* - z_i^0}{d_i}$, we have

$$z_i^0 + \alpha d_i + q_i \leq z_i^0 + \alpha d_i < z_i^0 + \frac{z_i^* - z_i^0}{d_i} d_i = z_i^*.$$

Since $z^* < z$ for all $z \in Z$ it follows that the constraint $z = z^0 + \alpha d + q$ is infeasible for any $\alpha > \frac{z_i^* - z_i^0}{d_i}$, so we have an upper bound for the first objective α.

Consider the second objective $\|q\|_p$. Since $q \in -\mathbb{R}^n_{\geqq}$, then $q_i \leq 0$ for all $i = 1, \ldots, n$. On the other hand, we have $q_i \geqq z_i^* - z_i^0$. It follows that $\|q\|_p \leqq \left\|z^* - z^0\right\|_p$. Therefore the second objective is bounded as well.

As we showed that (8) has a finite solution, it remains to prove that this solution is nondominated. Due to Pascoletti and Serafini (1984), the solution of the first optimization step is weakly nondominated, so it is sufficient to consider the second step. Assume $(\bar{\alpha}, \bar{z}, \bar{q})$ is a solution of the second step of (8) where $\bar{z}$ is weakly nondominated, but not nondominated. Thus there exists $\tilde{z} \in Z$ with $\tilde{z} \leq \bar{z}$.

Since $(\bar{\alpha}, \bar{z}, \bar{q})$ is also a solution of the first step of (8), Theorem 3.3 from Pascoletti and Serafini (1984) implies that $(\bar{\alpha}, \tilde{z}, \bar{q} + \tilde{q})$ is a solution of the first step as well where $\tilde{q} \in \partial(-\mathbb{R}^n_{\geqq})$ and $\tilde{q} \neq 0$. Then $\bar{q}_i \leq 0$ and $\tilde{q}_i \leq 0$ for all $i = 1, \ldots, n$ and $\tilde{q}_j < 0$ for some $j \in \{1, \ldots, n\}$ and thus $|\bar{q}_i| \leq |\bar{q}_i + \tilde{q}_i|$ for all $i = 1, \ldots, n$ and $|\bar{q}_j| < |\bar{q}_j + \tilde{q}_j|$. It follows that

$$\|\bar{q}\|_p = \left(\sum_{i=1}^{n} |\bar{q}_i|^p\right)^{1/p} < \left(\sum_{i=1}^{n} |\bar{q}_i + \tilde{q}_i|^p\right)^{1/p} = \|\bar{q} + \tilde{q}\|_p,$$

a contradiction to the optimality of $(\bar{\alpha}, \bar{z}, \bar{q})$ in step two of the optimization.

□

If the nondominated points found by the direction method generate a cone (or several cones) with empty interior, the algorithm has to be restarted with new or additional directions. Otherwise, the convex hull of the nondominated points and the reference point is constructed using the Beneath-Beyond Algorithm, see Edelsbrunner (1987). Given a set S of k points in $\mathbb{R}^n$, the Beneath-Beyond Algorithm constructs the convex hull of S by iteratively adding new points to partial convex hulls in $\mathrm{O}(k \log k + k^{\lfloor (n+1)/2 \rfloor})$ time and $\mathrm{O}(k^{\lfloor n/2 \rfloor})$ storage. It is among the most efficient algorithms for the construction of convex hulls, and it is particularly well-suited for an incorporation into our approximation algorithm.

The facets of the constructed convex hull not containing the reference point are used to define the cones of the initial approximation. Our goal is to approximate ∂Z in these cones which is, in the multicriteria case, not necessarily the same as approximating the set of nondominated solutions N. However, dominated points found during the approximation process can be easily removed after the approximation is completed.

Now a polyhedral gauge based on the current approximation is used to find a new point in each cone. Assuming that z^0 is the origin and that the current cone is defined by $k \geq n$ points $z^1, \ldots, z^k$, the *gauge method* solves the following problem (cf. Theorem 3.2):

$$\begin{array}{ll} \max & \sum_{i=1}^{k} \lambda_i \\ \text{s.t.} & z = \sum_{i=1}^{k} \lambda_i z^i \\ & \lambda_i \geq 0 \qquad i = 1, \ldots, k \\ & z \in Z. \end{array} \tag{9}$$

In contrast to the bicriteria case, this program does not necessarily generate a nondominated point in higher dimensional problems if the normal of the facet generating that point is not negative. The underlying gauge in (9) can not be extended to an oblique norm and Theorem 3.5 does not apply. For an example problem where facets with negative normals occur during the approximation process, consider the unit ball of the l_2 norm as the feasible set Z (see Figure 4). Obviously, the three points $z^1 = (-1, 0, 0)$,

$z^2 = (0, -1, 0)$ and $z^3 = (0, 0, -1)$ are (improperly) nondominated. It is impossible to find a fourth nondominated point for which all three resulting facets have negative outer normals; in fact, at least one of the normals always has a positive component. Nevertheless, the program generates the point of worst approximation in the cone even in this case.

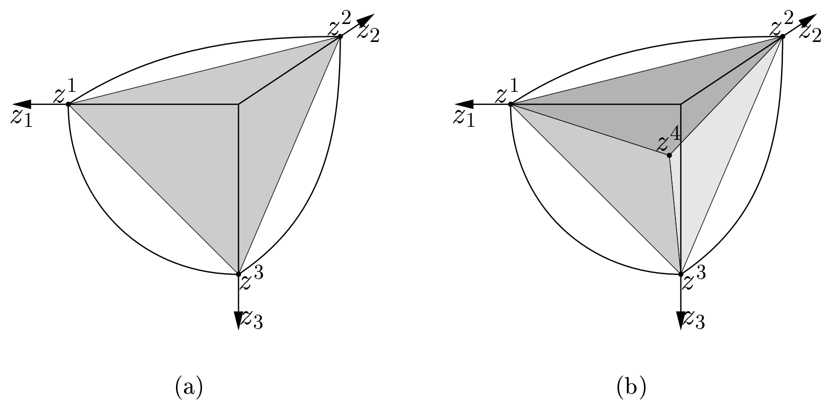


Figure 4: (a) Approximation of the unit sphere with negative normal, and (b) with nonnegative normals (z^4 is located on the sphere)

The *deviation* of a candidate $\bar{z} \in Z$ found by the gauge method (9) is defined as

$$\mathrm{dev}(\bar{z}) := |\gamma(\bar{z}) - 1| \,.$$

Thus $\mathrm{dev}(\bar{z})$ relates to the distance of the current approximation from the point $\bar{z}$ with respect to the gauge induced by the approximation itself. Using the result of Theorem 3.2, the objective value of (9) is the gauge of the candidate and thus the candidate's deviation is calculated as a by-product of problem (9). By adding a candidate $\bar{z}$ with the maximal deviation to the current approximation, we construct a new polyhedral gauge "induced by the problem".

Once a new candidate $\bar{z}$ of maximal deviation is identified, the convex hull of the candidate points generated so far is updated unless some stopping criterion is satisfied. Examples for possible stopping criteria are $\mathrm{dev}(\bar{z}) < \epsilon$ with some given error bound $\epsilon > 0$, or an upper bound `maxConeNo` on the number of constructed cones. The new convex hull can be constructed by applying one iteration of the Beneath-Beyond Algorithm, adding the point $\bar{z}$ to the previous approximation. Note that one of the central problems of the Beneath-Beyond Algorithm, the identification of a facet that is visible from the point being added to the convex hull and that needs to be changed or removed in the current iteration, is solved implicitly: The facet of the cone in which the new point $\bar{z}$ has been found is always a visible facet.

In the subsequent iterations of the approximation algorithm, we alternate between generating a new candidate point by the gauge method and adding it to the convex hull. The process of constructing the convex hull of the candidate points, although "interrupted" by the generation of new points, is exactly the same as if the convex hull was constructed all at once when all points are known. So either the problem-dependent complexity of the generation of nondominated points using the gauge method (9) or the overall complexity of the Beneath-Beyond Algorithm dominates the complexity of the complete approximation process.

Observe that it is not necessary to recalculate the gauge in each iteration of the approximation algorithm. As demonstrated in Theorem 3.2, the gauge can be evaluated cone by cone. The idea in our approach is to maximize the gauge separately in each cone. Thus we find a *candidate* $\bar{z}$ in every cone having a deviation $\operatorname{dev}(\bar{z}) := |\gamma(\bar{z}) - 1|$ associated with it. When a point is added to the convex hull, we have to keep track which facets (and therefore cones) are removed from the current approximation, and which facets are newly constructed. New candidates have to be generated only for all new cones.

Summarizing the discussion above, Figures 5 and 6 give an outline of the approximation algorithm for $\mathbb{R}^n_{\geqq}$-convex sets.

Using a gauge "induced by the problem" has several advantages. Usually the quality of an approximation is measured by some predefined gauge or norm, maybe a weighted l_p norm. The choice of the norm and the weights is very subjective and often difficult, especially if the problem criteria have different units, like time and distance. The choice of the weights additionally depends on the scaling of the criteria. In our approach, the quality of the

```
PROCEDURE: CONVEX APPROXIMATION
  Read/generate z^0, d^i, ε, maxConeNo
  for all d^i do
    Solve direction method
  end for
  Construct convex hull of the nondominated points and z^0
  Construct cones using the facets of the convex hull
  for all cones do
    Call CALCULATE CANDIDATE
  end for
  while #cones < maxConeNo and dev(next point) ≥ ε do
    Add next point using Beneath-Beyond technique
    Identify new and modified cones
    for all new or modified cones do
      Call CALCULATE CANDIDATE
    end for
  end while
  Output approximation
```

Figure 5: Pseudo code of the multicriteria approximation algorithm for an $\mathbb{R}^n_{\geqq}$-convex problem

```
PROCEDURE: CALCULATE CANDIDATE
  Solve gauge method to find z̄ and dev(z̄)
```

Figure 6: Finding a candidate in a cone for an $\mathbb{R}^n_{\geqq}$-convex problem

current approximation is estimated by the result of problem (7), or, more precisely, by the deviation $|\gamma(\hat{z}) - 1|$ of the next point to be added. As a stopping criterion, we check whether $|\gamma(\hat{z}) - 1| < \epsilon$ where $\epsilon > 0$. This condition does not depend on the scaling of the criteria; it depends only on the choice of the reference point z^0. There is no need for the decision maker to be concerned about the choice of norm, weights or scaling factors.

Additionally, the constructed gauge can be used to evaluate feasible points in $z^0 - \mathbb{R}^n_{\geqq}$. If we interpret a gauge of 1 as the maximal possible improvement over the point z^0, all feasible points in $z^0 - \mathbb{R}^n_{\geqq}$ have a gauge between 0 and 1 (assuming the current approximation is "good enough"). So the gauge of a point $\bar{z}$ can be interpreted as a measure of quality relative to the maximal achievable quality in the direction of $\bar{z}$.

5 Nonconvex Multicriteria Problems

In the case of an $\mathbb{R}^n_{\geqq}$-nonconvex problem the approximation algorithm given in Figure 5 generates an approximation of the convex hull of the nondominated set, see Figure 7 for an example. Note that the nondominated point $\bar{z}$ in Figure 7 cannot be found using the gauge method described in Figure 6.

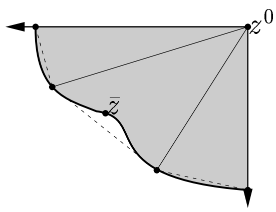


Figure 7: Finding a point in the nonconvex area

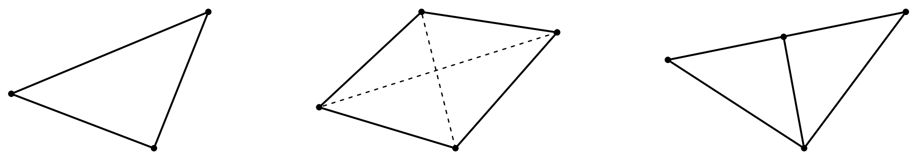
To overcome this difficulty in the $\mathbb{R}^n_{\geqq}$-nonconvex case, we switch to a different method, namely to the Tchebycheff method (see Steuer, 1986), in those cones where no significant improvement can be made with the gauge method. Moreover, while constructing the initial approximation and also in the updating phases in later stages of the algorithm, we do not use the Beneath-Beyond Algorithm since the generation of the convex hull of the candidate points is not suitable for the nonconvex areas of the feasible set Z. In conclusion, for $\mathbb{R}^n_{\geqq}$-nonconvex problems we proceed as follows.

Given a reference point z^0 (without loss of generality located at the origin), $k \geq n$ initial search directions and stopping parameters ϵ and `maxConeNo`, the direction method (8) is solved for each of the given directions. Then the cone(s) of the initial approximation can be constructed. For this purpose we use a projection of the generated points z^i, $i = 1, \ldots, k$

onto the facet of the l_1-norm in $-\mathbb{R}^n_{\geqq}$, i.e.

$$P(z) = -\frac{1}{\sum_{i=1}^{n} z_i} z, \qquad \forall i = 1, \dots, k.$$

Using this projection onto the $n-1$-dimensional hyperplane, initial cones can in general be defined in several different ways. Figure 8 shows different possibilities in $\mathbb{R}^2$. In order to avoid thin and elongated cones, we suggest the construction of the initial cones using Delauney tessellations of the projected point set (see, for example, Okabe *et al.* (1992)). The extreme points of the resulting Delauney simplices are used to define cones, which can equivalently be represented using the generated points $z^1, \dots, z^k$. Note that due to the fact that the cones are constructed based on the Delauney simplices, each cone has exactly n generators from the set $\{z^1, \dots, z^k\}$.



(a) One unique cone

(b) Two possible subdivisions

(c) Two unique cones

Figure 8: Constructing the initial cones

After constructing the initial cones, a new candidate has to be calculated for each cone. First, the gauge method (9) is used to search a candidate "outside" the current approximation. The deviation of the candidate is in this case implicitly given because the candidate's gauge is equal to the optimal objective value.

If no candidate with a deviation larger than the given stopping criterion $\epsilon > 0$ is found, the "inside" of the approximation is examined. Since using the direction method (8) with some search direction in the currently considered cone would further complicate the problem due to the fact that a point outside the cone may be found, we propose to use a heuristic based on the Tchebycheff method. To use the lexicographic Tchebycheff method, a local utopia point $\tilde{z}^*$ and a second point $\tilde{z}^\times$ defining the weights of the norm are needed, see Figure 9 for an example.

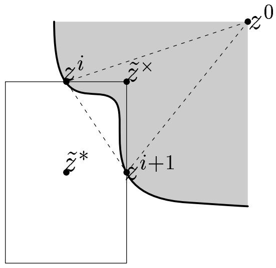


Figure 9: The Tchebycheff norm for a bicriteria example

Whereas in the bicriteria case $\tilde{z}^*$ and $\tilde{z}^\times$ can be chosen as the local utopia point and the local nadir point with respect to the generators z^i and z^{i+1} of the current cone, only the local utopia point $\tilde{z}^*$ directly generalizes to higher dimensional problems. If $\tilde{Z}^*$ is the set of generators of the cone and $|\tilde{Z}^*| = k$, the following points are reasonable choices for $\tilde{z}^\times$:

$$\tilde{z}_i^\times = \max\{z_i : z \in \tilde{Z}^*\} \tag{10}$$

$$\tilde{z}_i^\times = \min\{z_i : z \in \tilde{Z}^* \text{ and } \exists \bar{z} \in \tilde{Z}^* \text{ s.t. } \bar{z}_i < z_i\} \tag{11}$$

$$\tilde{z}_i^\times = \frac{1}{k} \sum_{z \in \tilde{Z}^*} z_i. \tag{12}$$

Choice (10) is similar to the choice of the default reference point, choice (11) selects the second smallest element for each coordinate, and choice (12) is the center of gravity of the generators of the current cone. Which of the points is the best to choose is probably related to the specific multicriteria problem. Numerical studies are needed to determine whether one of the choices is superior to the others. Another possibility is to solve the method for all points defined in (10) through (12), but this has the obvious drawback that a lot more calculations are necessary.

Given the point $\tilde{z}^{\times}$, the weights of the Tchebycheff norm are set to

$$w_i = \frac{1}{\tilde{z}_i^{\times} - \tilde{z}_i^*} \qquad i = 1, \dots, n,$$

and the lexicographic Tchebycheff method solved in the current cone is given by

$$\begin{aligned} \operatorname{lex\,min} \quad & \left(\|z - \tilde{z}^*\|_{\infty}^{w}, \|z - \tilde{z}^*\|_1\right) \\ \text{s. t.} \quad & z = \sum_{i=1}^{n} \lambda_i z^i \\ & \lambda_i \geq 0 \qquad i = 1, \dots, n \\ & z \in Z. \end{aligned} \tag{13}$$

The gauge (and therefore the deviation) of the candidate $\bar{z}$ is implicitly calculated because $\gamma(\bar{z}) = \sum_{i=1}^{n} \bar{\lambda}_i$ where $\bar{z} = \sum_{i=1}^{n} \bar{\lambda}_i z^i$ and $(\bar{z}, \bar{\lambda})$ is an optimal solution of (13).

After identifying a candidate in the cone by the gauge method (9) or the Tchebycheff method (13), we only know that the point is locally nondominated in the current cone, but there might be a dominating point outside of this cone. This is another complicating fact in the multicriteria case. Moreover, the generated point is in general not the point of worst approximation as this was the case with the gauge method in $\mathbb{R}^n_{\geqq}$-convex problems. For now, we simply accept such a candidate and proceed with the algorithm. When a stopping criterion is met, dominated points should be removed from the approximation before the output is given.

When candidates for all initial cones are found, we proceed with the algorithm as in the $\mathbb{R}^n_{\geqq}$-convex case, that is, the candidate with the largest deviation is added to the approximation by splitting the corresponding cone. The candidate and $n-1$ of the n generators of the cone are used to define a new cone. Given a cone with n generators, we thus have up to n possible new cones. For each of the new cones, we have to check whether the generators are linearly dependent in which case we omit the cone (it has an empty interior). Cones with linearly independent generators are added to the approximation, candidates for the new cones are calculated and the next iteration is started. Observe that the resulting approximation is not necessarily convex. However, calculating an approximation "similar to a norm" still yields the necessary information to evaluate the quality of the approximation in the considered cone.

The approximation algorithm is summarized in Figures 10 and 11.

```
PROCEDURE: NONCONVEX APPROXIMATION
  Read/generate z^0, d^i, ε, maxConeNo
  for all d^i do
    Solve direction method
  end for
  Transform points
  Construct Delauney tessellation
  Construct cones using the tessellation
  for all cones do
    Call CALCULATE CANDIDATE
  end for
  while #cones < maxConeNo and dev(next point) ≥ ε do
    Add next point
    Construct new cones
    for all new cones do
      Call CALCULATE CANDIDATE
    end for
  end while
  Output approximation
```

Figure 10: Pseudo code of the multicriteria approximation algorithm for an $\mathbb{R}^n_{\geqq}$-nonconvex problem

6 Discrete Multicriteria Problems

The general algorithm for the discrete multicriteria case follows the one for the $\mathbb{R}^n_{\geqq}$-nonconvex case given in Figure 10. Only for the procedure to calculate a candidate within a given cone (cf. Figure 11), an alternative to the lexicographic Tchebycheff method (13) can be given which is based on the generation of cutting planes.

PROCEDURE: CALCULATE CANDIDATE

Solve gauge method to find $\bar{z}$ and $\mathrm{dev}(\bar{z})$
if $\mathrm{dev}(\bar{z}) < \epsilon$ **then**
  Calculate $\tilde{z}^*$ and $\tilde{z}^\times$
  Use lexicographic Tchebycheff method to find $\bar{z}$
  Calculate $\mathrm{dev}(\bar{z})$
end if

Figure 11: Finding a candidate in a cone for
an $\mathbb{R}^n_{\geqq}$-nonconvex problem

The general idea of this approach is to cut off the generating points of the current cone by suitable cutting planes. Then the gauge method (9) can be applied in the new feasible set in order to find a new and possibly nondominated point inside the current cone. For this purpose, one cutting plane per generator of the current cone has to be constructed. Assume that the two hyperplanes of the cone containing a particular generator $\bar{z}$ are defined by the equations $\langle \mathfrak{n}_1, z \rangle = 0$ and $\langle \mathfrak{n}_2, z \rangle = 0$ where $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are the normals of the hyperplanes pointing to the outside of the cone. The right-hand side of both equations is zero, because both planes contain the reference point z^0 which is again assumed to be at the origin. We define an "intermediate" plane by using the normal $\bar{\mathfrak{n}} = \alpha \mathfrak{n}_1 + (1 - \alpha)\mathfrak{n}_2$ where $\alpha \in (0, 1)$, see Figure 12.

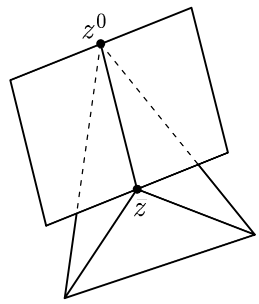
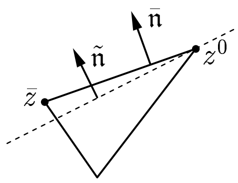


Figure 12: The intermediate plane containing the points z^0 and $\bar{z}$

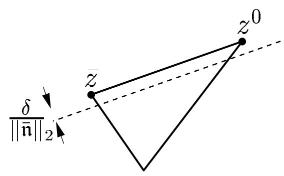
The plane defined by $\langle \bar{\mathfrak{n}}, z \rangle = 0$ contains the origin (and therefore the reference point z^0) and the point $\bar{z}$ because the right-hand side is zero and

$$\langle \bar{\mathfrak{n}}, \bar{z} \rangle = \alpha \langle \mathfrak{n}_1, \bar{z} \rangle + (1-\alpha) \langle \mathfrak{n}_2, \bar{z} \rangle = \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0.$$

In order to cut off $\bar{z}$, the intermediate plane must be either tilted or translated, see Figure 13. The plane can be tilted by using a normal $\tilde{\mathfrak{n}} = \delta \bar{z} + (1-\delta)\bar{\mathfrak{n}}$ where $\delta \in (0,1)$ is small enough not to cut off any other nondominated point, see Figure 13(a). The plane can be translated by changing the right-hand side of the defining equation to get $\langle \bar{\mathfrak{n}}, z \rangle = \delta$ where $\delta > 0$ is small enough not to cut off any other nondominated point, see Figure 13(b).



(a) Tilting the plane



(b) Translating the plane

Figure 13: Changing the cutting plane

None of the two methods has an obvious advantage and for both we have to estimate a suitable parameter δ. Obviously, this estimation is easier for the case where the variables are integer.

After constructing cutting planes for all generating points, the gauge method is applied in the cone with the cutting planes as additional constraints. To avoid cases where the gauge method searches "too deep" in the cone, an upper bound induced by one of the possible generalizations of the local nadir point given in (10) through (12) can be added. Assuming we construct n cutting planes using the "tilting method" and additionally use a point $\tilde{z}^{\times}$ as an upper bound, the problem to solve is

$$\begin{array}{lll} \max & \sum_{i=1}^{n} \lambda_i & \\ \text{s.t.} & z = \sum_{i=1}^{n} \lambda_i z^i & \\ & \lambda_i \geq 0 & i = 1, \ldots, n \\ & \langle \tilde{\mathrm{n}}^i, z \rangle \leq 0 & i = 1, \ldots, n \\ & z \leqq \tilde{z}^{\times} & \\ & z \in Z. & \end{array} \tag{14}$$

Similar to the lexicographic Tchebycheff method (13), the point generated by (14) is not necessarily globally nondominated. However, we can use this point as the candidate for the current cone and remove points that are not globally nondominated at the end of the approximation procedure.

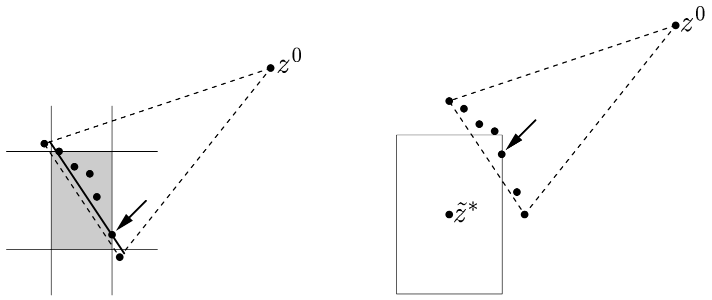
The candidate's deviation can be calculated directly from the optimal objective value of (14) which is equal to the candidate's gauge. The procedure to find a candidate using cutting planes is summarized in Figure 14.

Procedure: Calculate Candidate
Construct cutting plane for each generator
Calculate local nadir point
Construct new feasible set using the cone, the cutting planes and the local nadir point
Solve gauge method to find $\bar{z}$ and $\text{dev}(\bar{z})$

Figure 14: Finding a candidate in a cone for a discrete problem using cutting planes

If the procedure given in Figure 14 generates a point "outside" the current approximation, that is, a candidate with a deviation large enough, a point of worst approximation and a suitable point to add to the approximation is found. But if a point is obtained "inside" the approximation, it is actually a point of *best* approximation. Therefore it may happen that a cone is excluded from further consideration too early.

We can conclude that the approach using the Tchebycheff norm as well as the approach using cutting planes have disadvantages. In the former, two optimization problems have to be solved one of which is often NP-hard (see, for example, Warburton (1987) or Murthy and Her (1992)); the latter avoids NP-hard problems in some cases but cones might be excluded from further consideration prematurely because a point of best approximation is found when examining the "inside" of the approximation. While a premature termination is likely to take place using cutting planes whenever the algorithm examines the interior of the current approximation, such a termination is possible but less likely in the Tchebycheff approach in which, in general, we do not find a point of best approximation, but some other point. For an example see Figure 15 where the arrow points to the identified candidate.



(a) Cutting planes (b) Tchebycheff method

Figure 15: Finding a candidate in the interior of the current approximation

In effect, choosing one approach or the other is a problem-dependent task and has to be decided for the particular problem at hand.

7 Conclusions

In this paper we developed approximation algorithms for general multicriteria problems generalizing the ideas for the bicriteria case as given in Schandl *et al.* (1999). Due to numerous problems in higher dimensions which are present in general independently of the methodology, the developed approaches are more specialized than in the bicriteria case but preserve most of the properties of norm-based approximations in that case.

The presented approximation algorithms combine several desirable features. The most important and notable are:

- The approximation is improved in the area where "it is needed most" because in each iteration, the point of worst approximation is added whenever available.
- The algorithms are applicable even if the structure and convexity of the feasible set is unknown. Given the knowledge though that a problem is continuous and convex, more efficient versions can be applied.
- Using the approximation or a gauge induced by it to improve the approximation releases the decision maker from specifying preferences (in the form of weights, norms, or directions) to evaluate the quality of the approximation. Such preferences can be used in the initialization step (specifying the search directions) but apart from that the approximation is carried out in a very objective and neutral manner.

The algorithms yield a piecewise linear approximation of the nondominated set which can easily be visualized if not more than three criteria are present. For more criteria, plots of selected criteria against each other can be created. Such plots and the approximation in general should help the decision maker find a preferred solution within the nondominated set. While the approximation is carried out in an objective manner, the subjective preferences must be (and should be) applied to single out one (or several) final solution(s).

References

F. L. Bauer, J. Stoer, and C. Witzgall. Absolute and monotonic norms. *Numerische Mathematik* **3** 257–264 (1961).

H. P. Benson and S. Sayin. Towards finding global representations of the efficient set in multiple objective mathematical programming. *Naval Research Logistics* **44** 47–67 (1997).

E. Carrizosa, E. Conce, A. Pascual, and D. Romero-Morales. Closest solutions in ideal-point methods. In *Advances in Multiple Objective and Goal Programming*, (Edited by R. Caballero, F. Ruiz, and R. E. Steuer), pp. 274–281. Springer-Verlag, Berlin (1997).

P. Czyzak and A. Jaszkiewicz. Pareto Simulated Annealing—A metaheuristic technique for multiple-objective combinatorial optimization. *Journal of Multi-Criteria Decision Analysis* **7** 34–47 (1998).

I. Das. an improved technique for choosing parameters for pareto surface generation using normal-boundary intersection. In *Short Paper Proceedings of the Third World Congress of Structural and Multidisciplinary Optimization*, volume 2, pp. 411–413 (1999).

I. Das and J. Dennis. Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems. *SIAM Journal on Optimization* **8** 631–657 (1998).

H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Berlin (1987).

C. M. Fonseca and P. J. Fleming. An overview of evolutionary algorithms in multiobjective optimization. *Evolutionary Computation* **3** 1–16 (1995).

A. M. Geoffrion. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications* **22** (3) 618–630 (1968).

H. W. Hamacher and K. Klamroth. Planar location problems with barriers under polyhedral gauges. *Annals of Operations Research* (2000). To appear.

S. Helbig. Approximation of the efficient set by perturbation of the ordering cone. *ZOR – Methods and Models of Operations Research* **35** 197–220 (1991).

J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I.* Springer-Verlag, Berlin (1993a).

J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms II.* Springer-Verlag, Berlin (1993b).

J. Jahn and A. Merkel. Reference point approximation method for the solution of bicriterial nonlinear optimization problems. *Journal of Optimization Theory and Applications* **74** 87–103 (1992).

I. Kaliszewski. A modified weighted Tchebycheff metric for multiple objective programming. *Computers and Operations Research* **14** 315–323 (1987).

I. Kaliszewski. *Quantitative Pareto Analysis by Cone Separation Technique.* Kluwer Academic Publishers, Dordrecht (1994).

M. M. Kostreva, Q. Zheng, and D. Zhuang. A method for approximating solutions of multicriteria nonlinear optimization problems. *Optimization Methods and Software* **5** 209–226 (1995).

H. Minkowski. *Gesammelte Abhandlungen, Band 2.* Editor: D. Hilbert. Teubner Verlag, Leipzig und Berlin (1911). Also in: Chelsea Publishing Company, New York, 1967.

I. Murthy and S.-S. Her. Solving min-max shortest-path problems on a network. *Naval Research Logistics* **39** 669–683 (1992).

A. Okabe, B. Boots, and K. Sugihara. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* John Wiley & Sons Ltd., Chichester, England (1992).

A. Pascoletti and P. Serafini. Scalarizing vector optimization problems. *Journal of Optimization Theory and Applications* **42** 499–524 (1984).

E. Polak. On the approximation of solutions to multiple criteria decision making problems. In *Multiple Criteria Decision Making, Kyoto 1975*, (Edited by M. Zeleny), pp. 271–282. Springer-Verlag, Berlin, Germany (1976).

R. T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, NJ (1970).

B. Schandl. On some properties of gauges. Technical Report 662, Department of Mathematical Sciences, Clemson University, Clemson, SC (1998). Available at `http://www.math.clemson.edu/affordability/publications.html` (15.12.1999).

B. Schandl. *Norm-Based Evaluation and Approximation in Multicriteria Programming.* Ph.D. thesis, Clemson University, Clemson, SC (1999). Available at `http://www.math.clemson.edu/affordability/publications.html` (15.12.1999).

B. Schandl, K. Klamroth, and M. M. Wiecek. introducing oblique norms into multiple criteria programming. Technical Report 668, Department of Mathematical Sciences, Clemson University, Clemson, SC (1998). Submitted to *Journal of Global Optimization.*

B. Schandl, K. Klamroth, and M. M. Wiecek. Norm-based approximation in bicriteria programming. Technical report, Department of Mathematical Sciences, Clemson University, Clemson, SC (1999). Submitted to *Computational Optimization and Applications.*

I. M. Sobol' and Y. L. Levitan. Error estimates for the crude approximation of the trade-off curve. In *Multiple Criteria Decision Making*, (Edited by G. Fandel and T. Gal), pp. 83–92. Springer-Verlag, Berlin (1997).

R. B. Statnikov. Solution of multicriteria machines design problems on the basis of parameters space investigation. In *Multicriteria Decision-Making Problems*, (Edited by J. M. Gvishiani and S. V. Yemelyanov). Mashinostroyeniye, Moscow (1978). Russian.

R. E. Steuer. *Multiple Criteria Optimization: Theory, Computation, and Application.* Wiley, New York (1986).

R. E. Steuer and E. U. Choo. An interactive weighted Tchebycheff procedure for multiple objective programming. *Mathematical Programming* **26** 326–344 (1983).

E. L. Ulungu, J. Teghem, P. H. Fortemps, and D. Tuyttens. MOSA method: A tool for solving multiobjective combinatorial optimization problems. *Journal of Multi-Criteria Decision Analysis* **8** 221–236 (1999).

A. Warburton. Approximation of Pareto optima in multiple-objective, shortest-path problems. *Operations Research* **35** 70–79 (1987).

A. P. Wierzbicki. The use of reference objectives in multiobjective optimization. *Lecture Notes in Economics and Mathematical Systems* **177** 468–486 (1980).

P. L. Yu. A class of solutions for group decision problems. *Management Science* **19** 936–946 (1973).

M. Zeleny. Compromise programming. In *Multiple Criteria Decision Making*, (Edited by J. Cochrane and M. Zeleny), pp. 262–301. University of South Carolina, Columbia, SC (1973).